

Potentials for a Rectangular Electromagnetic Cavity

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(March 4, 2011; updated August 23, 2012)

1 Problem

Deduce scalar and vector potentials relevant to electromagnetic modes in a rectangular cavity of dimensions $d_x \geq d_y \geq d_z$, assuming the walls to be perfect conductors, and the interior of the cavity to be vacuum.

2 Solution

For the case of a cylindrical cavity, see [1].

2.1 E and B Fields of the Cavity Modes

The cavity has extent $0 < x < d_x$, $0 < y < d_y$, and $0 < z < d_z$. The electric field must be everywhere perpendicular to the (perfectly conducting) walls, such that for time dependence $e^{-i\omega t}$ there exists a set of modes with non-negative integer indices $\{l, m, n\}$ of the form

$$E_x = E_0 e_x \cos k_x x \sin k_y y \sin k_z z e^{-i\omega t}, \quad (1)$$

$$E_y = E_0 e_y \sin k_x x \cos k_y y \sin k_z z e^{-i\omega t}, \quad (2)$$

$$E_z = E_0 e_z \sin k_x x \sin k_y y \cos k_z z e^{-i\omega t}, \quad (3)$$

where $\hat{\mathbf{e}} = (e_x, e_y, e_z)$ is a unit vector, the wave vector \mathbf{k} is given by

$$\mathbf{k} = (k_x, k_y, k_z) = \pi \left(\frac{l}{d_x}, \frac{m}{d_y}, \frac{n}{d_z} \right), \quad (4)$$

and at most only one of indices l , m , or n is zero. These fields obey the free-space wave equation

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{\omega^2}{c^2} \mathbf{E}, \quad (5)$$

where c is the speed of light in vacuum, which implies that

$$\omega = kc = \pi c \sqrt{\frac{l^2}{d_x^2} + \frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}}. \quad (6)$$

The first (free-space) Maxwell equation, $\nabla \cdot \mathbf{E} = 0$ implies that $\hat{\mathbf{e}} \cdot \mathbf{k} = 0$, so that there are two orthogonal “polarizations” $\hat{\mathbf{e}}$ for each set of indices $\{l, m, n\}$.¹

The magnetic field is related to the electric field by Faraday’s law (in Gaussian units),

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial ct} = ik\mathbf{B}, \quad (8)$$

such that

$$B_x = iE_0 b_x \sin k_x x \cos k_y y \cos k_z z e^{-i\omega t}, \quad (9)$$

$$B_y = iE_0 b_y \cos k_x x \sin k_y y \cos k_z z e^{-i\omega t}, \quad (10)$$

$$B_z = iE_0 b_z \cos k_x x \cos k_y y \sin k_z z e^{-i\omega t}, \quad (11)$$

where $\hat{\mathbf{b}}$ is the unit vector

$$\hat{\mathbf{b}} = \hat{\mathbf{e}} \times \hat{\mathbf{k}} = \frac{1}{k}(e_y k_z - e_z k_y, e_z k_x - e_x k_z, e_x k_y - e_y k_x). \quad (12)$$

The magnetic field is everywhere tangential to the walls of the cavity (which motivated the use of the cosine functions in the electric field (1)-(3)). Thus, $\hat{\mathbf{b}} \cdot \mathbf{k} \propto \det(\hat{\mathbf{e}}, \mathbf{k}, \mathbf{k}) = 0$, consistent with the third Maxwell equation, $\nabla \cdot \mathbf{B} = 0$. Also, $\hat{\mathbf{e}} \cdot \hat{\mathbf{b}} \propto \det(\hat{\mathbf{e}}, \hat{\mathbf{e}}, \mathbf{k}) = 0$, such that for each mode the vectors \mathbf{E} , \mathbf{B} and \mathbf{k} form a mutually orthogonal triad, with

$$\hat{\mathbf{e}} = \hat{\mathbf{b}} \times \hat{\mathbf{k}} = \frac{1}{k}(b_y k_z - b_z k_y, b_z k_x - b_x k_z, b_x k_y - b_y k_x). \quad (13)$$

