Babinet’s Principle for Electromagnetic Fields
Zhong Ming Tan and Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(January 19, 2012)

1 Problem

In optics, Babinet’s principle [1] for complementary screens is that the sum of the wave transmitted through a screen (usually considered to be “black” except for its apertures), plus the wave transmitted through the complementary screen, is the same as if no screen were present. An electromagnetic version of this principle was given by Booker [2], who considered perfectly electrically conducting screens and argued that the electromagnetic fields in the case of the complementary screen (labeled with a ′) that appear in Babinet’s principle should be the dual fields −B′ and E′ rather than the nominal fields E′ and B′. That is, if the fields that would exist in the absence of the screen are labeled with the superscript i (for incident), the incident fields in the complementary case are taken to be E′i = −Bi and B′i = Ei.1 Then, the electromagnetic version of Babinet’s principle for the fields on the side of the screen away from the sources is

\[ E_{\text{away}} + B'_{\text{away}} = E^i_{\text{away}}, \quad B_{\text{away}} - E'_{\text{away}} = B^i_{\text{away}}. \] (1)

Also, the fields on the same side of the screen as the sources are related to the scattered/reflected fields Esr and Bsr if the perfectly conducting screen had no apertures by

\[ E_{\text{same side}} - B'_{\text{same side}} = E^sr_{\text{same side}}, \quad B'_{\text{same side}} + E'_{\text{same side}} = B^sr_{\text{same side}}. \] (2)

Justify these claims.

It suffices to consider the electromagnetic fields incident on the screen as being a plane electromagnetic wave of angular frequency \( \omega \), as a general wave field can be synthesized from such plane waves.2 Note that the fields dual to a plane electromagnetic wave with linear polarization are those obtained on rotating the direction of the polarization by 90°.

2 Solution

2.1 Dual Fields

Maxwell’s equations were extended by Heaviside [4, 5, 6], starting in 1885, to include the possibility of magnetic charges and currents,

\[ \nabla \cdot E = 4\pi \rho, \quad \nabla \cdot B = 4\pi \rho', \quad \nabla \times E = -\frac{4\pi}{c} J' - \frac{1}{c} \frac{\partial B}{\partial t}, \quad \nabla \times B = \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial E}{\partial t}. \] (3)

---

1We can also take the dual incident fields to have the opposite signs, in which case eqs. (1)-(2) hold on reversing the signs of the complementary fields.

2See, for example, [3].
where $\rho$ and $J$ are the densities of electrical charge and current, and $\rho'$ and $J'$ are the densities of hypothetical magnetic charge and current. The form of these equations is such that if $E$ and $B$ are solutions when only electrical charges and currents are present ($\rho' = 0 = J'$), then if only magnetic charges were present, and with values equal to the electrical charges and currents in the previous case, the electromagnetic fields $E'$ and $B'$ obey the duality relations

$$E' = -B, \quad B' = E. \quad (4)$$

### 2.2 No Screen vs. a Conducting Plane

A simple example is a plane electromagnetic wave traveling in empty space. There are no scattered fields in this case, and the incident fields are the total fields,

$$E = E^i = E_0 e^{i(kz-\omega t)} \hat{x}, \quad B = B^i = E_0 e^{i(kz-\omega t)} \hat{y}, \quad E^s = 0 = B^s. \quad (5)$$

A complementary situation is to have the entire plane $z = 0$ occupied by a perfectly electrically conducting screen. Then, the total fields vanish for $z > 0$, and the scattered fields in this regions are equal and opposite to the incident fields,

$$E'^i = E_0 e^{i(kz-\omega t)} \hat{x}, \quad E'(z > 0) = 0, \quad E'^s(z > 0) = -E'(z > 0), \quad (6)$$

$$B'^i = E_0 e^{i(kz-\omega t)} \hat{y}, \quad B'(z > 0) = 0, \quad B'^s(z > 0) = -B'(z > 0). \quad (7)$$

