

An Electrostatic Wave

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1 Problem

All electrostatic fields \mathbf{E} (*i.e.*, ones with no time dependence) can be derived from a scalar potential V ($\mathbf{E} = -\nabla V$) and hence obey $\nabla \times \mathbf{E} = 0$. The latter condition is sometimes considered to be a requirement for electrostatic fields. Show, however, that there can exist time-dependent electric fields for which $\nabla \times \mathbf{E} = 0$, which have been given the name “electrostatic waves”.

In particular, show that a plane wave with electric field \mathbf{E} parallel to the wave vector \mathbf{k} (a longitudinal wave) can exist in a medium with no time-dependent magnetic field if the electric displacement \mathbf{D} is zero. This cannot occur in an ordinary dielectric medium, but can happen in a plasma. (Time-independent electric and magnetic fields could, of course, be superimposed on the wave field.) Compare the potentials for the “electrostatic wave” in the Coulomb and Lorentz gauges. Discuss energy density and flow for such a wave.

Deduce the frequency ω of the longitudinal wave in a hot, collisionless plasma that propagates transversely to a uniform external magnetic field \mathbf{B}_0 in terms of the (electron) cyclotron frequency,

$$\omega_B = \frac{eB_0}{mc}, \quad (1)$$

(in Gaussian units), the (electron) plasma frequency,

$$\omega_P^2 = \frac{4\pi N e^2}{m}, \quad (2)$$

and the electron temperature T , where $e > 0$ and m are the charge and mass of the electron, c is the speed of light, and N is the electron number density.

For a simplified analysis, you may assume that the positive ions are at rest, that all electrons have the same transverse velocity $v_\perp = \sqrt{2KT/m}$, where K is Boltzmann’s constant, T is the temperature, and that the densities of the ions and unperturbed electrons are uniform. Then the discussion may proceed from an (approximate) analysis of the motion of an individual electron to the resulting polarization density and dielectric constant, *etc.*

Such waves are called electron Bernstein waves, following their prediction via an analysis based on the Boltzmann transport equation [1]. Bernstein waves were first produced in laboratory plasmas in 1964 [2], following possible detection in the ionosphere in 1963. They are now being applied in plasma diagnostics where it is desired to propagate waves below the plasma frequency [3].

2 Solution

2.1 General Remarks

We first verify that Maxwell's equations imply that when an electric field \mathbf{E} has no time dependence, then $\nabla \times \mathbf{E} = 0$.

If $\partial \mathbf{E} / \partial t = 0$, then the magnetic field \mathbf{B} obeys $\partial^2 \mathbf{B} / \partial t^2 = 0$, as follows on taking the time derivative of Faraday's law, $c \nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$ in Gaussian units. In principle, this is consistent with a magnetic field that varies linearly with time, $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r})t$. However, this leads to arbitrarily large magnetic fields at early and late times, and is excluded on physical grounds. Hence, $\partial \mathbf{E} / \partial t = 0$ implies that $\partial \mathbf{B} / \partial t = 0$ also, and $\nabla \times \mathbf{E} = 0$ according to Faraday's law.

We next consider some general properties of a longitudinal plane electric wave, taken to have the form

$$\mathbf{E} = E_x \hat{\mathbf{x}} e^{i(kx - \omega t)}. \quad (3)$$

This obeys $\nabla \times \mathbf{E} = 0$, and so can be derived from an electric potential, namely

$$\mathbf{E} = -\nabla V \quad \text{where} \quad V = i \frac{E_x}{k} e^{i(kx - \omega t)}. \quad (4)$$

The electric wave (3) has no associated magnetic wave, since Faraday's law tells us that

$$0 = \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (5)$$

and any magnetic field in the problem must be static.

