Avril's Radiation Problem

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1 Problem

Benoit Avril has posed the problem of a positive charge on a circular path of angular velocity ω spinning in the counterclockwise sense, and a negative charge on a circular path with the same angular velocity in the clockwise sense.

Deduce the electromagnetic fields, and the radiation, of this configuration supposing that the radius a of the circular path is small compared to the wavelength $\lambda = 2\pi c/\omega$, where c is the speed of light in vacuum, and that the velocity $v = a\omega$ is small compared to c.

2 Solution

2.1 $v \ll c$

In the stated approximation (first studied by Hertz [1]), we consider the electric dipole moment $\mathbf{p}$ of the configuration, taking the circular path to lie in the x-y plane with its center at the origin,

$$\mathbf{p} = qa(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\, e^{-i\omega t} - qa(\hat{\mathbf{x}} - i\hat{\mathbf{y}})\, e^{-i\omega t} = 2iqa\,\hat{\mathbf{y}}\, e^{-i\omega t} = ip\,\hat{\mathbf{y}}\, e^{-i\omega t}, \tag{1}$$

where the magnitude of the dipole moment is $p = 2qa$, and the particles are at "3 o'clock" $(x = a, y = 0)$ at time $t = 0$.

The electric and magnetic fields of an ideal, point Hertzian electric dipole $\mathbf{p}$ can be written (in Gaussian units) as[1]

$$\mathbf{E} = ik^2 p\,(\hat{\mathbf{r}} \times \hat{\mathbf{y}}) \times \hat{\mathbf{r}}\,\frac{e^{i(kr-\omega t)}}{r} + ip\,[3(\hat{\mathbf{y}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{y}}]\left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}, \tag{2}$$

$$\mathbf{B} = ik^2 p\,(\hat{\mathbf{r}} \times \hat{\mathbf{y}})\left(\frac{1}{r} - \frac{1}{ikr^2}\right) e^{i(kr-\omega t)}, \tag{3}$$

whose real parts are

$$\mathbf{E} = -k^2 p\,(\hat{\mathbf{r}} \times \hat{\mathbf{y}}) \times \hat{\mathbf{r}}\frac{\sin(kr-\omega t)}{r} + p\,[3(\hat{\mathbf{y}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}]\left[\frac{k\cos(kr-\omega t)}{r^2} - \frac{\sin(kr-\omega t)}{r^3}\right], \tag{4}$$

$$\mathbf{B} = -k^2 p\,(\hat{\mathbf{r}} \times \hat{\mathbf{y}})\left[\frac{\sin(kr-\omega t)}{r} + \frac{\cos(kr-\omega t)}{kr^2}\right], \tag{5}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer, $\mathbf{p} = p\sin\omega t\,\hat{\mathbf{y}}$ is the electric dipole moment vector, ω is the angular frequency, and $k = \omega/c = 2\pi/\lambda$ is the wave number.

