The Maximal Energy Attainable in a Betatron

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 19, 2000)

1 Problem

A betatron is a circular device of radius $R$ designed to accelerate electrons (charge $e$, mass $m$) via a changing magnetic flux $\Phi = \pi R^2 \dot{B}_{\text{ave}}$ through the circle.

Deduce the relation between the magnetic field $B$ at radius $R$ and the magnetic field $\dot{B}_{\text{ave}}$ averaged over the area of the circle needed for a betatron to function. Also deduce the maximum energy $\mathcal{E}$ to which an electron could be accelerated by a betatron in terms of $B$, $\dot{B}_{\text{ave}}$ and $R$.

Hints: The electrons in this problem are relativistic, so it is useful to introduce the factor $\gamma = \frac{\mathcal{E}}{mc^2}$ where $c$ is the speed of light. Recall that Newton’s second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is $\gamma m$. Recall also that for circular motion the rest frame acceleration is $\gamma^2$ times that in the lab frame.

2 Solution

This problem is due to Iwanenko and Pomeranchuk [1]. See also [2].

The electron is held in its circular orbit by the Lorentz force due to the field $B$. Newton’s law, $F = ma$, for this circular motion can be written (in Gaussian units)

$$ F = \gamma ma = \frac{\gamma mv^2}{R} = e\frac{v}{c}B. $$

(1)

For a relativistic electron, $v \approx c$, so we have

$$ \gamma \approx \frac{eRB}{mc^2}. $$

(2)

The electron is being accelerated by the electric field that is induced by the changing magnetic flux. Applying the integral form of Faraday’s law to the circle of radius $R$, we have (ignoring the sign)

$$ 2\pi RE_\phi = \frac{\dot{\Phi}}{c} = \frac{\pi R^2 \dot{B}_{\text{ave}}}{c}, $$

(3)

and hence,

$$ E_\phi = \frac{R\dot{B}_{\text{ave}}}{2c}, $$

(4)

The rate of change of the electron’s energy $\mathcal{E}$ due to $E_\phi$ is

$$ \frac{d\mathcal{E}}{dt} = F \cdot v \approx ecE_\phi = \frac{eR\dot{B}_{\text{ave}}}{2}, $$

(5)
Since $\mathcal{E} = \gamma mc^2$, we can write
\[
\dot{\gamma} mc^2 = \frac{eR \dot{B}_{\text{ave}}}{2}, \tag{6}
\]
which integrates to
\[
\gamma = \frac{eRB_{\text{ave}}}{2mc^2}. \tag{7}
\]
Comparing with eq. (2), we find the required condition on the magnetic field:
\[
B = \frac{B_{\text{ave}}}{2}. \tag{8}
\]

As the electron accelerates it radiates energy at rate given by the Larmor formula in the rest frame of the electron,
\[
\frac{d\mathcal{E}^*}{dt^*} = -\frac{2e^2p^*}{3c^3} = -\frac{2e^2a^*}{3c^3}, \tag{9}
\]
Because $\mathcal{E}$ and $t$ are both the time components of 4-vectors their transforms from the rest frame to the lab frame have the same form, and the rate $d\mathcal{E}/dt$ is invariant. However, acceleration at right angles to velocity transforms according to $a^* = \gamma^2 a$. Hence, the rate of radiation in the lab frame is
\[
\frac{d\mathcal{E}}{dt} = -\frac{2e^2\gamma^4 a^2}{3c^3} = -\frac{2e^4\gamma^2 B^2}{3m^2c^3}, \tag{10}
\]
using eq. (1) for the acceleration $a$.

The maximal energy of the electrons in the betatron obtains when the energy loss (10) cancels the energy gain (5), i.e., when
\[
\frac{eR \dot{B}_{\text{ave}}}{2} = \frac{2e^4\gamma_{\text{max}}^2 B^2}{3m^2c^3}, \tag{11}
\]
and
\[
\gamma_{\text{max}} = \sqrt{\frac{3m^2c^3 R \dot{B}_{\text{ave}}}{4e^3 B^2}} = \sqrt{\frac{3R B_{\text{ave}} B_{\text{crit}}}{4\alpha c B B_{\text{crit}}}} \approx \sqrt{\frac{3R B_{\text{crit}}}{4\alpha c \tau B_{\text{crit}}}}, \tag{12}
\]
where $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant, $B_{\text{crit}} = m^2c^3/\hbar = 4.4 \times 10^{13}$ G is the so-called QED critical field strength, and $\tau$ is the characteristic cycle time of the betatron such that $\dot{B}_{\text{ave}} = B/\tau$. For example, with $R = 1$ m, $\tau = 0.03$ sec (30 Hz), and $B = 10^4$ G, we find that $\gamma_{\text{max}} \approx 200$, or $\mathcal{E}_{\text{max}} \approx 100$ MeV.

We have ignored the radiation due to the longitudinal acceleration of the electron, since in the limiting case this acceleration ceases.

References

http://physics.princeton.edu/~mcdonald/examples/accel/iwanenko_pr_65_343_44.pdf