The fields (5)-(7) satisfy what we might naively expect Babinet’s principle to be for electromagnetism,

$$E(z > 0) + E'(z > 0) = E^i(z > 0), \quad B(z > 0) + B'(z > 0) = B^i(z > 0). \quad (8)$$

However, if we take the incident fields in the complementary case to be the duals of the original incident fields,

$$E'^i = -E_0 e^{i(kz-\omega t)} \hat{y}, \quad E'(z > 0) = 0, \quad E'^s(z > 0) = -E'(z > 0), \quad (9)$$

$$B'^i = E_0 e^{i(kz-\omega t)} \hat{x}, \quad B'(z > 0) = 0, \quad B'^s(z > 0) = -B'(z > 0), \quad (10)$$

the fields (5) and (9)-(10) rather trivially satisfy Booker’s version of Babinet’s principle for electromagnetism,

$$E(z > 0) + B'(z > 0) = E^i(z > 0), \quad B(z > 0) - E'(z > 0) = B^i(z > 0). \quad (11)$$

This example is too simple to clarify that eq. (11) rather than eq. (8) is the proper statement of Babinet’s principle for electromagnetism.

### 2.3 Solution via Smythe’s Diffraction Integrals

Booker [2] gave only a suggestive argument as to why the relations (1)-(2). An attempt at a more detailed argument for the electromagnetic version of Babinet’s principle was perhaps first given by Meixner [7, 8], and variants of this argument appear in [9, 10, 11] and in sec. 2.4 below.
In this section we follow the argument of sec. 10.8 of [11], which utilizes certain diffraction integrals due to Smythe [12, 13]. The conducting screen $S$ lies in the plane $z = 0$, the waves are incident from $z < 0$, and we use Gaussian units. Then, the fields (with time-dependence $e^{-i\omega t}$) for $z > 0$ can be computed from the electric fields in the apertures of the screen, according to
\[ E(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{\text{apertures of } S} \hat{z} \times E \frac{e^{ikr}}{r} d\text{Area}'', \quad (12) \]
\[ B(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{\text{apertures of } S} \hat{z} \times B \frac{e^{ikr}}{r} d\text{Area}'', \quad (13) \]
where $r = \sqrt{(x-x'')^2 + (y-y'')^2 + z^2}$, and $\hat{z} \times E = 0$ next to the perfectly conducting screen.

We can decompose the total electromagnetic fields $E$ and $B$ into the sum of “incident” and “scattered” fields,
\[ E = E^i + E^s, \quad B = B^i + B^s, \quad (14) \]
where the incident fields (defined for all $z$) are those associated with the sources at $z < 0$ in the absence of the screen, and the scattered field are those due only to the charges and currents on the screen. Relations of the forms (12)-(13) hold for both the incident and for the scattered fields.

We now consider the case when the incident fields are the duals of the original incident fields,
\[ E'^i = -B^i, \quad B'^i = E^i. \quad (15) \]
Only if the screen were made of a hypothetical perfect magnetic conductor in the case of the dual incident fields $E'^i$ and $B'^i$ would the scattered fields $E'^s$ and $B'^s$ be the duals of the scattered fields $E^s$ and $B^s$. Instead, we suppose the dual fields (15) are incident on the complementary screen $S'$ that is a perfect electrical conductor. The total fields in this case can then be written
\[ E' = E'^i + E'^s, \quad B' = B'^i + B'^s. \quad (16) \]

Smythe’s integral relations for the dual incident fields and the complementary screen are now
\[ E'(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{\text{apertures of } S'} \hat{z} \times E' \frac{e^{ikr}}{r} d\text{Area}'', \quad (17) \]
\[ B'(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{S'} \hat{z} \times B' \frac{e^{ikr}}{r} d\text{Area}'', \quad (18) \]
since the tangential component of $E'$ can be nonzero on the plane $z = 0$ only in the apertures of the complementary screen $S'$.