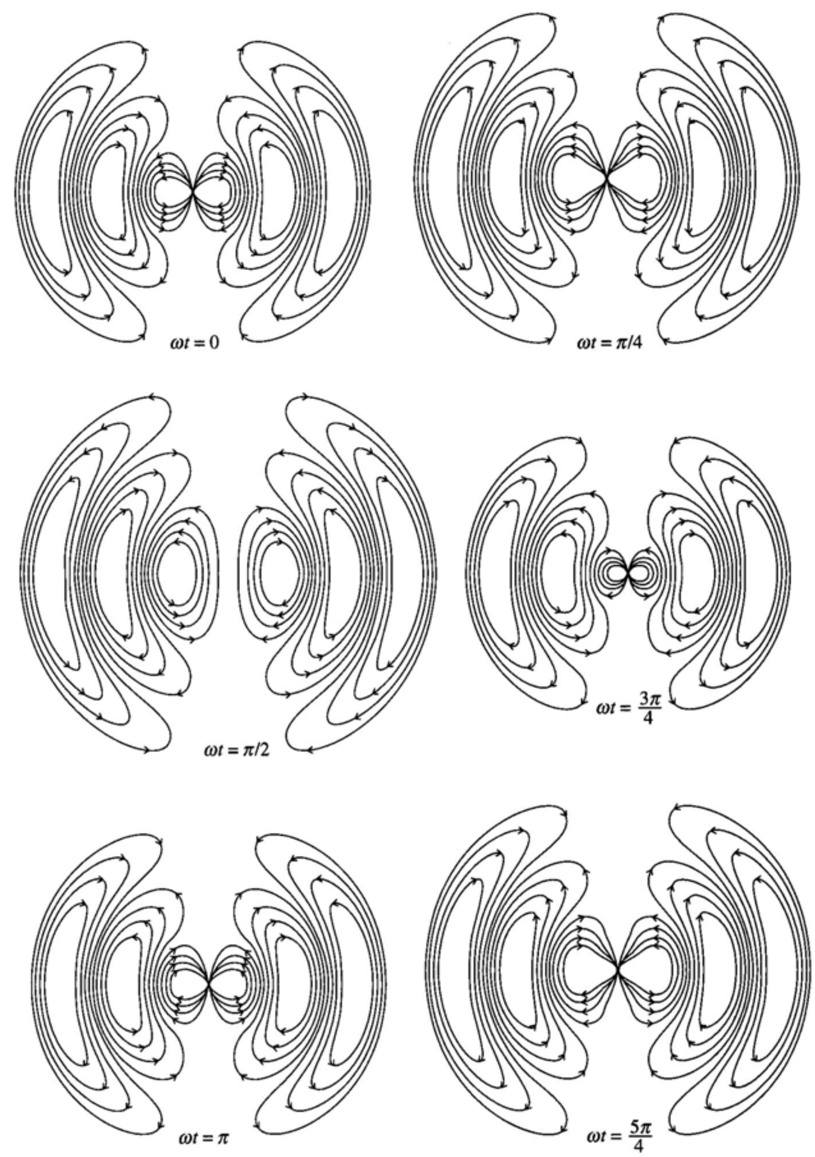
[1] See, for example, sec. 9.2 of [2].

We say that the radiation part of these fields are the terms that vary as $1/r$:

$$\mathbf{E}_{\rm rad} = k^2 p\,\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{y}}) \frac{\sin(kr - \omega t)}{r}\,, \tag{6}$$

$$\mathbf{B}_{\rm rad} = -k^2 p\,(\hat{\mathbf{r}} \times \hat{\mathbf{y}}) \frac{\sin(kr - \omega t)}{r}\,. \tag{7}$$

In the near zone of the dipole, where $kr \lesssim 1$, the radiation fields are smaller that the other components of $\mathbf{E}$ and $\mathbf{B}$. The most prominent feature of the fields in the near zone is that the electric field looks a lot like that of an electrostatic dipole, as shown in the figure below. Because field patterns that look like radiation are discernable only for $r \gtrsim \lambda$, there may be an impression that the radiation is created at some distance from an antenna, rather than at the antenna itself.



Since the radiated power comes from the antenna (from the power supply that drives the antenna), there must be a flow of energy out from the surface of the antenna into the surrounding space. The usual electrodynamic measure of energy flow is Poynting's vector [3] (in a medium with unit relative permeability),

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}. \tag{8}$$

When we use the fields (4)-(5) to calculate the Poynting vector we find six terms, some of which do not point along the radial vector $\hat{\mathbf{r}}$:

$$\begin{aligned}
\mathbf{S} &= \frac{c}{4\pi}\left\{k^4p^2\,[(\hat{\mathbf{r}}\times\hat{\mathbf{y}})\times\hat{\mathbf{r}}]\times(\hat{\mathbf{r}}\times\hat{\mathbf{y}})\left[\frac{\sin^2(kr-\omega t)}{r^2}+\frac{\cos(kr-\omega t)\sin(kr-\omega t)}{kr^3}\right]\right. \\
&\quad +k^2p^2\,[3(\hat{\mathbf{y}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\hat{\mathbf{y}}]\times(\hat{\mathbf{r}}\times\hat{\mathbf{y}})\left[\frac{\cos^2(kr-\omega t)-\sin^2(kr-\omega t)}{r^4}\right. \\
&\qquad \left.\left.+\cos(kr-\omega t)\sin(kr-\omega t)\left(\frac{k}{r^3}-\frac{1}{kr^5}\right)\right]\right\} \\
&= \frac{c}{4\pi}\left\{k^4p^2\,\sin^2\theta\,\hat{\mathbf{r}}\left[\frac{\sin^2(kr-\omega t)}{r^2}+\frac{\cos(kr-\omega t)\sin(kr-\omega t)}{kr^3}\right]\right. \\
&\quad +k^2p^2\,[(3\cos^2\theta-1)\,\hat{\mathbf{r}}-2\cos\theta\,\hat{\mathbf{y}}]\left[\frac{\cos^2(kr-\omega t)-\sin^2(kr-\omega t)}{r^4}\right. \\
&\qquad \left.\left.+\cos(kr-\omega t)\sin(kr-\omega t)\left(\frac{k}{r^3}-\frac{1}{kr^5}\right)\right]\right\},
\end{aligned} \tag{9}$$

