The Radiation Reaction during the Collapse of a Classical Electric Dipole

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1 Problem

Discuss the effect of the radiation reaction on the collapse of a classical electric dipole that is initially at rest.

For simplicity, consider electric charge e (which could be negative) together with a much heavier charge $-e$, which latter can be supposed to remain at rest at all times. You may also restrict the discussion to times when the velocity v of charge e is much less than the speed c of light in vacuum.

2 Solution

2.1 The Radiation Reaction

A survey by the author of the history of thoughts on the radiation reaction is at [1].

In sec. 120, p. 123 of [2], Lorentz (1892) approximated the retarded potentials of Lorenz [3] and Riemann [4] to order $1/c^3$ to deduce the self force on an accelerated, extended charge e with low velocity ($v \ll c$), finding in his eq. (111) the famous result (in Gaussian units),[1]

$$\mathbf{F}_{\rm self} = \frac{3e^2}{3c^3}\frac{d^2\mathbf{v}}{dt^2} = \frac{2e^2\ddot{\mathbf{v}}}{3c^2} \qquad (v \ll c). \tag{1}$$

That is, Lorentz considered the equation of motion for the charge, of mass m, when subject to an external force $\mathbf{F}_{\rm ext}$, to be

$$m\dot{\mathbf{v}} = \mathbf{F}_{\rm ext} + \mathbf{F}_{\rm self}. \tag{2}$$

Lorentz made no connection between this force and radiation at the time, which connection rather was made by Planck (1896) [5], who considered that there should be a damping force on an accelerated charge in reaction to its radiation, and by a clever transformation arrived at a "radiation-damping" force identical to eq. (1).[2] Subsequently, Abraham (1905), sec. 15

[1]Lorentz' form (1) is independent of the size of the classical charge. However, this form is only the first term in a series expansion that includes terms with even higher derivatives of the velocity together with powers of a characteristic length r.

[2]Planck's result was based on the approximation of electric-dipole radiation (that had been first analyzed by Hertz in 1889 [6]). In case of magnetic-dipole radiation (first analyzed by FitzGerald in 1883 [7, 8]), as for two like charges e moving in a circle of radius r, the radiation damping force is $\mathbf{F}_{\rm damping,M1} = -e^2 r^2 \dddot{\mathbf{v}}/6c^5$, as discussed in sec. 2.3 of [9].

of [10], extended Planck's argument to arbitrary velocity, find the radiation damping force to be

$$\mathbf{F}_{\rm damping} = \frac{2e^2}{3c^3}\left[\gamma^2\ddot{\mathbf{v}} + \frac{\gamma^4\mathbf{v}(\mathbf{v}\cdot\ddot{\mathbf{v}})}{c^2} + \frac{3\gamma^4\dot{\mathbf{v}}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2} + \frac{3\gamma^6\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^4}\right], \tag{3}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, which equals Lorentz' form (1) when $\mathbf{v} = 0$.

In 1908, Abraham, p. 387 of [11], and Von Laue [12] showed that a Lorentz transformation of the self force (1) from the rest frame of a charge to one in which it has velocity $\mathbf{v}$ yields the form (3), such that in general the self force and the radiation damping force on a classical charge are the same. In 1912, Schott, Appendix D of [13], extended Lorentz' derivation of the self force (via expansion of the retarded potentials) to arbitrary velocity, finding it to be the same as the damping force (3) in the leading approximation. Since the self force of Lorentz and Schott and the radiation-damping force of Planck and Abraham, appear to be the same (in suitable approximations), we now call both of them the radiation-reaction force.

In this note, we restrict our attention to the low-velocity radiation-reaction force, eq. (1).

Since the radiation-reaction force is very small compared to the "external" forces (in physically reasonable examples),[3] its effect on the motion is slight, and we approximate the velocity $\mathbf{v}$ that appears in eq. (1) by the velocity in the absence of the radiation reaction.

2.2 Freely Collapsing Dipole

Neglecting the radiation-reaction force, the low-velocity equation of motion for charge e, with mass m, of the electric dipole (with charge $-e$ assumed to be fixed at the origin) is,

$$m\dot{\mathbf{v}} = -\frac{e^2}{z^2}\,\hat{\mathbf{z}} \qquad (v \ll c), \tag{4}$$

taking charge e to lie along the positive z-axis.