It is well known that electromagnetic waves in vacuum are transverse. A longitudinal electric wave can only exist in a medium that can support a nonzero polarization density \mathbf{P} (volume density of electric dipole moments). The polarization density implies an effective charge density ρ given by

$$\rho = -\nabla \cdot \mathbf{P} \quad (6)$$

which is consistent with the first Maxwell equation,

$$\nabla \cdot \mathbf{E} = 4\pi \rho, \quad (7)$$

only if

$$\mathbf{P} = -\frac{\mathbf{E}}{4\pi}, \quad (8)$$

in which case the electric displacement \mathbf{D} of the longitudinal wave vanishes,

$$\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P} = 0. \quad (9)$$

Hence, the (relative) dielectric constant ϵ also vanishes

Strictly speaking, eq. (8) could read $\mathbf{P} = -\mathbf{E}/4\pi + \mathbf{P}'$, for any field \mathbf{P}' that obeys $\nabla \cdot \mathbf{P}' = 0$. However, since any magnetic field in the problem is static, the fourth Maxwell equation tells us that

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \left(\mathbf{J} + \frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} \right) \quad (10)$$

has no time dependence. Recalling that the polarization current is related by

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t}, \quad (11)$$

we again find relation (8) with the possible addition of a static field \mathbf{P}' that is associated with a truly electrostatic field \mathbf{E}' . In sum, a longitudinal electric wave described by eqs. (3), (8) and (9) can coexist with background electrostatic and magnetostatic fields of the usual type.

Maxwell's equations alone provide no relation between the wave number k and the wave frequency ω of the longitudinal wave, and hence the wave phase velocity ω/k is arbitrary. This suggests that purely longitudinal electric waves are best considered as limiting cases of more general waves, for which additional physical relations provide additional information as to the character of the waves.

2.2 Gauge Invariance

Since the electric wave (3) has no associated magnetic field, we can define its vector potential \mathbf{A} to be zero, which is certainly consistent with the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. Suppose, however, we prefer to work in the Lorentz gauge, for which

$$\nabla \cdot \mathbf{A} = -\frac{1}{c} \frac{\partial V}{\partial t}. \quad (12)$$

Then, the vector potential will be nonzero, and the electric field is related by

$$\mathbf{E} = -\nabla V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = E_x \hat{\mathbf{x}} e^{i(kx - \omega t)}. \quad (13)$$

Clearly the potentials have the forms

$$\mathbf{A} = A_x \hat{\mathbf{x}} e^{i(kx - \omega t)}, \quad V = V_0 e^{i(kx - \omega t)}, \quad (14)$$

which are consistent with $\mathbf{B} = \nabla \times \mathbf{A} = 0$. From the Lorentz gauge condition (12) we have

$$kA_x = \frac{\omega}{c} V_0, \quad (15)$$

and from eq. (13) we find

$$E_x = ikV_0 + i\frac{\omega}{c} A_x. \quad (16)$$

Hence,

$$\mathbf{A} = -i \frac{\omega c}{\omega^2 + k^2 c^2} E_x \hat{\mathbf{x}} e^{i(kx - \omega t)}, \quad V = -i \frac{k c^2}{\omega^2 + k^2 c^2} E_x e^{i(kx - \omega t)}. \quad (17)$$

We could also derive the wave (3) from the potentials

$$\mathbf{A} = -i \frac{c}{\omega} E_x \hat{\mathbf{x}} e^{i(kx - \omega t)}, \quad V = 0. \quad (18)$$

Thus, an ‘‘electrostatic wave’’ is not necessarily associated with an ‘‘electrostatic’’ scalar potential.

2.3 Energy Considerations

A common expression for the electric field energy density is $\mathbf{E} \cdot \mathbf{D}/8\pi$. However, this vanishes for longitudinal electric waves, according to eq. (9). Further, since the longitudinal electric wave can exist with zero magnetic field, there is no Poynting vector $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{H}$ or momentum density $\mathbf{p}_{\text{field}} = \mathbf{D} \times \mathbf{B}/4\pi c$, according to the usual prescriptions.

Let us recall the origins of the standard lore. Namely, the rate of work done by the field \mathbf{E} on current density \mathbf{J} is

$$\mathbf{J} \cdot \mathbf{E} = \frac{\partial \mathbf{P}}{\partial t} \cdot \mathbf{E} = -\frac{1}{4\pi} \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} = -\frac{\partial E^2/8\pi}{\partial t}, \quad (19)$$

using eqs. (8) and (11). This work is done at the expense of the electric field energy density u_{field} , which we therefore identify as

$$u_{\text{field}} = \frac{E^2}{8\pi} = \frac{E_x^2}{8\pi} \cos^2(kx - \omega t), \quad (20)$$

for the longitudinal wave (3). We readily interpret this energy density as moving in the $+x$ direction at the phase velocity $v_p = \omega/k$, even though the derivation of eq. (19) did not lead to a Poynting vector.

