Radial Dependence of Radiation from a Bounded Source

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(January 25, 2004)

1 Problem

Show that the amplitude of radiation from a bounded three-dimensional source falls off as $f(\theta, \phi)/r$ in the far zone, where the angular factor f is independent of the radius r from a characteristic point in the source.

2 Solution

The stated result is likely considered to be "obvious" by most physicists [1]. Yet, there have been recent claims [2, 3, 4, 5, 6] that electromagnetic waves from a bounded source can be generated whose amplitude falls off as $1/\sqrt{r}$ over portions of solid angle in far zone.

Mathematically, there exist cylindrical waves whose amplitude falls off as $1/\sqrt{\rho}$, where ρ is the radial distance in a cylindrical coordinate system whose axis is the axis of symmetry of the source, which latter has infinite extent along the axis. An example is Čerenkov radiation; however, the amplitude of Čerenkov radiation from a finite path length falls off as $1/r$ for r large compared to the path length [7]. Also, mathematical plane waves, whose amplitude is independent of distance only their direction of propagation, can be generated by sources of infinite extent in the plane perpendicular to the direction of propagation. An example of plane waves for which misconceptions abound is the case of the so-called Bessel beam [8, 9, 10].

However, when the source of the waves is localized to a bounded three-dimensional region, there are restrictions on the character of the waves. One aspect of waves from a bounded source that is too often overlooked is that such waves cannot be unipolar [12]. Here, we reconfirm that the amplitude of waves from a bounded source fall off as $1/r$ as distances large compared to the size of the source.[1]

We will make the desired demonstration in the context of scalar diffraction theory, which gives a prescription for calculation of the amplitude of a wave of a pure frequency ω based on knowledge of the amplitude of the wave on a surface that encloses the observation point, provided that there are no charges or currents within the enclosed volume [1].

We suppose that the source of the waves is in the vicinity of the origin of a spherical coordinate system (r, θ, ϕ), and that the source lies entirely within a sphere of radius r_0. We take this sphere (plus the "sphere at infinity") to be the surface that encloses the point of observation at radius $r \gg r_0$. Since the amplitude of the wave on the "sphere at infinity"

[1]A different argument that the amplitude falls off as $1/r$ for a charge in uniform circular motion with speed $v > c/n$ in a medium of index of refraction n is given on p. 4 of [11].

is negligible, the prescription of the Huygens-Kirchoff diffraction integral for the Fourier component of frequency $\omega = kc$ of the wave is

$$\psi(\mathbf{r}) = \frac{k}{2\pi i} \int r_0^2 \; d\cos\theta' d\phi \; \psi(r_0, \theta', \phi') \frac{e^{ikR}}{R}, \tag{1}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ is the vector distance between the source and observation points. For $r \gg r_0$, we can approximate distance R as

$$R = r - r_0 \cos\alpha, \tag{2}$$

where $\alpha(\theta, \phi, \theta', \phi')$ is the angle between vectors $\mathbf{r}$ and $\mathbf{r}_0$. Making the usual approximation that in the far zone the factor R in the denominator of eq. (1) can be approximated by r, we have

$$\psi(\mathbf{r}) \approx \frac{e^{ikr}}{r} \frac{kr_0^2}{2\pi i} \int d\cos\theta' d\phi \; \psi(r_0, \theta', \phi') e^{-ikr_0 \cos\alpha} \equiv \frac{e^{ikr}}{r} F(r_0, \theta, \phi). \tag{3}$$

That is, the amplitude of the wave at a point (r, θ, ϕ) in the far zone is a spherical wave modulated by an angular factor $F(r_0, \theta, \phi)$. The amplitude falls off as $1/r$ for all angles (θ, ϕ) as claimed, independent of the details of the source inside radius r_0. This result holds even if amplitude $\psi(r_0, \theta', \phi')$ is "singular" for some angles (θ', ϕ') as might happen formally if the source generates a kind of Čerenkov radiation.

3 Discussion

The Huygens-Kirchoff integral (1) is only an approximation. In what ways might it give an inaccurate representation of waves in the far zone?

The replacement of distance R by r in the denominator of eq. (1) is not strictly correct. It would be more proper to expand $1/R$ is a power series of terms $(1/r)^n$ where $n = 1,2,3,...$ This reminds us that the amplitude of a field can fall off *more rapidly* than $1/r$, and that even in the far zone there is some evidence of near-zone behavior where the fields fall off as $1/r^2$ (or faster).

Further, there can be debate as to whether eq. (1) should be modified by the inclusion of an "obliquity factor" (see, for example, sec. 10.5 of [1]), which changes slightly the character of the angular integration. However, the details of the angular obliquity factor do not affect the radial dependence of the calculation.

Hence, the usual approximations in the Huygens-Kirchoff theory do not lead one to doubt that wave amplitudes fall off as $1/r$ in the far zone.

Ultimately, this behavior can be ascribed to conservation of energy. In the far zone, the wave energy that lies within some element of solid angle $d\Omega$ propagates only within that solid angle. Therefore, conservation of the field energy, $U \propto E^2 r^2 \; d\Omega$, in that solid angle requires that $E \propto 1/r$.

Hence it is very hard to understand what Ardavan *et al.* mean by their claim [2, 6] that "The focused wave packets that embody the non spherically decaying pulses are constantly dispersed and reconstructed out of other waves, so that the constructive interference of their constituent waves takes place within different solid angles on spheres of different radii r. The integral of the flux of energy across a large sphere centered on the source is the same as the integral of the flux of energy across any other sphere that encloses the source. The strong fields that occur in focal regions are compensated by weaker fields elsewhere, so that the distribution of the flux of energy across such spheres is highly nonuniform and r dependent." Such wording might apply to considerations in the near zone, *i.e.*, within a focal length of the wave pattern possibly created by "lenslike" structures within the sphere of radius r_0. But at distances larger than that focal length, the wave is diverging and field energy neither enters nor leaves any given element of solid angle with the consequence that the field falls off as $1/r$ for all angles.

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