1 Problem

Some biological systems consist of two “phases” of nearly square fiber bundles of differing thermal and electrical conductivities. Consider a circular region of radius $a$ near a corner of such a system as shown below.

Phase 1, with electrical conductivity $\sigma_1$, occupies the “bowtie” region of angle $\pm \alpha$, while phase 2, with conductivity $\sigma_2 \ll \sigma_1$, occupies the remaining region.

Deduce the approximate form of lines of current density $J$ when a background electric field is applied along the symmetry axis of phase 1. What is the effective conductivity $\sigma$ of the system, defined by the relation $I = \sigma \Delta \phi$ between the total current $I$ and the potential difference $\Delta \phi$ across the system?

It suffices to consider the case that the boundary arc ($r = a, |\theta| < \alpha$) is held at electric potential $\phi = 1$, while the arc ($r = a, \pi - \alpha < |\theta| < \pi$) is held at electric potential $\phi = -1$, and no current flows across the remainder of the boundary.

Hint: When $\sigma_2 \ll \sigma_1$, the electric potential is well described by the leading term of a series expansion.

2 Solution

The series expansion approach is unsuccessful in treating the full problem of a “checkerboard” array of two phases if those phases meet in sharp corners as shown above. However, an analytic form for the electric potential of a two-phase (and also a four-phase) checkerboard can be obtained using conformal mapping of certain elliptic functions [1]. If the regions of one phase are completely surrounded by the other phase, rather lengthy series expansions
for the potential can be given [2]. The present problem is based on work by Grimvall [3] and Keller [4].

In the steady state, the electric field obeys $\nabla \times \mathbf{E} = 0$, so that $\mathbf{E}$ can be deduced from a scalar potential $\phi$ via $\mathbf{E} = -\nabla \phi$. The steady current density obeys $\nabla \cdot \mathbf{J} = 0$, and is related to the electric field by Ohm’s law, $\mathbf{J} = \sigma \mathbf{E}$. Hence, within regions of uniform conductivity, $\nabla \cdot \mathbf{E} = 0$ and $\nabla^2 \phi = 0$. Thus, we seek solutions to Laplace’s equations in the four regions of uniform conductivity, subject to the stated boundary conditions at the outer radius, as well as the matching conditions that $\phi$, $E_\parallel$, and $j_\perp$ are continuous at the boundaries between the regions.

We analyze this two-dimensional problem in a cylindrical coordinate system $(r, \theta)$ with origin at the corner between the phases and $\theta = 0$ along the radius vector that bisects the region whose potential is unity at $r = a$. The four regions of uniform conductivity are labeled $I$, $II$, $III$ and $IV$ as shown below.

\[
\begin{align*}
\frac{\partial \phi}{\partial r} &= 0 \\
\phi &= -1 \\
\phi &= 1
\end{align*}
\]

Since $\mathbf{J}_\perp = J_r = \sigma E_r = -\sigma \partial \phi / \partial r$ at the outer boundary, the boundary conditions at $r = a$ can be written

\[
\begin{align*}
\phi_I(r = a) &= 1, \\
\frac{\partial \phi_{II}(r = a)}{\partial r} &= \frac{\partial \phi_{IV}(r = a)}{\partial r} = 0, \\
\phi_{III}(r = a) &= -1.
\end{align*}
\]

Likewise, the condition that $j_\perp = j_\theta = \sigma E_\theta = -(\sigma / r) \partial \phi / \partial \theta$ is continuous at the boundaries between the regions can be written

\[
\begin{align*}
\sigma_1 \frac{\partial \phi_I(\theta = \alpha)}{\partial \theta} &= \sigma_2 \frac{\partial \phi_{II}(\theta = \alpha)}{\partial \theta}, \\
\sigma_1 \frac{\partial \phi_{III}(\theta = \pi - \alpha)}{\partial \theta} &= \sigma_2 \frac{\partial \phi_{II}(\theta = \pi - \alpha)}{\partial \theta},
\end{align*}
\]

etc.

From the symmetry of the problem we see that

\[
\phi(-\theta) = \phi(\theta),
\]
\[ \phi(\pi - \theta) = -\phi(\theta), \]  

(7)

and in particular \( \phi(r = 0) = 0 = \phi(\theta = \pm \pi/2) \).

We recall that two-dimensional solutions to Laplace’s equations in cylindrical coordinates involve sums of products of \( r^{\pm k} \) and \( e^{\pm ik\theta} \), where \( k \) is the separation constant that in general can take on a sequence of values. Since the potential is zero at the origin, the radial function is only \( r^k \). The symmetry condition (6) suggests that the angular functions for region I be written as \( \cos k\theta \), while the symmetry condition (7) suggests that we use \( \sin k(\pi/2 - |\theta|) \) in regions II and IV and \( \cos k(\pi - \theta) \) in region III. That is, we consider the series expansions

\[
\begin{align*}
\phi_I &= \sum A_k r^k \cos k\theta, \\
\phi_{II} = \phi_{IV} &= \sum B_k r^k \sin \left( \frac{\pi}{2} - |\theta| \right), \\
\phi_{III} &= -\sum A_k r^k \cos (\pi - \theta).
\end{align*}
\]