2.2 Potentials

The electromagnetic fields \mathbf{E} and \mathbf{B} can be related to scalar and vector potentials V and \mathbf{A} according to

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla V + ik\mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (14)$$

In sec. 2.1 we deduced the electromagnetic fields inside the cavity, but did not comment on their values outside it. While it seems most reasonable to consider the case that \mathbf{E} and \mathbf{B} are zero outside the cavity, we can also imagine the case of periodic boundary conditions, in which all space is filled with an infinite collection of cavities similar to the specified one.

¹The electric field can be regarded as the superposition of eight plane waves,

$$\begin{aligned} \mathbf{E} = & -\frac{E_0}{8} \left[(e_x, e_y, e_z) e^{i(k_x x + k_y y + k_z z - \omega t)} + (-e_x, e_y, e_z) e^{i(-k_x x + k_y y + k_z z - \omega t)} \right. \\ & - (e_x, -e_y, e_z) e^{i(k_x x - k_y y + k_z z - \omega t)} + (-e_x, -e_y, e_z) e^{i(-k_x x - k_y y + k_z z - \omega t)} \\ & - (e_x, e_y, -e_z) e^{i(k_x x + k_y y - k_z z - \omega t)} - (-e_x, e_y, -e_z) e^{i(-k_x x + k_y y - k_z z - \omega t)} \\ & \left. + (e_x, -e_y, -e_z) e^{i(k_x x - k_y y - k_z z - \omega t)} + (-e_x, -e_y, -e_z) e^{i(-k_x x - k_y y - k_z z - \omega t)} \right], \quad (7) \end{aligned}$$

with a similar relation holding for the magnetic field.

2.2.1 Hamiltonian Gauge

A simple option for the potentials is to adopt the so-called Hamiltonian gauge in which the scalar potential is everywhere zero (see, for example, sec. 8 of [2]),^{2,3}

$$V = 0, \quad \mathbf{A} = -\frac{i\mathbf{E}}{k} = \begin{cases} -\frac{iE_0}{k} J_0(kr) e^{-i\omega t} \hat{\mathbf{z}} & \text{(inside),} \\ 0 & \text{(outside).} \end{cases} \quad (15)$$

Then, $\nabla \times \mathbf{A} = \mathbf{B}$ is confirmed by use of Faraday's law, eq. (8).

This vector potential is not continuous on the planar faces of the cavity. However, this is not a formal problem in that the computation $\mathbf{B} = \nabla \times \mathbf{A}$ next to the surface does not involve derivatives normal to that surface.⁴

2.2.2 Poincaré Gauge

In cases where the \mathbf{E} and \mathbf{B} fields are known, we can compute the potentials in the so-called Poincaré gauge (see sec. 9A of [2] and [6, 7]),⁵

$$V(\mathbf{x}, t) = -\mathbf{x} \cdot \int_0^{u_0=1} du \mathbf{E}(u\mathbf{x}, t), \quad \mathbf{A}(\mathbf{x}, t) = -\mathbf{x} \times \mathbf{I}(\mathbf{B}), \quad (16)$$

where

$$\mathbf{I}(\mathbf{F}) = \int_0^{u_0=1} u du \mathbf{F}(u\mathbf{x}, t) = u_0 \mathbf{G}(u_0\mathbf{x}, t) - \int_0^{u_0=1} du \mathbf{G}(u\mathbf{x}, t) \quad (17)$$

and

$$\mathbf{F}(u\mathbf{x}, t) = \frac{d\mathbf{G}(u\mathbf{x}, t)}{du}. \quad (18)$$

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.⁶

²This gauge appears to have been first used by Gibbs in 1896 [3].

³For a static electric field the Hamiltonian-gauge vector potential is $\mathbf{A} = -c(t - t_0)\mathbf{E}$, while for a static magnetic field the vector potential is the same as that in the Coulomb gauge (and also in the Lorenz gauge).