We should also note that energy is stored in the medium in the form of kinetic energy of the electrons (and, in general, ions as well) that contribute to the polarization,

$$\mathbf{P} = Ne(\mathbf{x} - \mathbf{x}_0) = -\frac{\mathbf{E}}{4\pi}. \quad (21)$$

Thus, the velocity of an electron is given by

$$\mathbf{v} = \mathbf{v}_0 - \frac{\dot{\mathbf{E}}}{4\pi Ne} = \mathbf{v}_0 - \frac{\omega E_x \hat{\mathbf{x}}}{4\pi Ne} \sin(kx - \omega t). \quad (22)$$

In squaring this to get the kinetic energy, we neglect the term in $\mathbf{v}_0 \cdot \hat{\mathbf{x}}$, assuming its average to be zero as holds for a medium that is at rest on average (and also holds for a plasma in a tokamak when x is taken as the radial coordinate in a small volume). Then, we find the mechanical energy density to be

$$u_{\text{mech}} = \frac{1}{2} N m v^2 = \frac{1}{2} N m v_0^2 + \frac{E_x^2}{8\pi} \frac{\omega^2 m}{4\pi N e^2} \sin^2(kx - \omega t) = u_{\text{mech},0} + \frac{\omega^2}{\omega_P^2} \frac{E_x^2}{8\pi} \sin^2(kx - \omega t), \quad (23)$$

where ω_P is the (electron) plasma frequency (2). We again can interpret the additional term as an energy density that flows in the $+x$ direction at the phase velocity.

The total, time-averaged energy density associated with the longitudinal wave is

$$\langle u_{\text{wave}} \rangle = \frac{\omega^2 + \omega_P^2}{2\omega_P^2} \frac{E_x^2}{8\pi}. \quad (24)$$

If the wave frequency is less than the plasma frequency, as is the case for examples of Bernstein waves discussed in the sec. 2.5, the longitudinal electric field energy density is larger than that of the mechanical energy density of the wave.

2.4 Longitudinal Waves in a Cold, Unmagnetized Plasma

As a preliminary exercise we consider the case of a longitudinal wave,

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(kx - \omega t), \quad (25)$$

in a cold, unmagnetized plasma. An electron at \mathbf{x}_0 in the absence of the wave has coordinate $\mathbf{x} = \mathbf{x}_0 + \delta\mathbf{x}$ when in the wave, where only the x component of the equation of motion is nontrivial:

$$m\delta\ddot{x} = -eE_x \cos(kx - \omega t) \approx -eE_x \cos(kx_0 - \omega t). \quad (26)$$

The approximation in eq. (26) is that the oscillations are small. Then we find,

$$\delta x \approx \frac{e}{m\omega^2} E_x \cos(kx_0 - \omega t). \quad (27)$$

The resulting electric dipole moment density \mathbf{P} is

$$\mathbf{P} = -Ne\delta x \hat{\mathbf{x}} = -\frac{Ne^2}{m\omega^2} \mathbf{E} = -\frac{\omega_P^2}{4\pi\omega^2} \mathbf{E}, \quad (28)$$

where ω_P is the (electron) plasma frequency (2).

For a longitudinal wave, the electric displacement must vanish according to eq. (9), so we find

$$0 = \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \left(1 - \frac{\omega_P^2}{\omega^2}\right) \mathbf{E}, \quad (29)$$

which requires that

$$\omega = \omega_P. \quad (30)$$

That is, the frequency of longitudinal electric waves can only be the plasma frequency in a cold, unmagnetized plasma.

