

Expansion of an Axially Symmetric, Static Magnetic Field in Terms of Its Axial Field

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1 Problem

Deduce a series expansion of an axially symmetric, static magnetic field in terms of its axial field $B_z(0, 0, z)$ in cylindrical coordinates (r, ϕ, z) . Also give an expansion for the vector potential of this field. The azimuthal currents that produce this field are at very large radius r .

2 Solution

This problem is a peculiar kind of boundary value problem in which a field is specified only along a line. In case the on-axis field has transverse components there is no a unique solution, as discussed in [1]. Here we obtain a unique solution under the assumption that the field off-axis is azimuthally symmetric. See sec. 13.4.2 of [2] for a multipole expansion for fields without azimuthal symmetry.¹

2.1 Expansion of the Field

Suppose a magnetic field in a current-free region is rotationally symmetric about the z -axis. Then,

$$\mathbf{B} = B_r(r, z)\hat{\mathbf{r}} + B_z(r, z)\hat{\mathbf{z}} \quad (1)$$

in cylindrical coordinates (r, ϕ, z) . If we write

$$B_z(r, z) = \sum_{n=0}^{\infty} a_n(z)r^n, \quad \text{and} \quad B_r(r, z) = \sum_{n=0}^{\infty} b_n(z)r^n, \quad (2)$$

then $a_0(z) = B_z(0, z)$. Since the divergence of the magnetic field vanishes, the proposed expansions (2) obey

$$\nabla \cdot \mathbf{B} = \frac{1}{r} \frac{\partial B_r}{\partial r} + \frac{\partial B_z}{\partial z} = \sum_n \left[(n+1)b_n r^{n-1} + a_n^{(1)} r^n \right] = 0, \quad (3)$$

where $a^{(m)}(z) \equiv d^m a / dz^m$. For this to be true at all r , the coefficients of r^n must separately vanish for all n . Hence,

$$b_0 = 0, \quad (4)$$

$$b_n = -\frac{a_{n-1}^{(1)}}{n+1}. \quad (5)$$

¹The function $a_0(z) = a_0^{(0)}(z)$ used here is the same as $C_0^{[1]}(z)$ in [2].

Since the curl of the magnetic field also vanishes (outside the source currents),

$$(\nabla \times \mathbf{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \sum_n \left(b_n^{(1)} r^n - n a_n r^{n-1} \right) = 0, \quad (6)$$

Again, the coefficient of r^n must vanish for all n , so that

$$b_n^{(1)} = (n+1)a_{n+1}. \quad (7)$$

Using eq. (7) in eq. (5), we find

$$b_n = -\frac{b_{n-2}^{(2)}}{(n+1)(n+3)}. \quad (8)$$

Since b_0 vanishes, b_{2n} vanishes for all n , and from eq. (7), a_{2n+1} vanishes for all n . Then, using eq. (8) in eq. (7), we find

$$a_{2n} = -\frac{a_{2n-2}^{(2)}}{4n^2}. \quad (9)$$

Repeatedly applying this to itself gives

$$a_{2n} = (-1)^n \frac{a_0^{(2n)}}{2^{2n}(n!)^2}. \quad (10)$$

Inserting this in eq. (5), we get

$$b_{2n+1} = (-1)^{n+1} \frac{a_0^{(2n+1)}}{2^{2n+1}(n+1)(n!)^2}. \quad (11)$$

Combining eqs. (10)-(11) with eq. (2), we arrive at the desired forms,

$$B_z(r, z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n!)^2} \left(\frac{r}{2}\right)^{2n}, \quad (12)$$

and

$$B_r(r, z) = \sum_n (-1)^{n+1} \frac{a_0^{(2n+1)}(z)}{(n+1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1}, \quad (13)$$

for the field components, where

$$a_0^{(n)} = \frac{d^n a_0}{dz^n}. \quad (14)$$

These results are overly detailed for some purposes. If one is interested only in the leading behavior at small r , then eqs. (12)-(13) simplify to

$$B_z(r, z) \approx B_z(0, z), \quad B_r(r, z) \approx -\frac{r}{2} \frac{\partial B_z(0, z)}{\partial z}. \quad (15)$$

The result for B_r also follows quickly from $\nabla \cdot \mathbf{B} = 0$, according to eq. (3),

$$B_r(r, z) = -\int_0^r r \frac{\partial B_z(r, z)}{\partial z} dr \approx -\int_0^r r \frac{\partial B_z(0, z)}{\partial z} dr = -\frac{r}{2} \frac{\partial B_z(0, z)}{\partial z}. \quad (16)$$