⁴In Hamiltonian dynamics of a particle with charge q the normal (z) component of the canonical momentum $\mathbf{p} = \mathbf{p}_{\text{mech}} + q\mathbf{A}/c$ takes a discontinuous step when a particle enters or exits the rf cavity through the planar faces. This undesirable feature can be mitigated by switching from coordinates (x, y, z) with independent variable t to coordinates (x, y, t) with independent variable z , in which case the canonical momentum of the t -coordinate is $p_t = -E_{\text{mech}} - qV$ (see, for example, sec. 1.6 of [5]), which is just $-E_{\text{mech}}$ in the Hamiltonian gauge. Then, if the faces of the cavity traversed by particles are at constant z , all three canonical momenta p_x , p_y and p_t are continuous. *Only if the particles are muons would it be considered practical to used closed rf cavities for their acceleration.*

⁵The Poincaré gauge is also called the multipolar gauge [8].

⁶The potentials in the Poincaré gauge depend on the choice of origin. If the origin is inside the region of electromagnetic fields, then the Poincaré potentials are nonzero throughout all space. If the origin is to one side of the region of electromagnetic fields, then the Poincaré potentials are nonzero only inside that region, and in the region on the "other side" from the origin.

For points \mathbf{x} outside the cavity, we restrict the calculation to the case that the vector \mathbf{x} passes through the cavity wall at $z = d_z$, along which $\mathbf{E}(u\mathbf{x})$ and $\mathbf{B}(u\mathbf{x})$ are nonzero only for $u < u_0 = d_z/z$.

Using integrals 2.533.4-5 of [9] we have that

$$\begin{aligned} -x \int_0^{u_0} du E_x &= -xE_0 e_x e^{-i\omega t} \int_0^{u_0} du \cos k_x u x \sin k_y u y \sin k_z u z \\ &= \frac{x E_0 e_x}{4} e^{-i\omega t} \left[\frac{\sin(k_x x + k_y y + k_z z) u_0}{k_x x + k_y y - k_z z} + \frac{\sin(-k_x x + k_y y + k_z z) u_0}{-k_x x + k_y y + k_z z} \right. \\ &\quad \left. - \frac{\sin(k_x x - k_y y + k_z z) u_0}{k_x x - k_y y + k_z z} - \frac{\sin(k_x x + k_y y - k_z z) u_0}{k_x x + k_y y - k_z z} \right], \end{aligned} \quad (19)$$

where $u_0 = 1$ for \mathbf{x} inside the cavity, and $u_0 = d_z/z$ for \mathbf{x} outside the cavity such that vector \mathbf{x} passes through the cavity face at $z = d_z$. Similar expressions hold for the terms $-yI_y(\mathbf{E})$ and $-zI_z(\mathbf{E})$ of the scalar potential V . At large $|\mathbf{x}|$ these terms fall off as $1/|\mathbf{x}|^2$, so the scalar potential, $V = -\mathbf{x} \cdot \mathbf{I}(\mathbf{E})$, has a dipole character.

For the vector potential we have that

$$\begin{aligned} I_x(\mathbf{B}) &= iE_0 b_x e^{-i\omega t} \int_0^{u_0} u du \sin k_x u x \cos k_y u y \cos k_z u z \\ &= \frac{iE_0 b_x}{4} e^{-i\omega t} \left[-\frac{u_0 \cos(k_x x + k_y y + k_z z) u_0}{k_x x + k_y y + k_z z} + \frac{u_0 \cos(-k_x x + k_y y + k_z z) u_0}{-k_x x + k_y y + k_z z} \right. \\ &\quad - \frac{u_0 \cos(k_x x - k_y y + k_z z) u_0}{k_x x - k_y y + k_z z} - \frac{u_0 \cos(k_x x + k_y y - k_z z) u_0}{k_x x + k_y y - k_z z} \\ &\quad + \frac{\sin(k_x x + k_y y + k_z z) u_0}{(k_x x + k_y y + k_z z)^2} - \frac{\sin(-k_x x + k_y y + k_z z) u_0}{(-k_x x + k_y y + k_z z)^2} \\ &\quad \left. + \frac{\sin(k_x x - k_y y + k_z z) u_0}{(k_x x - k_y y + k_z z)^2} + \frac{\sin(k_x x + k_y y - k_z z) u_0}{(k_x x + k_y y - k_z z)^2} \right], \end{aligned} \quad (20)$$