2.5 Longitudinal Waves in a Hot, Magnetized Plasma

Turning now to the problem of plane waves in a magnetized plasma, we consider waves whose propagation vector \mathbf{k} is transverse to the external magnetic field \mathbf{B}_0 , and seek a solution where electric field vector \mathbf{E} is parallel to \mathbf{k} .

We adopt a rectangular coordinate system in which the external magnetic field \mathbf{B}_0 is along the $+z$ axis and the plane electric wave propagates along the $+x$ axis:

$$\mathbf{E} = E_x \hat{\mathbf{x}} \cos(kx - \omega t). \quad (31)$$

The unperturbed ($E = 0$) motion of an electron is on a helix of radius

$$r_B = \frac{v_\perp}{\omega_B}, \quad (32)$$

where $v_\perp = \sqrt{2KT/m}$ for all electrons in our simplified analysis. Hence, we can write the general (nonrelativistic) motion as

$$x = x_0 + r_B \cos(\omega_B t + \phi_0) + \delta x, \quad (33)$$

$$y = y_0 + r_B \sin(\omega_B t + \phi_0) + \delta y, \quad (34)$$

$$z = z_0 + v_z t + \delta z, \quad (35)$$

noting that the circular motion of a negatively charged electron is counterclockwise in the x - y plane for an external magnetic field along the $+z$ axis. For an electron in the collisionless plasma, we consider the Lorentz force only from the wave electric field and the external magnetic field, $-e(\mathbf{E} + \mathbf{v}/c \times \mathbf{B}_0)$. The equations of motion are then

$$m[-\omega^2 r_B \cos(\omega_B t + \phi_0) + \delta\ddot{x}] = -eE_x \cos(kx - \omega t) - \frac{eB_0}{c} [\omega_B r_B \cos(\omega_B t + \phi_0) + \delta\dot{y}] \quad (36)$$

$$m[-\omega^2 r_B \sin(\omega_B t + \phi_0) + \delta\dot{y}] = -\frac{eB_0}{c} [\omega_B r_B \sin(\omega_B t + \phi_0) - \delta\dot{x}] \quad (37)$$

$$m\delta\ddot{z} = 0. \quad (38)$$

Recalling eq. (1) for the cyclotron frequency, the equations of motion reduce to

$$\delta\ddot{x} + \omega_B \delta\dot{y} = -\frac{eE_x}{m} \cos(kx - \omega t), \quad (39)$$

$$\delta\dot{y} - \omega_B \delta\dot{x} = 0, \quad (40)$$

$$\delta\ddot{z} = 0. \quad (41)$$

Equation (41) has the trivial solution $\delta z = 0$, while eq. (40) integrates to

$$\delta\dot{y} = \omega_B \delta x. \quad (42)$$

With this, the remaining equation of motion becomes

$$\delta\ddot{x} + \omega_B^2 \delta x = -\frac{eE_x}{m} \cos(kx - \omega t), \quad (43)$$

To proceed, we must expand the factor $\cos(kx - \omega t)$, which we do as follows:

$$\begin{aligned} \cos(kx - \omega t) &= \cos(kx_0 - \omega t + kr_B \cos(\omega_B t + \phi_0) + k\delta x) \\ &\approx \cos(kx_0 - \omega t + kr_B \cos(\omega_B t + \phi_0)) \\ &\approx \cos(kx_0 - \omega t) \cos(kr_B \cos(\omega_B t + \phi_0)) - kr_B \cos(\omega_B t + \phi_0) \sin(kx_0 - \omega t) \\ &\approx \cos(kx_0 - \omega t) \left(1 - \frac{1}{2} k^2 r_B^2 \cos^2(\omega_B t + \phi_0)\right) \\ &\approx \cos(kx_0 - \omega t) \left(1 - \frac{1}{4} k^2 r_B^2\right) = \cos(kx_0 - \omega t) \left(1 - \frac{k^2 v_1^2}{4\omega_B^2}\right). \end{aligned} \quad (44)$$

In the above we have supposed that $\delta x \ll r_B$ in going from the first line to the second, that $r_B \ll x_0$ in going from the second line to the third, that $kr_B \ll 1$ and $\langle \cos(\omega_B t + \phi_0) \sin(kx_0 - \omega t) \rangle = 0$ in going from the third line to the fourth, and that $\langle \cos^2(\omega_B t + \phi_0) \rangle = 1/2$ in going from the fourth line to the fifth. Perhaps the most doubtful assumption is that $kr_B \ll 1$.