\[ ^3 \text{See also Appendix A.3 of [18].} \]
As mentioned in Appendix A.3 of [18], Smythe’s diffraction integrals (12)-(13) and (17)-(18) hold separately for the incident and scattered fields (when the region of integration is the entire plane \( z' = 0 \)). Hence, we can write

\[
E^s(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{\text{apertures of } S} \hat{z} \times E^s e^{ikr} \frac{d\text{Area}}{r},
\]

(19)

\[
B^s(z > 0) = \frac{1}{2\pi} \nabla \times \oint_S \hat{z} \times B^s e^{ikr} \frac{d\text{Area}}{r},
\]

(20)

\[
E'^s(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{S'} \hat{z} \times E'^s e^{ikr} \frac{d\text{Area}}{r},
\]

(21)

\[
B'^s(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{S'} \hat{z} \times B'^s e^{ikr} \frac{d\text{Area}}{r}.
\]

(22)

The forms (19)-(22) are mathematically consistent with the scattered fields in the complementary case being the duals of the scattered fields in the original case, but we cannot expect this to be true as the complementary screen is a perfect electrical conductor, not a perfect magnetic conductor.

To go further, it appears necessary that the scattered fields obey the symmetries

\[
E^x_s(x, y, -z) = E^x_s(x, y, z), \quad B^x_s(x, y, -z) = -B^x_s(x, y, z),
\]

(23)

\[
E^y_s(x, y, -z) = E^y_s(x, y, z), \quad B^y_s(x, y, -z) = -B^y_s(x, y, z),
\]

(24)

\[
E^z_s(x, y, -z) = -E^z_s(x, y, z), \quad B^z_s(x, y, -z) = B^z_s(x, y, z),
\]

(25)

as assumed in all “proofs” of Babinet’s principle in the literature. For the symmetries (23)-(25) to hold there must be no currents the flow from one side of the screen to the other, such that the vector potential due to currents on the screen have no \( z \)-component, i.e. \( A^s_z = 0 \).

As first noted in [14], and reviewed in [15], there can be no currents on the edges of a plane conducting screen as otherwise the magnetic field energy would be infinite in a finite volume surrounding a portion of the edge, and the relations (23)-(24) do hold in general.

According to the (anti)symmetries (23)-(24), the transverse components of the magnetic field vanish in the apertures of the screen (while equal and opposite nonzero transverse magnetic fields can exist close to the two sides of the conductor). In this case, the region of integration in relations (20) and (22) can be restricted to the conductors of the screens, whose locations correspond to the apertures of the complementary screens

\[
B^s(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{\text{apertures of } S'} \hat{z} \times B^s e^{ikr} \frac{d\text{Area}}{r},
\]

(26)

\[
B'^s(z > 0) = \frac{1}{2\pi} \nabla \times \oint_{S'} \hat{z} \times B'^s e^{ikr} \frac{d\text{Area}}{r}.
\]

(27)

It is claimed in sec. 10.8 of [11] that the similarity of the forms of eqs. (12) and (27), and that of eqs. (17) and (26) permit us to conclude that

\[
E(z > 0) = -B'^s(z > 0), \quad E'(z > 0) = B^s(z > 0),
\]

\footnote{In [11] the complementary incident fields are taken to be \( E'^i = B^i \) and \( B'^i = -E^i \) rather than those of eq. (15), which leads to various reversals of signs compared to those in this note.}
\[ B(z > 0) = B^i(z > 0) + B^s(z > 0) = -E^i(z > 0) + E'(z > 0) = E^s(z > 0), \]  
\[ B'(z > 0) = B'^i(z > 0) + B'^s(z > 0) = E'(z > 0) - E(z > 0) = -E^s(z > 0). \]

The total fields \( E' \) and \( B' \) are not the duals of the total fields \( E \) and \( B \), but of the scattered fields \( E^s \) and \( B^s \). If so, we finally obtain Booker’s electromagnetic version (1) of Babinet’s principle,

\[ E(z > 0) + B'(z > 0) = E(z > 0) - E^s(z > 0) = E^i(z > 0), \]
\[ B(z > 0) - E'(z > 0) = B(z > 0) - B^s(z > 0) = B^i(z > 0). \]

### 2.4 Jones’ Argument

A slightly different argument is given in sec. 9.3 of [10], following [8].

