“Hidden” Momentum in an Isolated Brick?

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1 Problem

The term “hidden” momentum was popularized by Shockley [1] in considerations of an electromechanical example, and essentially all subsequent use of this term has been for such examples, where one considers the system to consist of matter plus electromagnetic fields.

A definition of “hidden” momentum has been proposed by Daniel Vanzella [2] (see also [3]) which can be applied to mechanical systems as well, where a subsystem has a specified volume and can interact with the rest of the system via contact forces and/or transfer of mass/energy across its surface (which can be in motion),

\[ P_{\text{hidden}} \equiv P - M v_{\text{cm}} - \int_{\text{boundary}} (x - x_{\text{cm}}) (p - \rho v_b) \cdot d\text{Area} = - \int f^\mu_0 \cdot (x - x_{\text{cm}}) d\text{Vol}, \]  

(1)

where \( P \) is the total momentum of the subsystem, \( M = U/c^2 \) is its total “mass”, \( U \) is its total energy, \( c \) is the speed of light in vacuum, \( x_{\text{cm}} \) is its center of mass/energy, \( v_{\text{cm}} = dx_{\text{cm}}/dt \), \( p \) is its momentum density, \( \rho = u/c^2 \) is its “mass” density, \( u \) is its energy density, \( v_b \) is the velocity (field) of its boundary, and

\[ f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}, \]

(2)
is the 4-force density due to the subsystem, with \( T^{\mu\nu} \) being the stress-energy-momentum 4-tensor of the subsystem.

Consider an isolated (unstressed) brick in an inertial frame where it has constant velocity \( v \hat{x} \) parallel to, say, its longest dimension \( l \). In this frame the brick has mass \( m \). The total “hidden” momentum of this brick is zero, so consider a partition of the brick into two subsystems by an imaginary, moving surface, \( x_{\text{boundary}} = a + ut \) for \( 0 < a < l \), while the brick extends over \( vt < x < l + vt \). What is the “hidden” momentum in the portion of the brick at \( 0 < x < a \) at time \( t = 0 \)?

2 Solution

Labeling the subsystem at \( vt < x < a + ut \) by the subscript \( a \), its time-dependent mass is,

\[ m_a = \frac{a + (u - v)t}{l} m. \]

(3)
The time-dependent momentum of this subsystem is,

\[ P_a = m_a v \hat{x}. \]

(4)
The center of mass of the subsystem has $x$-coordinate (for times when the boundary surface is within the brick),

$$ x_{a,cm} = vt + \frac{a + (u - v)t}{2}, \quad (5) $$

and the velocity of the center of mass is,

$$ v_{a,cm} = \frac{u + v}{2}. \quad (6) $$

Thus,

$$ P_a - m_a v_{a,cm} = m_a \left( v - \frac{u + v}{2} \right) = \frac{m a (v - u)}{2l}, \quad (7) $$

which is nonzero for unless $u = v$.

We now evaluate the “hidden” momentum at time $t = 0$ according to the first form of eq. (1). The boundaries in $x$ of subsystem $a$ at this time are at $x = 0$ and $a$, with velocities $v$ and $u$, while $x_{a,cm}(0) = a/2$.

$$ P_{a,hidden} = P_a - m_a v_{a,cm} + \left( 0 - \frac{a}{2} \right) \frac{m}{l} (v - v) - \left( a - \frac{a}{2} \right) \frac{m}{l} (v - u) $$

$$ = \frac{m a}{2l} (v - u) - \frac{m a}{2l} (v - u) = 0. \quad (8) $$

To use the second form of eq. (1), we note that $T^{00} = mc^2/Al$ and $T^{0x} = mcv/Al$ are constant within subsystem $a$, whose cross-sectional area is $A$, while $T^{0y} = 0 = T^{0z}$ everywhere. The time component $f^0$ of the 4-force density (2) is,

$$ f^0 = \partial_0 T^{00} + \partial_1 T^{0i} = \frac{\partial T^{00}}{\partial ct} + \frac{\partial T^{0x}}{\partial x}. \quad (9) $$

The time dependence of $T^{00}$ of subsystem $a$ is due only to the presence of its moving boundaries, $x = vt$ and $x = a + ut$. Near the “left” boundary, $T^{00}(x, t) = T^{00}(x - vt)$, such that $\partial T^{00}/\partial ct = -(v/c) \partial T^{00}/\partial x$, while near the “right” boundary, $T^{00}(x, t) = T^{00}(x - a - ut)$, such that $\partial T^{00}/\partial ct = -(u/c) \partial T^{00}/\partial x$.