The potential must be continuous at the boundaries between the regions, which requires

\[ A_k \cos k\alpha = B_k \sin k\left( \frac{\pi}{2} - \alpha \right) . \]

(11)

The normal component of the current density is also continuous across these boundaries, so eq. (4) tells us that

\[ \sigma_1 A_k \sin k\alpha = \sigma_2 B_k \cos \left( \frac{\pi}{2} - \alpha \right). \]

(12)

On dividing eq. (12) by eq. (11) we find that

\[ \tan k\alpha = \frac{\sigma_2}{\sigma_1} \cot k\left( \frac{\pi}{2} - \alpha \right) . \]

(13)

There is an infinite set of solutions to this transcendental equation. When \( \sigma_2/\sigma_1 \ll 1 \) we expect that only the first term in the expansions (8)-(9) will be important, and in this case we expect that both \( k\alpha \) and \( k(\pi/2 - \alpha) \) are small. Then eq. (13) can be approximated as

\[ k\alpha \approx \frac{\sigma_2/\sigma_1}{k(\pi/2 - \alpha)} ; \]

(14)

and hence

\[ k^2 \approx \frac{\sigma_2/\sigma_1}{\alpha(\pi/2 - \alpha)} \ll 1. \]

(15)

Equation (11) also tells us that for small \( k\alpha \),

\[ A_k \approx B_k k \left( \frac{\pi}{2} - \alpha \right) . \]

(16)

Since we now approximate \( \phi_I \) by the single term \( A_k r^k \cos k\theta \approx A_k r^k \), the boundary condition (1) at \( r = a \) implies that

\[ A_k \approx \frac{1}{a^k} , \]

(17)
and eq. (16) then gives

$$B_k \approx \frac{1}{ka^k(\frac{\pi}{2} - \alpha)} \gg A_k. \quad (18)$$

The boundary condition (2) now becomes

$$0 = kB_k a^{r-1} \sin k\left(\frac{\pi}{2} - \theta\right) \approx k\frac{\pi}{2} - \theta \frac{a^{\theta}}{a^{\frac{\pi}{2} - \alpha}}, \quad (19)$$

which is approximately satisfied for small $k$.

So we accept the first terms of eqs. (8)-(10) as our solution, with $k, A_k$ and $B_k$ given by eqs. (15), (17) and (18).

In region $I$ the electric field is given by

$$E_r = -\frac{\partial \phi}{\partial r} \approx -k\frac{r^{k-1}}{a^k} \cos k\theta \approx -k\frac{r^{k-1}}{a^k}, \quad (20)$$

$$E_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} \approx k\frac{r^{k-1}}{a^k} \sin k\theta \approx k^2 r^{k-1} \frac{r^{k-1}}{a^k}. \quad (21)$$

Thus, in region $I$, $E_\theta/E_r \approx k\theta \ll 1$, so the electric field, and the current density, is nearly radial. In region $II$ the electric field is given by

$$E_r = -\frac{\partial \phi_{II}}{\partial r} \approx -k\frac{r^{k-1}}{ka^k(\frac{\pi}{2} - \alpha)} \sin k\left(\frac{\pi}{2} - \theta\right) \approx -k\frac{r^{k-1}}{a^k} \frac{\pi}{2} - \theta, \quad (22)$$

$$E_\theta = -\frac{1}{r} \frac{\partial \phi_{II}}{\partial \theta} \approx k\frac{r^{k-1}}{ka^k(\frac{\pi}{2} - \alpha)} \cos k\left(\frac{\pi}{2} - \theta\right) \approx k^2 r^{k-1} \frac{\pi}{2} - \alpha. \quad (23)$$

Thus, in region $II$, $E_r/E_\theta \approx k(\pi/2 - \theta) \ll 1$, so the electric field, and the current density, is almost purely azimuthal.

The current density $\mathbf{J}$ follows the lines of the electric field $\mathbf{E}$, and therefore behaves as sketched below:

The total current can be evaluated by integrating the current density at $r = a$ in region $I$:

$$I = 2a \int_0^\alpha J_r d\theta = 2a \sigma_1 \int_0^\alpha E_r(r = a) d\theta \approx -2k\sigma_1 \int_0^\alpha d\theta = -2k\sigma_1 \alpha \approx -2 \sqrt{\frac{\sigma_1 \sigma_2 \alpha}{\pi/2 - \alpha}}. \quad (24)$$
In the present problem the total potential difference $\Delta \phi$ is $-2$, so the effective conductivity is

$$\sigma = \frac{I}{\Delta \phi} = \sqrt{\frac{\sigma_1 \sigma_2 \alpha}{\frac{\pi}{2} - \alpha}}. \tag{25}$$

For a square checkerboard, $\alpha = \pi/4$, and the effective conductivity is $\sigma = \sqrt{\sigma_1 \sigma_2}$. It turns out that this result is independent of the ratio $\sigma_2/\sigma_1$, and holds not only for the corner region studied here but for the entire checkerboard array [5].

References