and similarly for $I_y(\mathbf{B})$ and $I_z(\mathbf{B})$. These integrals fall off at large $|\mathbf{x}|$ as $1/|\mathbf{x}|^2$, so the vector potential, $\mathbf{A} = -\mathbf{x} \times \mathbf{I}(\mathbf{B})$, falls off as $1/|\mathbf{x}|$.

2.2.3 Lorenz Gauge

In the Lorenz gauge the potentials are related by

$$\nabla \cdot \mathbf{A} = \frac{1}{c} \frac{\partial V}{\partial t} = ikV, \quad (21)$$

where the latter form holds for time dependence $e^{-i\omega t}$. Then, the potentials obey the Helmholtz wave equations

$$(\nabla^2 + k^2)\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}, \quad (\nabla^2 + k^2)V = -\frac{4\pi}{c}\rho, \quad (22)$$

whose solutions are the retarded potentials in the frequency domain,

$$V(\mathbf{x}) = \int \frac{\rho(\mathbf{x}') e^{ikr}}{r} d\text{Vol}', \quad \mathbf{A}(\mathbf{x}) = \int \frac{\mathbf{J}(\mathbf{x}') e^{ikr}}{cr} d\text{Vol}'. \quad (23)$$

For a cavity with perfectly conducting walls, the only charge and current densities reside on these walls, so the retarded potentials have the form

$$V(\mathbf{x}) = \int \frac{\sigma(\mathbf{x}') e^{ikr}}{r} d\text{Area}' = \int \frac{\mathbf{E}(\mathbf{x}') \cdot \hat{\mathbf{n}}' e^{ikr}}{4\pi r} d\text{Area}', \quad (24)$$

$$\mathbf{A}(\mathbf{x}) = \int \frac{\mathbf{K}(\mathbf{x}') e^{ikr}}{cr} d\text{Area}' = \int \frac{\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}') e^{ikr}}{4\pi r} d\text{Area}', \quad (25)$$

where σ and \mathbf{K} are the surface charge and current densities, and $\hat{\mathbf{n}}'$ is the inward unit vector normal to the bounding surface. These potentials are nonzero both inside and outside of the cavity.

It does not seem possible to give analytic expressions for these potentials in the present example.

An argument in Prob. 14.2 of [4] leads to different form for the potentials than eqs. (24)-(25).

Inside a rectangular cavity the wave equation (21) for the vector potential is simply

$$(\nabla^2 + k^2)\mathbf{A} = 0, \quad (26)$$

which permits solutions of the form

$$A_j = \begin{Bmatrix} \cos k_x x \\ \sin k_x x \end{Bmatrix} \begin{Bmatrix} \cos k_y y \\ \sin k_y y \end{Bmatrix} \begin{Bmatrix} \cos k_z z \\ \sin k_z z \end{Bmatrix}. \quad (27)$$

We can consider the special case that the vector potential has only a z -component. Then,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{1}{c} \frac{\partial V}{\partial t} = ikV, \quad V = -\frac{i}{k} \frac{\partial A_z}{\partial z}, \quad (28)$$

and the electric and magnetic fields follow from the potentials as

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = \frac{i}{k} \nabla \frac{\partial A_z}{\partial z} + ikA_z \hat{\mathbf{z}}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad (29)$$

$$E_x = \frac{i}{k} \frac{\partial^2 A_z}{\partial x \partial z}, \quad E_y = \frac{i}{k} \frac{\partial^2 A_z}{\partial y \partial z}, \quad E_z = \frac{i}{k} \left(\frac{\partial^2 A_z}{\partial z^2} + k^2 A_z \right), \quad (30)$$

$$B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x}, \quad B_z = 0. \quad (31)$$