It suffices to complete the calculation of $P_{a,hidden}$ for $t = 0$, as the result should be independent of time. At $t = 0$, $f^0$ is nonzero only at/near the boundaries $x = 0$ and $x = a$, so we can split the integration to that over $0 < x < b < a$ and $b < x < a$,

$$ P_{hidden,x}(t = 0) = -\int c f^0 b (x - x_{cm}) dVol = -\frac{A}{c} \int_0^a dx T^0_0 \left( x - \frac{a}{2} \right) $$

$$ = \frac{Av}{c^2} \int_0^b dx \frac{\partial T^{00}}{\partial x} \left( x - \frac{a}{2} \right) + \frac{Au}{c^2} \int_0^a dx \frac{\partial T^{00}}{\partial x} \left( x - \frac{a}{2} \right) - \frac{A}{c} \int_0^a dx \frac{\partial T^{0x}}{\partial x} \left( x - \frac{a}{2} \right) $$

$$ = \frac{Av}{c^2} \left[ x T^{00} \bigg|_0^b - \int_0^b dx T^{00} - \frac{a}{2} [T^{00}(b) - T^{00}(0)] \right] $$

$$ + \frac{Au}{c^2} \left[ x T^{0x} \bigg|_0^a - \int_0^a dx T^{0x} - \frac{a}{2} [T^{0x}(a) - T^{0x}(0)] \right] $$

$$ - \frac{A}{c} \left[ x T^{0x} \bigg|_0^a - \int_0^a dx T^{0x} - \frac{a}{2} [T^{0x}(a) - T^{0x}(0)] \right] = 0. \quad (10) $$
in agreement with eq. (8), taking $T^{00}$ and $T^{0x}$ to have values their nonzero, constant values within the interval $0 \leq x \leq a$. and zero outside this.\footnote{An analysis of eq. (10) which invokes Heaviside step functions $\Theta$, and Dirac delta functions $\delta$, notes that in the frame where the rod has velocity $v$, the nonzero components of $T^{0\mu}$ can be written as,}

Thus, according to the calculations (9) and (10), there is no “hidden” momentum in the “all-mechanical” example of an isolated brick, or in subsystems of it defined by moving partitions.

As noted in sec. VI of \cite{4}, “hidden” momentum is associated with (sub)systems that have internal motion when “at rest”, which is not the case for an isolated brick.

References

\begin{itemize}
\item \cite{1} W. Shockley and R.P. James, “Try Simplest Cases” Discovery of “Hidden Momentum” Forces on “Magnetic Currents”, Phys. Rev. Lett. 18, 876 (1967), \url{http://physics.princeton.edu/~mcdonald/examples/EM/shockley_prl_18_876_67.pdf}
\item \cite{2} D. Vanzella, Private communication, (June 29, 2012).
\item \cite{3} K.T. McDonald, On the Definition of “Hidden” Momentum (July 9, 2012), \url{http://physics.princeton.edu/~mcdonald/examples/hiddendef.pdf}
\item \cite{4} D. Babson \textit{et al.}, Hidden momentum, field momentum, and electromagnetic impulse, Am. J. Phys. 77, 826 (2009), \url{http://physics.princeton.edu/~mcdonald/examples/EM/babson_ajp_77_826_09.pdf}
\end{itemize}

\footnote{An analysis of eq. (10) which invokes Heaviside step functions $\Theta$, and Dirac delta functions $\delta$, notes that in the frame where the rod has velocity $v$, the nonzero components of $T^{0\mu}$ can be written as,}

\begin{equation}
T^{00} = \frac{mc^2}{Al} [\Theta(x - vt) - \Theta(x - a - ut)],
\end{equation}

\begin{equation}
T^{0x} = \frac{mcv}{Al} [\Theta(x - vt) - \Theta(x - a - ut)],
\end{equation}

where $\Theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$. Then,

\begin{equation}
\frac{\partial T^{00}}{\partial ct} = -\frac{mc^2}{Al} \left[ \frac{v}{c} \delta(x - vt) - \frac{u}{c} \delta(x - a - ut) \right],
\end{equation}

\begin{equation}
\frac{\partial T^{0x}}{\partial x} = \frac{mcv}{Al} \left[ \delta(x - vt) - \delta(x - a - ut) \right].
\end{equation}

Then,

\begin{align}
P_{\text{hidden},x}(t = 0) &= -\int \frac{f^0}{c} (x - x_{cm}) \, d\text{Vol} = -\frac{A}{c} \int_0^a dx \left( \frac{\partial T^{00}}{\partial ct} + \frac{\partial T^{0x}}{\partial x} \right) (x - x_{cm}) \\
&= \frac{A}{c} \int_0^a dx \frac{mc^2}{Al} \left[ \frac{v}{c} \delta(x) - \frac{u}{c} \delta(x) \right] (x - a/2) - \frac{A}{c} \int_0^a dx \frac{mcv}{Al} \left[ \delta(x) - \delta(x - a) \right] (x - a/2) \\
&= \frac{ma}{l} (u - v) + \frac{mav}{l} = \frac{ma(v - u)}{2l}.
\end{align}

Note that the result of eq. (15) is the same as $P_a - m_a V_{a,cm}$ of eq. (7). Hence, if the boundary integral in the first form of eq. (1) were ignored, the two forms of that expression, according to calculations using delta functions in $f^0$, would both lead to the same, nonzero “hidden” momentum in present example.

This author finds the delta functions in the expressions (13)-(14) for the 4-force density $f^\mu$ very unappealing physically, and so prefers the analysis in the main text that avoids them, with the implication that there is zero “hidden” momentum in the present example.

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