Jones begins with a justification of the symmetries (23)-(25), and then proceeds assuming that these symmetries hold. He invokes the spirit of the image method for the case of a perfectly electrically conducting plane with no apertures. In the case the fields for \( z < 0 \) are the same as if there were no conducting plane but image charge and current densities existed for \( z > 0 \), with \( \rho^{\text{image}}(x, y, z > 0) = -\rho(x, y, z < 0) \), \( J^{\text{image}}_{x,y}(x, y, z > 0) = -J_{x,y}(x, y, z < 0) \), and \( J^{\text{image}}_z(x, y, z > 0) = J_z(x, y, z < 0) \). Jones supposes these relations also hold if the conducting plane has apertures.

As noted in sec. 2.3, the symmetries (23)-(25) imply that certain components of the scattered fields vanish in the apertures of the screen at \( z = 0 \), namely,

\[ E^s_z = B^s_{x,y} = 0, \quad E_z = E^i_z, \quad B_{x,y} = B^i_{x,y}, \] 

in the apertures at \( z = 0 \). (33)

Likewise, next to the material of the screen we can write

\[ E_z(0^+) + E_z(0^-) = 2E^i_z(0), \quad B_{x,y}(0^+) + B_{x,y}(0^-) = 2B^i_{x,y}(0), \] 

next to conductor. (34)

The claim is that in the case of the complementary, electrically conducting screen the following fields are solutions to Maxwell’s equations,

\[ E'(z) = \begin{cases} -B(z) + B^i_{\parallel}(-z) - B^s_i(-z)\hat{z} & (z \leq 0), \\ B(z) - B^i(z) = B^s(z) & (z \geq 0), \end{cases} \]
\[ B'(z) = \begin{cases} E(z) + E^i_{\parallel}(-z) - E^s_i(-z)\hat{z} & (z \leq 0), \\ -E(z) + E^i(z) = -E^s(z) & (z \geq 0). \end{cases} \]

By considering the case that the complementary screen is vacuum, for which the total fields are the same as the incident fields, we see that the incident fields are the dual fields given by eq. (15).\(^5\)

\(^5\)In this case the original screen fills the entire plane \( z = 0 \) and reflects the incident wave. Here, the
Next to the complementary screen the fields (35)-(36) are, noting that the incident fields are continuous at \( z = 0 \),

\[
E'(z = 0^\pm) = \begin{cases}
-B_z^i(0^-) - (2B_z^i(0) + B_s^s(0^+) \hat{z}) & (z = 0^-), \\
B(0^+) - B^i(0) = B^s(0^+) & (z = 0^+),
\end{cases} \tag{40}
\]

\[
B'(z = 0^\pm) = \begin{cases}
2E_0^i(0) + E_0^i(0^-) + E_s^s(0^-) \hat{z} & (z = 0^-), \\
-E(0^+) + E^i(0) = -E^s(0^+) & (z = 0^+). 
\end{cases} \tag{41}
\]

Since the conductor of the complementary screen corresponds to the apertures in the original screen, eq. (33) tells us that the tangential electric field \( E_\parallel' \) and the normal magnetic field \( B'_z \) vanish next to the conductor of the complementary screen. Thus, the fields (35)-(36) satisfy the boundary conditions at the screen in the complementary case. It is therefore plausible that these fields are indeed solutions to Maxwell’s equations for the dual incident fields (15) and the complementary electrically conducting screen.\(^6\) Then, the representation of Babinet’s principle by eqs. (31)-(32) follows at once.