The resulting velocity, for motion beginning at rest at position $z_0 > 0$, follows from conservation of energy (neglecting both radiation and the radiation reaction) as,

$$\mathbf{v}(z) = -\left|e\right|\sqrt{\frac{2}{m}\left(\frac{1}{z} - \frac{1}{z_0}\right)}\,\hat{\mathbf{z}}. \tag{5}$$

The time derivative of eq. (4) yields

$$\ddot{\mathbf{v}} = \frac{2e^2\,\mathbf{v}}{mz^3}\,, \tag{6}$$

which points toward charge $-e$ at the origin.

Approximating the radiation-reaction force as eq. (1) with $\ddot{\mathbf{v}}$ of eq. (6), we have

$$\mathbf{F}_{\rm radreact} = \frac{4e^4\,\mathbf{v}}{3mc^2z^3} = -\frac{4\left|e\right|^5}{3mc^2z^3}\sqrt{\frac{2}{m}\left(\frac{1}{z} - \frac{1}{z_0}\right)}\,\hat{\mathbf{z}} \qquad (v \ll c). \tag{7}$$

[3]We follow Planck, sec. 6 of [5], in regarding "runaway" solutions, that can occur formally when the radiation-reaction force exceeds the "external" force, as having no physical meaning (*keine Bedeutung*).

This force adds to the Coulomb force $\mathbf{F}_{\rm Coulomb} = -e^2\,\hat{\mathbf{x}}/x^2$, rather than opposing it as might have been expected from the interpretation of the radiation-reaction force as a damping force.

Is there something wrong with the above analysis (or with the concept of the radiation-reaction force)?

To help answer this question, it is useful to consider the electromagnetic fields, and the Poynting vector [16], of the electric dipole as it collapses.

For this, we approximate the initial size z_0 of the dipole as "small", and use the fields of a "point" electric dipole with arbitrary time dependence,[4]

$$\mathbf{E}(\mathbf{r},t) = \frac{([\ddot{\mathbf{p}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c^2 r} + \frac{3([\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - [\dot{\mathbf{p}}]}{c r^2} + \frac{3([\mathbf{p}]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - [\mathbf{p}]}{r^3}, \tag{8}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{[\ddot{\mathbf{p}}]\times\hat{\mathbf{r}}}{c^2 r} + \frac{[\dot{\mathbf{p}}]\times\hat{\mathbf{r}}}{c r^2}, \tag{9}$$

where $[\mathbf{p}(t)] = \mathbf{p}(t - r/c)$ is the retarded dipole moment. In the present example, the electric dipole moment, $\mathbf{p}(t) = ez(t)\,\hat{\mathbf{z}} = p(t)\,\hat{\mathbf{z}}$, and its time derivatives are all along the z-axis, so the fields (8)-(9) can be written as,

$$\mathbf{E}(\mathbf{r},t) = \frac{[\ddot{p}]\sin\theta}{c^2 r}\,\hat{\boldsymbol{\theta}} + (2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\left(\frac{[\dot{p}]}{c r^2} + \frac{[p]}{r^3}\right), \tag{10}$$

$$\mathbf{B}(\mathbf{r},t) = \left(\frac{[\ddot{p}]}{c^2 r} + \frac{[\dot{p}]}{c r^2}\right)\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{11}$$

The retarded derivatives $[\dot{p}]$ and $[\ddot{p}]$ are nonzero only for $t > r/c$, assuming that the collapse starts at time $t = 0$. The radial component of the Poynting vector is,

$$S_r(\mathbf{r},t) = \frac{cE_\theta B_\phi}{4\pi} = \frac{\sin^2\theta}{4\pi}\left(\frac{[\ddot{p}]^2}{c^3 r^2} + \frac{2[\dot{p}\,\ddot{p}]}{c^2 r^3} + \frac{[p\,\ddot{p} + \dot{p}^2]}{c r^4} + \frac{[p\,\dot{p}]}{r^5}\right). \tag{12}$$

Close to the dipole, the last term of eq. (12) dominates,

$$S_r(\mathbf{r}_{\rm small}, t > r/c) \approx \frac{[p\,\dot{p}]\sin^2\theta}{4\pi r^5} = -\frac{|e|^3\sin^2\theta}{4\pi r^5}\sqrt{\frac{2[z]}{m}\left(1 - \frac{[z]}{z_0}\right)} < 0. \tag{13}$$

Thus, close to the collapsing dipole, the flow of electromagnetic field energy is onto the dipole, rather than away from it as one might naïvely have expected.

