An Electric Bottle: 
Charged Particle Orbiting a Charged Needle

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

A “magnetic bottle” is a quasisolenoidal magnetic field, perhaps created by a pair of coils as in the figure below, such that a charged particle of sufficiently low velocity is “trapped” in the region between the coils, which act as “magnetic mirrors.” See, for example, sec. 12.5 of [1].

Show that a charged particle orbiting a charged needle with sufficiently low velocity is similarly “trapped” in an “electric bottle.” A delightful demonstration of this effect in the NASA Space Station is at [2].

2 Solution

We take the needle to be a uniform line of length 2a and total charge Q, centered on the origin and along the z-axis of a cylindrical coordinate system (r, \( \theta \), z).

![Diagram of a charged needle and its electric field](image)

The electric scalar potential \( V \) at a point \((r, \theta, z)\) is, in Gaussian units,

\[
V(r, z) = \int_{-a}^{a} \frac{Q/2a}{R} \, dz' = \frac{Q}{2a} \int_{-a}^{a} \frac{dz'}{\sqrt{r^2 + (z-z')^2}} = -\frac{Q}{2a} \ln \left[ z - z' + \sqrt{r^2 + (z-z')^2} \right]_{-a}^{a}
\]

A related example of a charged needle inside a coaxial conductor cylinder is discussed in [3]. Such traps were first considered by Kingdon in 1923 [4]. A trap with a potential \( U(r, \theta, z) = A(z^2 - r^2/2 + B \ln r) \) was discussed in [5], and is now commercialized as the Orbitrap [6, 7].

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\[
V = \frac{Q}{2a} \ln \frac{z + a + R_1}{z - a + R_2}, \quad \text{where} \quad R_{1,2} = \sqrt{r^2 + (z \pm a)^2}. 
\] (1)

If we write
\[
u = \frac{R_1 + R_2}{2}, \quad \nu = \frac{R_1 - R_2}{2}, \quad \text{with} \quad -\infty < u < \infty, \quad -a \leq v \leq a,
\] (2)
then surfaces of constant \(u\) are prolate spheroids, surfaces of constant \(v\) are hyperboloids,
\[
R_1 = u + v, \quad R_2 = u - v, \quad az = uv,
\] (3)
and
\[
\frac{z + a + R_1}{z - a + R_2} = \frac{uv + a^2 + ua + va}{uv - a^2 + ua - va} = \frac{(u + a)(v + a)}{(u - a)(v + a)} = \frac{u + a}{u - a},
\] (4)
such that the electric potential is constant on surfaces of constant \(u\),
\[
V = \frac{Q}{2a} \ln \frac{u + a}{u - a},
\] (5)
and the electric field lines lie on surfaces of constant \(v\) with no azimuthal component \(E_\theta\).

The coordinates \((u, \theta, v)\) form a prolate spheroidal coordinate system.
The electric field in the symmetry plane \((v = 0 = z)\) has only a radial component,
\[
E_r(r, z = 0) = -\frac{\partial V(r, z = 0)}{\partial r} = \frac{Q}{2a} \frac{\partial}{\partial r} \ln \frac{\sqrt{r^2 + a^2 + a}}{\sqrt{r^2 + a^2} - a} = \frac{Q}{r \sqrt{r^2 + a^2}},
\] (6)
which falls off as \(1/r^2\) at large radii.

An electric charge \(-q\) with mass \(m\) can have a circular orbit in the plane \(z = 0\) with any radius \(r_0\) if its azimuthal speed \(v_{0\theta} = r_0 \omega_0\) is related by
\[
\omega_0^2 = \frac{qQ}{mr_0^2 \sqrt{r_0^2 + a^2}}, \quad v_{0\theta}^2 = \frac{qQ}{m \sqrt{r_0^2 + a^2}} = \frac{qQ}{am \sqrt{1 + r_0^2/a^2}}.
\] (7)
It is useful to note the angular-momentum \( L = r \times m \mathbf{v} \) of the moving charge/mass obeys

\[
\frac{dL}{dt} = \mathbf{r} \times \mathbf{F} = (r \dot{\mathbf{r}} + z \dot{\mathbf{z}}) \times (F_r \mathbf{r} + F_z \mathbf{z}) = (zF_r - rF_z) \dot{\theta}, \tag{8}
\]
as the force \( \mathbf{F} = -q \mathbf{E} = q \nabla V(r, z) \) has no \( \theta \)-component. Recalling that

\[
\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \dot{r} \mathbf{r} + r \dot{\theta} \mathbf{\theta}, \quad \frac{d\mathbf{\theta}}{dt} = \dot{\theta} \mathbf{r}, \tag{9}
\]
we have