It seems reasonable to complete the solution by enforcing the boundary conditions that the electric field must be normal, and the magnetic field must be tangential, to the perfectly conducting walls of the cavity. Note, however, that these boundary conditions involve combinations of derivatives of V and components of \mathbf{A} , so it is not obvious that the solution for the potentials obtained using them will be unique, even if the \mathbf{E} and \mathbf{B} are satisfactory.

The potentials

$$A_z = -\frac{iE_0}{\sqrt{k_x^2 + k_y^2}} \sin k_x x \sin k_y y \cos k_z z, \quad V = E_0 \frac{k_z}{k\sqrt{k_x^2 + k_y^2}} \sin k_x x \sin k_y y \sin k_z z, \quad (32)$$

generate the fields (3) and (11) for which $b_z = 0$. The unit vectors $\hat{\mathbf{e}}$ and $\hat{\mathbf{b}}$ have components

$$\begin{aligned} e_x &= -\frac{k_x k_z}{k\sqrt{k_x^2 + k_y^2}}, & e_y &= -\frac{k_y k_z}{k\sqrt{k_x^2 + k_y^2}}, & e_z &= \frac{\sqrt{k_x^2 + k_y^2}}{k}, \\ b_x &= \frac{k_y}{\sqrt{k_x^2 + k_y^2}}, & b_y &= -\frac{k_x}{\sqrt{k_x^2 + k_y^2}}, & b_z &= 0. \end{aligned} \quad (33)$$

However, when $k_z = 0$ we have that $V = 0$ (inside the cavity) according to eq. (32), whereas the surface charge density σ has opposite signs on the walls at $z = 0$ and $z = d_z$, and is independent of d_z , such that the retarded potential (24) is nonzero. For example, consider a high mode with odd indices l and m of a long, thin cavity where $kd_z = \sqrt{k_x^2 + k_y^2} d_z \gg 1$ and $d_x, d_y \ll d_z$. Then, for a point inside the cavity far from both ends ($z \gg 1/k$ and $d_z - z \gg 1/k$) the retarded potential (24) is

$$V(\mathbf{x}) \approx Q \left(\frac{e^{ikz}}{z} - \frac{e^{ik(d_z-z)}}{d_z - z} \right) \neq 0, \quad (34)$$

where $Q = E_0 d_x d_y / \pi^2 l m$ is the peak charge on the face $z = 0$. This potential also holds for points outside the cavity such that $\sqrt{x^2 + y^2}$ is small compared to $z \gg 1/k$ and $d_z - z \gg 1/k$. Only on the plane $z = d_z/2$ does $V = 0$. Then, the Lorenz-gauge vector potential in this region follows as

$$\begin{aligned} A_z(\mathbf{x}) &= -\frac{i}{k} \left(\frac{\partial V}{\partial z} + E_z \right) \approx -\frac{iQ}{k} \left[\left(ik - \frac{1}{z} \right) \frac{e^{ikz}}{z} + \left(ik - \frac{1}{d_z - z} \right) \frac{e^{ik(d_z-z)}}{d_z - z} \right] \\ &\quad + \begin{cases} -\frac{iE_0}{k} \sin k_x x \sin k_y y & (\text{inside}), \\ 0 & (\text{outside}). \end{cases} \end{aligned} \quad (35)$$

Similar approximations can be given for the (nonzero) Lorenz-gauge potentials outside the cavity for $z \ll 1/k$ and $z - d_z \gg 1/k$.

While the potentials (32) do generate the fields (3) and (11), it is not clear that they can be regarded as the Lorenz-gauge potentials. This brings into question their use in sec. 14.2 of [4] in computing the strength of excitation of the fundamental cavity mode by a passing charged particle.