In contrast, far from the dipole, where the first term of eq. (12) dominates, the flow of electromagnetic energy is outward, as expected for radiation into the far zone.

While one can say that the collapsing dipole "radiates" energy, this energy does not come from the dipole itself, but rather from the energy previously stored (at times $t < 0$) in the static field of the dipole. While the flow of energy in the far zone is associated with the acceleration of charge e, that acceleration does not "cause" the "radiation", in the sense that the radiated energy does not come from the accelerated charge.[5]

[4]Equations (8)-(9) are derived in the Appendix of [14]. See also [15].

[5]This disconcerting behavior of collapsing dipoles was first noted in [17] for the case of exponential decay of the dipole moment. See also [18] and the examples in [14].

Close to the collapsing dipole, where electromagnetic-field energy flows onto it, we now expect the reaction to this "radiation" to be an increase in the kinetic energy of the charge e, due to a radiation-reaction force that points toward the origin and adds to the inward Coulomb force, rather than opposing it.[6]

2.3 Comments

This example also illustrates an issue in the derivation of the radiation-reaction force.

As noted by Abraham in sec. 15 of [10], it seems natural to consider the radiation-reaction force to be equal and opposite to the rate of momentum radiated by an accelerated charge,[7]

$$\mathbf{F}_{\rm rad} = -\frac{d\mathbf{P}_{\rm rad}}{dt} = -\frac{dU_{\rm rad}}{dt}\frac{\mathbf{v}}{c^2} = - - \frac{2e^2\gamma^4\mathbf{v}}{3c^5}\left[\dot{\mathbf{v}}^2 + \gamma^2\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right]. \tag{14}$$

However, this goes to zero at low velocity (and does not depend on $\ddot{\mathbf{v}}$ as does eq. (1) of Lorentz and Planck). Abraham argued that via integrating by parts of the energy $\int \mathbf{F}_{\rm rad}\cdot\mathbf{v}\,dt$ delivered to an accelerated charge by $\mathbf{F}_{\rm rad}$, one finds the radiation-damping force (3), which reduces to eq. (1) in the low-velocity limit.[8]

Thus, it appears that a transformation of $\mathbf{F}_{\rm rad}$ of eq. (14) into eq. (3), rather than $\mathbf{F}_{\rm rad}$ itself, is actually the radiation-reaction force. This result has always been somewhat disconcerting. The present example reminds us that the quantities $dU_{\rm rad}/dt$ and $d\mathbf{P}_{\rm rad}/dt$ refer to energy and momentum in the far zone, and do not necessarily reflect the changes in electromagnetic-field energy and momentum close to the accelerated source charge.[9]

Hence, we should not expect that eq. (14) correctly describes the reaction of the charge to the radiation process.

It remains impressive that the "trick" of integration by parts (first applied to the radiation reaction by Planck [5]) does seem to yield a valid result, starting from a doubtful one.

Perhaps we should close by noting with Wigner [24] "the unreasonable effectiveness of mathematics in the natural sciences".

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[6] This behavior constitutes a violation of Lenz' law [23], if one considers the radiation reaction to be an "induced effect".

[7] The expression for $dU_{\rm rad}/dt$, for the total power radiated into the far zone, had been deduced by Liénard (1898) in eq. (21) of [19], and by Heaviside (1902) in eq. (10) of [20].

[8] Abraham's argument is transcribed into English in sec. 14 of [1].

[9] Another such example is the famous case of a uniformly accelerated charge (first studied by Born [21], and greatly clarified by Schott [13, 22]), for which the radiation-reaction force (3) vanishes, while $dU_{\rm rad}/dt$ crossing a sphere surrounding the charge does not. Since no electromagnetic field energy flows away from the charge itself, one might say that the charge does not "radiate", but a distant observer at rest detects "radiation" = flow of electromagnetic into his/her detector and therefore says that the accelerated charge is associated with "radiation".

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