The approximate equation of motion is now

$$\delta\ddot{x} + \omega_B^2 \delta x = -\frac{eE_x}{m} \left(1 - \frac{k^2 v_1^2}{4\omega_B^2}\right) \cos(kx_0 - \omega t). \quad (45)$$

The solution to this is

$$\delta x = -\frac{e}{m(\omega_B^2 - \omega^2)} \left(1 - \frac{k^2 v_\perp^2}{4\omega_B^2}\right) E_x \cos(kx_0 - \omega t). \quad (46)$$

The resulting electric dipole moment density \mathbf{P} is

$$\mathbf{P} = -Ne\delta x \hat{\mathbf{x}} = \frac{Ne^2}{m(\omega_B^2 - \omega^2)} \left(1 - \frac{k^2 v_\perp^2}{4\omega_B^2}\right) \mathbf{E} = \frac{\omega_P^2}{4\pi(\omega_B^2 - \omega^2)} \left(1 - \frac{k^2 v_\perp^2}{4\omega_B^2}\right) \mathbf{E}, \quad (47)$$

where ω_P is the (electron) plasma frequency (2).

For a longitudinal wave, the electric displacement must vanish according to eq. (9), so we find

$$0 = \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \left[1 + \frac{\omega_P^2}{\omega_B^2 - \omega^2} \left(1 - \frac{k^2 v_\perp^2}{4\omega_B^2}\right)\right] \mathbf{E}, \quad (48)$$

which requires that

$$\omega^2 = \omega_B^2 + \omega_P^2 \left(1 - \frac{k^2 v_\perp^2}{4\omega_B^2}\right) = \omega_B^2 + \omega_P^2 \left(1 - \frac{k^2 KT}{2m\omega_B^2}\right). \quad (49)$$

This result corresponds to keeping only the first term of Bernstein's series expansion, eq. (50) of [1].

In the limit of a cold plasma, where $v_\perp = 0$, the frequency of the longitudinal wave is $\sqrt{\omega_B^2 + \omega_P^2}$, which is the so-called upper hybrid resonance frequency. (This result is well-known to follow from the assumption of a cold plasma.)

In our model, the effect of nonzero temperature is to lower the frequency of the longitudinal wave, bringing it closer to the cyclotron frequency, ω_B . The effect is greater for shorter wavelengths (larger wave number k). Our approximation implies that for wavelengths small compared to r_\perp , the characteristic radius of the electron cyclotron motion at temperature T , the frequency of the wave approaches zero. However, our approximation becomes doubtful for $kr_\perp \gg 1$. Bernstein finds that the wave frequency is restricted to a band around ω_B , which result is only hinted at by our analysis.

If we evaluate the dispersion relation (47) at the cyclotron frequency, $\omega = \omega_B$, then we find the following representative values for parameters of a Bernstein wave:

$$k = \frac{2\omega_B}{v_\perp} = \frac{2}{r_\perp}, \quad \lambda = \pi r_\perp, \quad \text{and} \quad v_p = \frac{\omega_B}{k} = \frac{v_\perp}{2} \ll c. \quad (50)$$

While our analysis does not constrain the phase velocity, $v_p = \omega/k$, of the longitudinal wave, we do find a relation between v_p and the group velocity, $v_g = d\omega/dk$:

$$v_g = \frac{d\omega}{dk} = -\frac{\omega_P^2}{\omega_B^2} \frac{KT}{2mv_p}. \quad (51)$$

The longitudinal electric waves are negative group velocity waves! We have written elsewhere on a paradox associated with this latter phenomenon [4], where we found that a negative group velocity can have any magnitude without contradicting the insight of Einstein that signals must propagate at velocities less than or equal to c . Hence, the lack of a constraint on v_p is not a fundamental flaw in the analysis.

The author thanks Brent Jones for bringing this problem to his attention. For a related discussion of magnetostatic waves, see [5].

3 References

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