The Maximal Energy Attainable in a Betatron

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(November 19, 2000)

1 Problem

A betatron is a circular device of radius R designed to accelerate electrons (charge e, mass m) via a changing magnetic flux $\dot{\Phi} = \pi R^2 \dot{B}_{\rm ave}$ through the circle.

Deduce the relation between the magnetic field B at radius R and the magnetic field $B_{\rm ave}$ averaged over the area of the circle needed for a betatron to function. Also deduce the maximum energy $\mathcal{E}$ to which an electron could be accelerated by a betatron in terms of B, $\dot{B}_{\rm ave}$ and R.

Hints: The electrons in this problem are relativistic, so it is useful to introduce the factor $\gamma = \mathcal{E}/mc^2$ where c is the speed of light. Recall that Newton's second law has the same form for nonrelativistic and relativistic electrons except that in the latter case the effective mass is γm. Recall also that for circular motion the rest frame acceleration is γ^2 times that in the lab frame.

2 Solution

This problem is due to Iwanenko and Pomeranchuk [1]. See also [2].

The electron is held in its circular orbit by the Lorentz force due to the field B. Newton's law, $F = ma$, for this circular motion can be written (in Gaussian units)

$$F = \gamma m a = \frac{\gamma m v^2}{R} = e\frac{v}{c}B. \tag{1}$$

For a relativistic electron, $v \approx c$, so we have

$$\gamma \approx \frac{eRB}{mc^2}. \tag{2}$$

The electron is being accelerated by the electric field that is induced by the changing magnetic flux. Applying the integral form of Faraday's law to the circle of radius R, we have (ignoring the sign)

$$2\pi R E_\phi = \frac{\dot{\Phi}}{c} = \frac{\pi R^2 \dot{B}_{\rm ave}}{c}, \tag{3}$$

and hence,

$$E_\phi = \frac{R\dot{B}_{\rm ave}}{2c}, \tag{4}$$

The rate of change of the electron's energy $\mathcal{E}$ due to E_ϕ is

$$\frac{d\mathcal{E}}{dt} = \mathbf{F}\cdot\mathbf{v} \approx ecE_\phi = \frac{eR\dot{B}_{\rm ave}}{2}, \tag{5}$$

Since $\mathcal{E} = \gamma mc^2$, we can write

$$\dot{\gamma} mc^2 = \frac{eR\dot{B}_{\rm ave}}{2}, \tag{6}$$

which integrates to

$$\gamma = \frac{eRB_{\rm ave}}{2mc^2}. \tag{7}$$

Comparing with eq. (2), we find the required condition on the magnetic field:

$$B = \frac{B_{\rm ave}}{2}. \tag{8}$$

As the electron accelerates it radiates energy at rate given by the Larmor formula in the rest frame of the electron,

$$\frac{d\mathcal{E}^\star}{dt^\star} = -\frac{2e^2\ddot{p}^{\star 2}}{3c^3} = -\frac{2e^2 a^{\star 2}}{3c^3} \tag{9}$$

Because $\mathcal{E}$ and t are both the time components of 4-vectors their transforms from the rest frame to the lab frame have the same form, and the rate $d\mathcal{E}/dt$ is invariant. However, acceleration at right angles to velocity transforms according to $a^\star = \gamma^2 a$. Hence, the rate of radiation in the lab frame is

$$\frac{d\mathcal{E}}{dt} = -\frac{2e^2\gamma^4 a^2}{3c^3} = -\frac{2e^4\gamma^2 B^2}{3m^2c^3}, \tag{10}$$

using eq. (1) for the acceleration a.

The maximal energy of the electrons in the betatron obtains when the energy loss (10) cancels the energy gain (5), *i.e.*, when

$$\frac{eR\dot{B}_{\rm ave}}{2} = \frac{2e^4\gamma_{\rm max}^2 B^2}{3m^2c^3}, \tag{11}$$

and

$$\gamma_{\rm max} = \sqrt{\frac{3m^2c^3R\dot{B}_{\rm ave}}{4e^3B^2}} = \sqrt{\frac{3R}{4\alpha c}\frac{\dot{B}_{\rm ave}}{B}\frac{B_{\rm crit}}{B}} \approx \sqrt{\frac{3R}{4\alpha c\tau}\frac{B_{\rm crit}}{B}}, \tag{12}$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine structure constant, $B_{\rm crit} = m^2c^3/e\hbar = 4.4 \times 10^{13}$ G is the so-called QED critical field strength, and τ is the characteristic cycle time of the betatron such that $\dot{B}_{\rm ave} = B/\tau$. For example, with $R = 1$ m, $\tau = 0.03$ sec (30 Hz), and $B = 10^4$ G, we find that $\gamma_{\rm max} \approx 200$, or $\mathcal{E}_{\rm max} \approx 100$ MeV.

We have ignored the radiation due to the longitudinal acceleration of the electron, since in the limiting case this acceleration ceases.

References

[1] D. Iwanenko and I. Pomeranchuk, *On the Maximal Energy Attainable in a Betatron*, Phys. Rev. **65**, 343 (1944),
http://physics.princeton.edu/~mcdonald/examples/accel/iwanenko_pr_65_343_44.pdf

[2] K.T. McDonald and C.G. Tully, *Maximum Energy of Circular Colliders*, (Dec. 10, 2001),
http://physics.princeton.edu/~mcdonald/examples/lep.pdf