It is also instructive that the approximation (16) can be deduced quickly from the integral form of Gauss' law (without the need to recall the form of $\nabla \cdot \mathbf{B}$ in cylindrical coordinates). Consider a Gaussian pillbox of radius r and thickness dz centered on $(r = 0, z)$. Then,

$$\begin{aligned} 0 &= \int \mathbf{B} \cdot d\mathbf{S} \approx \pi r^2 [B_z(0, z + dz) - B_z(0, z)] + 2\pi r dz B_r(r, z) \\ &\approx \pi r^2 dz \frac{\partial B_z(0, z)}{\partial z} + 2\pi r dz B_r(r, z), \end{aligned} \quad (17)$$

which again implies eqs. (15).

2.2 Expansion of the Vector Potential

The magnetic field can be generated by (distant) currents that are purely azimuthal, so that a purely azimuthal vector potential A_ϕ suffices. Then,

$$B_r = -\frac{\partial A_\phi}{\partial z}, \quad \text{and} \quad B_z = \frac{1}{r} \frac{\partial(rA_\phi)}{\partial r}. \quad (18)$$

Hence, $A_\phi = -\int B_r dz$, are recalling eq. (13) we find

$$A_\phi(r, z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n+1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1}. \quad (19)$$

This result also follows from $A_\phi = (1/r) \int r B_z dr$ and eq. (12).

3 B Deduced from Its Value on a Surface

A more typical boundary value problem is to determine the field \mathbf{B} from its value on a bounding surface.

One approach for this is to recall the results of vector diffraction theory, particularly as formulated by Kottler [3, 4] for fields with time dependence $e^{-i\omega t}$ in vacuum,²

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \int_V \left(\frac{ik}{c} \mathbf{J}(\mathbf{x}') \frac{e^{ikr}}{r} + \rho(\mathbf{x}') \nabla' \frac{e^{ikr}}{r} \right) d\text{Vol}' + \frac{i}{\omega} \oint_S (\mathbf{J} \cdot \hat{\mathbf{n}}') \nabla' \frac{e^{ikr}}{r} d\text{Area}' \\ &\quad - \frac{1}{4\pi} \nabla \times \oint_S \left\{ [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')] \frac{e^{ikr}}{r} + \frac{i}{k} \nabla \times [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')] \frac{e^{ikr}}{r} \right\} d\text{Area}', \end{aligned} \quad (20)$$

$$\begin{aligned} \mathbf{B}(\mathbf{x}) &= \frac{1}{c} \int_V \mathbf{J}(\mathbf{x}') \times \nabla' \frac{e^{ikr}}{r} d\text{Vol}' \\ &\quad - \frac{1}{4\pi} \nabla \times \oint_S \left\{ [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')] \frac{e^{ikr}}{r} - \frac{i}{k} \nabla \times [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')] \frac{e^{ikr}}{r} \right\} d\text{Area}', \end{aligned} \quad (21)$$

where $\hat{\mathbf{n}}'$ is the outward unit vector normal to surface S , $r = |\mathbf{x} - \mathbf{x}'|$, c is the speed of light in vacuum, $k = \omega/c$, and Gaussian units are employed. See the Appendix of [5] for derivations and discussion of these forms.

²The operations involving ∇ , which act only on the factor r , should be performed before the surface integrations in eqs. (20)-(21).

For a region with no currents the magnetic field can be related to a vector potential that follows from eq. (21) as

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \oint_S \left\{ \mathbf{B}(\mathbf{x}') \times \hat{\mathbf{n}}' \frac{e^{ikr}}{r} - \frac{i}{k} \nabla \times [\mathbf{E}(\mathbf{x}') \times \hat{\mathbf{n}}'] \frac{e^{ikr}}{r} \right\} d\text{Area}', \quad (22)$$

assuming that we can take the curl after performing the integrations. If \mathbf{E} and \mathbf{B} are zero everywhere on the surface of a region then \mathbf{A} is zero in its interior, according to eq. (22). The prescription of eq. (22) cannot be extended to all of space since there must be currents somewhere if \mathbf{B} is nonzero somewhere.

