Green’s Function for a Conducting Plane with a Hemispherical Boss

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1 Problem

What is the electric potential in rectangular coordinates \((x, y, z)\) when a charge \(q\) is located at \((x_0, y_0, 0)\) and there is a grounded conducting plane at \(y = 0\) that has a (conducting) hemispherical boss of radius \(a < b = \sqrt{x_0^2 + y_0^2}\) whose center is at the origin? What is the electrostatic force on the charge \(q\) for the case that \(x_0 = 0\)?

Consider also the case of a grounded conducting plane with a half-circular, conducting ridge of radius \(a\).

2 Solution

2.1 Hemispherical Boss

This example is posed as prob. 23, p. 284 of [2], prob. 13, p. 224 of [3], and as prob. 17 p. 232 of [4].

We use the image method [1].

First, we bring the hemispherical boss to zero potential by imagining that a charge \(q' = -qa/b\) is placed at distance \(a^2/b\) along the line from the origin to charge \(q\). The rectangular coordinates of charge \(q'\) are \((a^2/b^2)(x_0, y_0, 0)\). Next, to bring the plane \(y = 0\) to zero potential, we add images charges for both \(q\) and \(q'\). Namely, we imagine charge \(q'' = -q\) at \((x_0, -y_0, 0)\), and charge \(q''' = -q' = qa/b\) at \((a^2/b^2)(x_0, -y_0, 0)\). Then, both the plane \(y = 0\) and the spherical shell of radius \(a\) about the origin are at zero potential.
The electric scalar potential \( V \) at an arbitrary point \((x, y, z)\) outside the conductor is therefore
\[
V = \frac{q}{r_1} - \frac{q}{r_2} - \frac{qa}{br_3} + \frac{qa}{br_4},
\]
where
\[
r_{1,2} = \sqrt{(x - x_0)^2 + (y \mp y_0)^2 + z^2}, \quad r_{3,4} = \sqrt{(x - a^2 x_0/b^2)^2 + (y \mp a^2 y_0/b^2)^2 + z^2}.
\]

When \( x_0 = 0 \), then \( y_0 = b \) and the force on charge \( q \) is in the \(-y\) direction, with magnitude
\[
F = \frac{q^2}{4b^2} + \frac{q^2a/b}{(b - a^2/b)^2} - \frac{q^2a/b}{(b + a^2/b)^2} = \frac{q^2}{4b^2} + \frac{4q^2a^3b^3}{(b^4 - a^4)^2}.
\]
The electric field at the origin in the absence of the boss would be \( E_0 = 2q/y_0^2 = 2q/b^2 \). With the boss present, the electric potential along the \( y\)-axis is
\[
V(0, y > a, 0) = \frac{q}{b - y} - \frac{q}{b + y} - \frac{qab}{by - a^2} + \frac{qab}{by + a^2},
\]
so the electric field at the pole of the boss, \((0, a, 0)\) has magnitude
\[
|E_y(0, a, 0)| = \left| -\frac{dV(0, a, 0)}{dy} \right| = \frac{2q(2b^2 + a^2)}{(b^2 - a^2)^2} \approx \frac{4q}{b^2} = 2E_0,
\]
where the approximation holds for \( b \gg a \). The field at the pole of the boss is roughly twice that at the origin in its absence.

If the conducting plane with the hemispherical boss of radius \( a \) were part of a parallel-plate capacitor, with separation \( b \gg a \) between the plates, the above results indicate that the peak electric field at the pole of the boss would be \( \approx 2E_0 \), where \( E_0 \) is the field inside the capacitor in the absence of the boss.\(^1\)

### 2.2 Half-Cylindrical Ridge

We now consider the case of a conducting plane \( y = 0 \) with a conducting, half-cylindrical ridge of radius \( a \) and axis \((0, 0, z)\), together with a line charge \( q \) per unit length in the \( z\)-direction, located at \((x_0, y_0, z)\). Again, we use an image method, now for 2-dimensional conductors.\(^2\)