The fields (35)-(36) are defined for \( z < 0 \), and we find that Babinet’s principle for this region is that stated in eq. (2),

\[
E(z < 0) - B'(z < 0) = -E_\parallel^i(z > 0) + E_z^i(z > 0) \hat{z} = E_s^s(z < 0), \tag{42}
\]

\[
B(z > 0) + E'(z < 0) = B_\parallel^i(z > 0) - B_z^i(z > 0) \hat{z} = B_s^s(z < 0), \tag{43}
\]

recalling eqs. (37)-(39) for the scattered fields in the case of total reflection by the electrically conducting plane \( z = 0 \).\(^7\)

### 2.5 Solution via an Integral Equation for a Scalar Field Component

The arguments presented in secs. 2.3-4 appear to the authors to have logical gaps, so it may be worthwhile to consider other arguments. Already in 1897 Rayleigh gave an argument involving an integral equation for a scalar component of the electromagnetic fields [21, 22], and came very close to enunciating the electromagnetic version of Babinet’s principle. Some 50 years later, Fox [23, 24] (who considered only sound waves) and Copson [25, 26, 27, 28, 29] revived this theme. For the electromagnetic fields to be characterized by a single scalar field

---

\(^6\) Although Jones’ argument is not completely compelling, it seems more convincing than that of sec. 2.3.  
\(^7\) The relations (42)-(43) also follow from the argument of sec. 2.3 in that eqs. (35)-(36) can be obtained from eq. (28) and the relations (23)-(25).
component, the problem must be restricted to the (interesting) case of incident fields normal
to the screen, and with the apertures in the screen having edges all along one axis, say, the
$y$-axis, and finally with the incident fields having either $\mathbf{E}_i$ or $\mathbf{B}_i$ along the $y$-axis. Then,
following lengthy preliminary arguments, one finds the electromagnetic versions (1)-(2) of
Babinet’s principle to hold.

2.6 Examples of Babinet’s Principle

The only two cases for which “exact” results have been obtained which obey the electro-
magnetic form of Babinet’s principle are Sommerfeld’s famous solution [30, 31, 32] for the
diffraction of electromagnetic waves by a conducting half plane, and the transmission of
waves through a conducting planar grating [33, 34], although this principle was not noticed
in the original studies.

References


[2] H. Booker, Slot Aerials and Their Relation to Complementary Wire Aerials (Babinet’s
Principle), J. I.E.E. 93, 620 (1946),


genous Isotropic Medium, especially in regard to the Derivation of special Solutions,
and the Formula for Plane Waves, Phil. Mag. 27, 29 (1889),
http://physics.princeton.edu/~mcdonald/examples/EM/heaviside_pm_27_29_89.pdf

http://physics.princeton.edu/~mcdonald/examples/EM/heaviside_electrical_papers_2.pdf

http://physics.princeton.edu/~mcdonald/examples/EM/meixner_zn_1_496_46.pdf

[8] J. Meixner, Streng Thierie der Beugung elektromagnetisher Wellen an der vollkomen
leitenden Kreisscheibe, Z. Naturf. 3a, 506 (1948),


Wellen an vollkommen leitenden ebenen Schirmen*, Ann. Phys. 441, 1 (1950),
http://physics.princeton.edu/~mcdonald/examples/EM/meixner_ap_441_1_50.pdf

a Plane Conducting Screen* (Jan. 14, 2012),


[21] Lord Rayleigh, *On the Passage of Waves through Apertures in Plane Screens, and Allied Problems*, Phil. Mag. 43, 259 (1897),

[22] Lord Rayleigh, *On the Incidence of Aerial and Electric Waves upon Small Obstacles in the form of Ellipsoids or Elliptic Cylinders, and on the Passage of Electric Waves through a circular Aperture in a Conducting Screen*, Phil. Mag. 44, 28 (1897),