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} = \frac{d}{dt}(r \dot{\mathbf{r}} + z \dot{\mathbf{z}}) = \dot{r} \mathbf{r} + r \dot{\theta} \mathbf{\theta} + \dot{z} \mathbf{z}, \tag{10}
\]

\[
\begin{align*}
L = r \times m \mathbf{v} &= m(r \dot{\mathbf{r}} + z \dot{\mathbf{z}}) \times (\dot{r} \mathbf{r} + r \dot{\theta} \mathbf{\theta} + \dot{z} \mathbf{z}) = -mr z \dot{\theta} \mathbf{r} + m(\dot{r}z - r \dot{z}) \dot{\theta} + mr^2 \dot{\theta} \mathbf{z}, \tag{11}
\end{align*}
\]

and hence from eq. (8), both the \( r \) and \( z \)-components of eq. (12) lead to

\[
2 \dot{r} \dot{\theta} + r \ddot{\theta} = 0, \quad \text{and} \quad L_z = mr^2 \dot{\theta} = \text{constant}. \tag{13}
\]

We now consider perturbations about such circular orbits, first in the symmetry plane, and then (more interestingly) along the \( z \)-axis.

### 2.1 Perturbed Motion in the Symmetry Plane

For motion in the plane \( z = 0 \), we have that

\[
\dot{r} - r \dot{\theta}^2 = \dot{r} - \frac{L^2}{mr^3} = F_r = -\frac{qQ}{r \sqrt{r^2 + a^2}}. \tag{14}
\]

The equilibrium circular orbit has \( \dot{r} = 0 \), and hence the equilibrium radius \( r_0 \) and angular velocity \( \omega_0 = \dot{\theta}_0 \) are related by

\[
\frac{L^2}{r_0^2} = \frac{qQm}{\sqrt{r_0^2 + a^2}}, \quad \omega_0^2 = \frac{L^2}{mr_0^4} = \frac{qQ}{mr_0^2 \sqrt{r_0^2 + a^2}}. \tag{15}
\]

For small perturbations about the equilibrium orbit, we expand the radial equation of motion as

\[
\dot{r} = \frac{L_z^2}{mr^3} - \frac{qQ}{r \sqrt{r^2 + a^2}} \approx -\frac{qQ}{mr_0^2 \sqrt{r_0^2 + a^2}} \frac{r_0^2 + 2a^2}{r_0 + a^2} (r - r_0), \tag{16}
\]

which has oscillatory solutions of the form

\[
r \approx r_0(1 + \epsilon \sin \omega t) \quad \text{where} \quad \omega^2 = \frac{\omega_0^2 r_0^2 + 2a^2}{r_0 + a^2} > \omega_0^2, \tag{17}
\]

with period shorter than that of the equilibrium orbit. The angular velocity is given by

\[
\dot{\theta} \approx \omega_0(1 - \epsilon \cos \omega t). \tag{18}
\]

The perturbed orbits are ellipse-like with retrograde precession of the pericharge (to coin a phrase).
2.2 Perturbed Motion Perpendicular to the Symmetry Plane

We now consider motion that includes nonzero velocity \( \dot{z} \) parallel to the axis of the charged rod. In particular, we consider motion with constant angular momentum \( \dot{L}_z = mr^2 \dot{\theta} \), such that when \( z = 0 \), then \( r = r_0 \), \( \dot{r} = 0 \), and \( r \dot{\theta} = \dot{v}_{0\theta} \) according to eq. (15), \( L_z = mr_0 v_{0\theta} \), and \( \dot{z} = v_{0z} \). The subsequent motion is qualitatively helical, and the total energy \( U \) is conserved,

\[
U = \frac{m}{2} \left( r^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) - \frac{qQ}{2a} \ln \frac{u + a}{u - a} = \frac{m}{2} \left( \dot{r}^2 + \frac{L_z^2}{m^2 r^2} + \dot{z}^2 \right) - \frac{qQ}{2a} \ln \frac{u + a}{u - a}
\]

where \( u_0 = \sqrt{r_0^2 + a^2} \), recalling the electric potential (5) and that

\[
u = \frac{\sqrt{r^2 + (z + a)^2} + \sqrt{r^2 + (z - a)^2}}{2}.
\]

An interesting question is whether there exists a maximal value \( z_1 \) for the helical motion, such that the charge \(-q\) is “trapped” in an “electric bottle,” oscillating axially between “turning points” \( \pm z_1 \). The video [2] suggests that such “trapping” is possible for some values of \( v_{0z} \).