Here, the image of the line charge at distance \( b = \sqrt{x_0^2 + y_0^2} \) from the \( z\)-axis is a line charge \( q' = -q \) per unit length at distance \( a^2/b \) from that axis, with coordinates \((a^2/b^2)(x_0, y_0, z)\). The solution is completed by the image line charges \( q'' = -q \) and \( q''' = q \) at coordinates

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1\(^{\text{The potential difference between the capacitor plates is } V \approx E_0 b. \text{ In contrast, an isolated conducting sphere of radius } a \text{ at potential } V = E_0 b \text{ has electric field of strength } V/a = E_0 b/a \gg E_0 \text{ at its surface.}}\)

2\(^{\text{Note that for large } b, \text{ the potential takes the form } V = E_0 (r - a^2/r^2) \cos \theta = E_0 y (1 - a^2/r^3), \text{ where angle } \theta \text{ is measured with respect to the } y\text{-axis, and } r = \sqrt{x^2 + y^2 + z^2}. \text{ Compare also to the case of a conducting sphere in an otherwise uniform external field } E_0, \text{ where the peak field at the surface of the sphere is } 3E_0. \text{ See, for example, sec. 2.3 of [5].}}\)

2\(^{\text{See, for example, prob. 11(a) of [6].}}\)
\((x_0, -y_0, z)\) and \((a^2/b^2)(x_0, -y_0, z)\), respectively. The electric scalar potential \(V\) at an arbitrary point \((x, y, z)\) outside the conductor is therefore (to within a constant),

\[ V = -2q(\ln r_1 - \ln r_2 - \ln r_3 + \ln r_4), \quad (6) \]

where

\[ r_{1,2} = \sqrt{(x - x_0)^2 + (y \mp y_0)^2}, \quad r_{3,4} = \sqrt{(x - a^2x_0/b^2)^2 + (y \mp a^2y_0/b^2)^2}. \quad (7) \]

When \(x_0 = 0\), then \(y_0 = b\) and the force per unit length on charge \(q\) (per unit length) is in the \(-y\) direction, with magnitude

\[ F = q^2b + 2q^2b/b^4 - a^4. \quad (8) \]

The electric field strength at the origin in the absence of the boss would be \(E_0 = 4q/y_0 = 4q/b\). With the boss present, the electric potential in the plane \(x = 0\) is (to within a constant),

\[ V(0, y > a, z) = -2q \left[ \ln |b - y| - \ln |b + y| - \ln |by - a^2| + \ln |by + a^2| \right], \quad (9) \]

so the electric field long the peak of the ridge, \((0, a, z)\) has magnitude

\[ |E_y(0, a, 0)| = \left| \frac{-dV(0, a, z)}{dy} \right| = \frac{8q}{b^2 - a^2} \approx \frac{8q}{b} \approx 2E_0, \quad (10) \]

where the approximation holds for \(b \gg a\). The peak field along the ridge is roughly twice that at the origin in its absence.

If the conducting plane with the half-cylindrical ridge of radius \(a\) were part of a parallel-plate capacitor, with separation \(b \gg a\) between the plates, the above results indicate that the peak electric field at the pole of the boss would be \(\approx 2E_0\), where \(E_0\) is the field inside the capacitor in the absence of the boss.\(^3\)

References


\(^3\)The potential difference between the capacitor plates is \(V \approx E_0b\). In contrast, an isolated conducting cylinder of radius \(a\) at potential \(V = E_0b\) (with \(V = 0\) at distance \(b\) from its axis) has charge \(Q = E_0b/(2\ln b/a)\) per unit length, and electric field of strength \(2Q/a = E_0b/(a \ln b/a) \gg E_0\) at its surface.

Note that for large \(b\), the potential takes the form \(V = E_0(r - a^2/r) \cos \theta = E_0y(1 - a^2/r^2)\), where angle \(\theta\) is measured with respect to the \(y\)-axis, and \(r = \sqrt{x^2 + y^2}\). See prob. 5, p. 229 of [4].

Compare also to the case of a conducting cylinder in an otherwise uniform external field \(E_0\), where the peak field at the surface of the sphere is \(2E_0\). See, for example, sec. 2.2 of [5].