A Appendix: Patch Antenna ‘‘Cavity’’

A ‘‘patch’’ antenna consists of a rectangular (or circular, *etc.*) conductor (the ‘‘patch’’) above a more-or-less infinite conducting plane. The gap between these conductors is filled with a dielectric whose thickness is typically smaller than the diagonal of the rectangle.

An approximate analysis (see, for example, sec. 14.2.2 of [10]) of these devices supposes them to be a kind of rf cavity whose sides are “magnetic conductors” (which support no tangential magnetic field). The cavity model (see below) yields expressions for the electric currents on the “top” and “bottom”, and the “magnetic currents” on the “sides,” from which the far-zone radiation pattern can be deduced.

Of interest here is that the fields of this “cavity” can be deduced from a vector potential that obeys eq. (26, having the general form (27). For simplicity, we assume that the (relative) permittivity of the dielectric space is unity.

We take the “top” and “bottom” of the cavity to be the surfaces at $z = 0$ and d_z , such that the z -direction has a different physical significance to the x - and y -directions. This suggests that we seek solutions where the vector potential has only a z -component.

Of course, the scalar potential V in the Lorenz gauge is related to the vector potential by

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{1}{c} \frac{\partial V}{\partial t} = ikV, \quad V = -\frac{i}{k} \frac{\partial A_z}{\partial z}. \quad (36)$$

The electric and magnetic fields follow from the potentials according to eqs. (29)-(31). The boundary conditions are that the tangential electric field vanish on the electrical conductors,

$$E_x(x, y, 0) = E_x(x, y, d_z) = E_y(x, y, 0) = E_y(x, y, d_z) = 0, \quad (37)$$

and that the tangential magnetic field vanish on the “magnetic conductors,”

$$\begin{aligned} B_y(0, y, z) &= B_y(d_x, y, z) = B_z(0, y, z) = B_z(d_x, y, z) = 0, \\ B_x(x, 0, z) &= B_x(x, d_y, z) = B_z(x, 0, z) = B_z(x, d_y, z) = 0. \end{aligned} \quad (38)$$

The satisfactory form of eq. (27) inside the “cavity” is⁷

$$A_z = \frac{E_0}{i\sqrt{k_x^2 + k_y^2}} \cos k_x x \cos k_y y \cos k_z z, \quad (39)$$

and the fields there are

$$\begin{aligned} E_x &= E_0 \frac{k_x k_z}{k\sqrt{k_x^2 + k_y^2}} \sin k_x x \cos k_y y \sin k_z z, \\ E_y &= E_0 \frac{k_y k_z}{k\sqrt{k_x^2 + k_y^2}} \cos k_x x \sin k_y y \sin k_z z, \\ E_z &= E_0 \frac{\sqrt{k_x^2 + k_y^2}}{k} \cos k_x x \cos k_y y \cos k_z z, \\ B_x &= iE_0 \frac{k_y}{\sqrt{k_x^2 + k_y^2}} \cos k_x x \sin k_y y \cos k_z z, \\ B_y &= -iE_0 \frac{k_x}{\sqrt{k_x^2 + k_y^2}} \sin k_x x \cos k_y y \cos k_z z, \\ B_z &= 0. \end{aligned} \quad (40)$$

$$B_z = 0. \quad (41)$$

⁷As in sec. 2.2.3, it is not clear that this vector potential is the Lorenz-gauge potential, despite the use of Lorenz-gauge condition in its derivation.

The scalar potential inside the “cavity” is

$$V = E_0 \frac{k_z}{k \sqrt{k_x^2 + k_y^2}} \cos k_x x \cos k_y y \cos k_z z. \quad (42)$$

The mode with indices $(l, m, n) = (0, 0, 0)$ has zero frequency, zero magnetic field, and corresponds to the electric field of the “cavity” considered as a DC capacitor. Supposing that $d_x > d_y > d_z$, the lowest-frequency mode can be labeled TM_{100}^z (in that the magnetic field of all modes from potential (39) is perpendicular to the z -axis).