If the turning point \( z_1 \) of the motion exists, then here \( r = r_1, \dot{r} = 0, \dot{z} = 0 \), and the energy equation (19) becomes

\[
\frac{L_z^2}{m^2 r_1^2} - \frac{qQ}{am} \ln \frac{u_1 + a}{u_1 - a} = \frac{L_z^2}{m^2 r_0^2} + v_{0z}^2 - \frac{qQ}{am} \ln \frac{u_0 + a}{u_0 - a}.
\]

The value of \( z_1 \) increases with increasing \( v_{0z} \), and is infinite for the maximum axial speed \( v_{0z, max} \) for which a turning point exists. At infinite \( z_1 \) the electrical potential energy is zero, and eq. (21) becomes

\[
\frac{L_z^2}{m^2 r_1^2} = \frac{L_z^2}{m^2 r_0^2} + v_{0z, max}^2 - \frac{qQ}{am} \ln \frac{u_0 + a}{u_0 - a}.
\]

Since \( r_1 < r_0 \) we have, recalling eq. (7),

\[
v_{0z, max} > \frac{qQ}{am} \ln \frac{u_0 + a}{u_0 - a} = \dot{v}_{0\theta} \sqrt{1 + \frac{r_0^2}{a^2} \ln \frac{\sqrt{r_0^2 + a^2 + a}}{\sqrt{r_0^2 + a^2 - a}}} \approx 2 \dot{v}_{0\theta} \ln \frac{2a}{r_0},
\]

where the approximation holds for \( r_0 \ll a \). Thus, turning points \( \pm z_1 \) exist; for \( r_0 \ll a \), the ratio \( v_{0z, max}/v_{0\theta} \) can be large compared to unity, and the case \( v_{0z, max} = v_{0\theta} \) holds for \( r_0 \approx a \). The latter result is qualitatively consistent with the NASA video [2].

2.2.1 Frequency of Small Axial Oscillations

Although the main interest in this problem is the existence of large-amplitude axial perturbations to the orbital motion, we also consider small axial perturbations for which an approximate calculation of the frequency can be given.
The axial equation of motion is

\[ m\ddot{z} = -qE_x = \frac{q}{2a} \frac{\partial V}{\partial z} \ln \frac{u+a}{u-a} = -\frac{qQ}{u^2-a^2} \frac{\partial u}{\partial z} \]

\[ = -\frac{qQ}{2(u^2-a^2)} \left( \frac{z+a}{\sqrt{r^2+(z+a)^2}} + \frac{z-a}{\sqrt{r^2+(z-a)^2}} \right). \tag{24} \]

For \(|z| \ll \sqrt{r_0^2+a^2}\) we approximate

\[ u = \frac{\sqrt{r^2+(z+a)^2} + \sqrt{r^2+(z-a)^2}}{2} \approx \sqrt{r_0^2+a^2}, \quad \text{so} \quad u^2 - a^2 \approx r_0^2, \tag{25} \]

and

\[ \frac{z+a}{\sqrt{r^2+(z+a)^2}} + \frac{z-a}{\sqrt{r^2+(z-a)^2}} \approx \frac{1}{\sqrt{r_0^2+a^2}} \left[ (z+a) \left( 1 - \frac{za}{r_0^2+a^2} \right) + (z-a) \left( 1 + \frac{za}{r_0^2+a^2} \right) \right] = \frac{2r_0^2 z}{(r_0^2+a^2)^{3/2}}. \tag{26} \]

Hence,

\[ \ddot{z} \approx -\frac{qQz}{mr_0^2 \sqrt{r_0^2+a^2}} = -\omega_0^2 \frac{r_0^2}{r_0^2+a^2} z, \tag{27} \]

recalling eq. (7). Thus, for \(|z| \ll \sqrt{r_0^2+a^2}\), we have simple harmonic motion with angular frequency \(\omega_z\) given by

\[ \omega_z^2 = \omega_0^2 \frac{r_0^2}{r_0^2+a^2}. \tag{28} \]

For equilibrium orbits with \(r_0 \gg a\), where the charged needle appears to be a point charge, we have that \(\omega_z \approx \omega_0\) as expected for small oscillations, while for small \(r_0\) the frequency of small axial oscillations is smaller than the equilibrium orbital frequency \(\omega_0\); the charge \(-q\) can complete several spiral turns per each cycle of axial oscillation, as seen in the video [2].

References


http://physics.princeton.edu/~mcdonald/examples/EM/nasa_needle.mp4