In the static limit, $\omega = 0 = k$, the electric field does not depend the current density \mathbf{J} or the magnetic field, and the magnetic field does not depend on the electric field. Noting that $\nabla'(1/r) = \hat{\mathbf{r}}/r^2 = -\nabla(1/r)$, we obtain

$$\mathbf{E}(\mathbf{x}) = \int_V \rho(\mathbf{x}') \frac{\hat{\mathbf{r}}}{r^2} d\text{Vol}' + \frac{1}{4\pi} \oint_S \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{n}}' \times \mathbf{E}(\mathbf{x}')]}{r^2} d\text{Area}', \quad (23)$$

$$\mathbf{B}(\mathbf{x}) = \frac{1}{c} \int_V \frac{\mathbf{J}(\mathbf{x}') \times \hat{\mathbf{r}}}{r^2} d\text{Vol}' + \frac{1}{4\pi} \oint_S \frac{\hat{\mathbf{r}} \times [\hat{\mathbf{n}}' \times \mathbf{B}(\mathbf{x}')]}{r^2} d\text{Area}', \quad (24)$$

If there are no currents within the volume of integration, the static magnetic field there can be deduced from the vector potential

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi} \oint_S \frac{\mathbf{B}(\mathbf{x}') \times \hat{\mathbf{n}}'}{r} d\text{Area}' \quad (\text{static limit}), \quad (25)$$

recalling eq. (21). The example of a static, toroidal magnetic field (for which $\mathbf{B} = 0$ outside the torus but $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{Area}$ is nonzero for loops that link the torus) suggests that eqs. (22) and (25) are restricted to simply connected regions.

3.1 Uniform Axial Field

As an example, consider a uniform axial field, $\mathbf{B} = B_0 \hat{\mathbf{z}}$ that is generated by azimuthal currents about the z -axis. The associated vector potential has only the azimuthal component

$$A_\phi = \frac{\rho B_0}{2}. \quad (26)$$

in a cylindrical coordinate system (ρ, ϕ, z) .

We take the point of observation to be $(\rho, 0, 0)$. As the surface of integration for eq. (25) we consider a cylinder of radius $a > \rho$ with faces at $-z_1$ and z_2 . Then, $\mathbf{B} \times \hat{\mathbf{n}}' = B_0 \hat{\phi}$ and

$$\begin{aligned} A_\phi &= A_y = \frac{1}{4\pi} \int_0^{2\pi} a d\phi \int_{-z_1}^{z_2} dz \frac{B_0 \cos \phi}{\sqrt{z^2 + a^2 + \rho^2 - 2a\rho \cos \phi}} \\ &= \frac{aB_0}{4\pi} \int_0^{2\pi} \cos \phi d\phi \ln \frac{z_2 + \sqrt{z_2^2 + a^2 + \rho^2 - 2a\rho \cos \phi}}{-z_1 + \sqrt{z_1^2 + a^2 + \rho^2 - 2a\rho \cos \phi}} \\ &= \frac{aB_0}{4\pi} \int_0^{2\pi} \cos \phi d\phi \left[\ln \left(z_2 + \sqrt{z_2^2 + a^2 + \rho^2 - 2a\rho \cos \phi} \right) \right. \\ &\quad \left. + \ln \left(z_1 + \sqrt{z_1^2 + a^2 + \rho^2 - 2a\rho \cos \phi} \right) - \ln \left(a^2 + \rho^2 - 2a\rho \cos \phi \right) \right] \\ &= -\frac{aB_0}{4\pi} \int_0^{2\pi} \cos \phi d\phi \ln \left(1 + \frac{\rho^2}{a^2} - 2\frac{\rho}{a} \cos \phi \right) = \frac{\rho B_0}{2}, \end{aligned} \quad (27)$$

using 4.397.6 of [6]. A delicacy is our assumption that

$$\int_0^{2\pi} \cos \phi \, d\phi \ln \left(z + \sqrt{z^2 + a^2 + \rho^2 - 2a\rho \cos \phi} \right) = 0, \quad (28)$$

for nonzero values of z . This integral clearly goes to zero for large z , and the calculation (27) of A_ϕ must be independent of the values of z_1 and z_2 .

3.2 Other Formulations

Section 14.3-4 of [2] gives a formalism by which \mathbf{B} can be computed from knowledge of its normal component, $\mathbf{B} \cdot \hat{\mathbf{n}}$, on elliptical cylindrical surfaces, and sec. 18.2 describes the use of the tangential component $\mathbf{B} \times \hat{\mathbf{n}}$ on circular cylinders. Expansions in terms of a magnetic scalar potential can also be given [7].

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