Both \mathbf{E} and \mathbf{B} have nonzero tangential components on the “sides” of the “cavity”, so the fields and the potentials are nonzero outside the “cavity.” Indeed, the purpose of the “cavity” is to radiate energy. In the present approximation, \mathbf{E} and \mathbf{B} are 90° out of phase inside the cavity, so the time-average Poynting vector $\langle \mathbf{S} \rangle = c \text{Re}(\mathbf{E} \times \mathbf{B}^*)/8\pi$ is zero inside the “cavity,” including at its surface, so it is not obvious that the “cavity” radiates. However, further approximations, taking into account that $z = 0$ is effectively an infinite conducting plane, lead to useful estimates of the radiation [10, 11].

For completeness, we note that all modes of the patch antenna “cavity” are of the form

$$\begin{aligned} E_x &= E_0 e_x \sin k_x x \cos k_y y \sin k_z z, \\ E_y &= E_0 e_y \cos k_x x \sin k_y y \sin k_z z, \\ E_z &= E_0 e_z \cos k_x x \cos k_y y \cos k_z z, \end{aligned} \quad (43)$$

$$\begin{aligned} B_x &= iE_0 b_x \cos k_x x \sin k_y y \cos k_z z, \\ B_y &= iE_0 b_y \sin k_x x \cos k_y y \cos k_z z, \\ B_z &= iE_0 b_z \sin k_x x \sin k_y y \sin k_z z. \end{aligned} \quad (44)$$

References

- [1] J. Gallardo and K.T. McDonald, *Potentials for a Cylindrical Electromagnetic Cavity*, (Mar. 9, 2011), <http://physics.princeton.edu/~mcdonald/examples/cavity.pdf>
- [2] J.D. Jackson, *From Lorenz to Coulomb and other explicit gauge transformations*, Am. J. Phys. **70**, 917 (2002), http://physics.princeton.edu/~mcdonald/examples/EM/jackson_ajp_70_917_02.pdf
- [3] J.W. Gibbs, *Velocity of Propagation of Electrostatic Forces*, Nature **53**, 509 (1896), http://physics.princeton.edu/~mcdonald/examples/EM/gibbs_nature_53_509_96.pdf
- [4] K.A. Milton and J. Schwinger, *Electromagnetic Radiation: Variational Methods, Waveguides and Accelerators* (Springer, 2006), http://physics.princeton.edu/~mcdonald/examples/EM/milton_chap14.pdf
- [5] A.J. Dragt, *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics* (Feb. 27, 2011), <http://www.physics.umd.edu/dsat/>
- [6] W. Brittin *et al.*, *Poincaré gauge in electrodynamics*, Am. J. Phys. **50**, 693 (1982), http://physics.princeton.edu/~mcdonald/examples/EM/brittin_ajp_50_693_82.pdf

- [7] B.-S.K. Skagerstam *et al.*, *A note on the Poincaré gauge*, Am. J. Phys. **51**, 1148 (1983), http://physics.princeton.edu/~mcdonald/examples/EM/skagerstam_ajp_51_1148_83.pdf
- [8] R.G. Woolley, *The Electrodynamics of Atoms and Molecules*, Adv. Chem. Phys. **33**, 153 (1975), http://physics.princeton.edu/~mcdonald/examples/EM/woolley_acp_33_153_75.pdf
- [9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. (Academic Press, 1994).
- [10] C.A. Balanis, *Antenna Theory*, 3rd ed. (Wiley, New York, 2005).
- [11] W.F. Richards, Y.T. Lo and D.D. Harrison, *An Improved Theory for Microstrip Antennas and Applications*, IEEE Trans. Ant. Prop. **29**, 33 (1981), http://physics.princeton.edu/~mcdonald/examples/EM/richard_ieeetap_29_38_81.pdf