where θ is the angle between vectors $\mathbf{r}$ and $\mathbf{p}$. As well as the expected radial flow of energy, there is a flow in the direction of the dipole moment $\mathbf{p} = ip\,\hat{\mathbf{y}}$. Since the product $\cos(kr-\omega t)\sin(kr-\omega t)$ can be both positive and negative, part of the energy flow is inwards at times, rather than outwards as expected for pure radiation.

However, we obtain a simple result if we consider only the time-averaged Poynting vector, $\langle\mathbf{S}\rangle$. Noting that $\langle\cos^2(kr-\omega t)\rangle = \langle\sin^2(kr-\omega t)\rangle = 1/2$ and $\langle\cos(kr-\omega t)\sin(kr-\omega t)\rangle = (1/2)\,\langle\sin 2(kr-\omega t)\rangle = 0$, eq (9) leads to

$$\langle\mathbf{S}\rangle = \frac{ck^4p^2\sin^2\theta}{8\pi r^2}\,\hat{\mathbf{r}}. \tag{10}$$

The time-average Poynting vector is purely radially outwards, and falls off as $1/r^2$ at all radii, as expected for a flow of energy that originates in the oscillating point dipole. The time-average angular distribution $d\langle P\rangle/d\Omega$ of the radiated power is related to the Poynting vector by

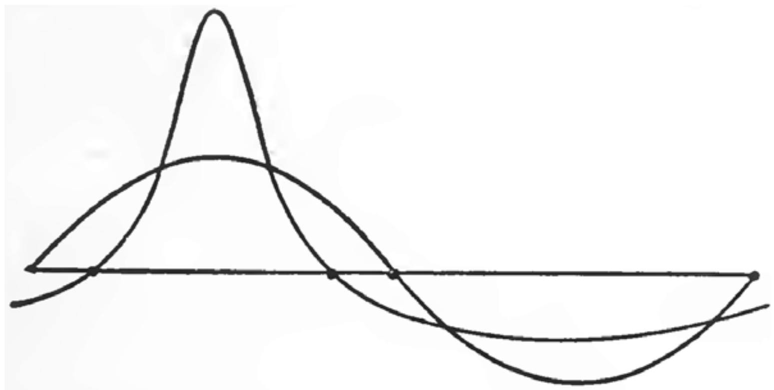
$$\frac{d\langle P\rangle}{d\Omega} = r^2\,\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle = \frac{ck^4p^2\sin^2\theta}{8\pi} = \frac{p^2\omega^4\sin^2\theta}{8\pi c^3}, \tag{11}$$

which is the expression usually derived for dipole radiation in the far zone. Here we see that this expression holds in the near zone as well.

We conclude that radiation, as measured by the time-averaged Poynting vector, exists in the near zone of the present example (and of all antennas) as well as in the far zone.

2.2 $v \approx c$

When the charges move with velocity close to that of light the fields are considerably different from those in sec. 2.1, as first studied by Heaviside in sec. 534, pp. 432-498 of [4]. A graphical method of determining the "radiation" field (which falls off as $1/r$) of a single charge according to a distant observer was given on p. 445, which was also discussed by Feynman in chap. 34 of [5].



While the electric field would be sinusoidal with time for $v \ll c$, if $v \approx c$ the electric field is very large when the charge is heading directly towards the observer (at the retarded time $t' = t - r/c$). This large field occurs once a revolution of the charge in its orbit. This large field can be thought of as a "searchlight" beam of angular extent $1/\gamma = \sqrt{1 - v^2/c^2}$ that rotates with angular velocity ω.[2]

For the case of two opposite charges moving oppositely around a circle as in sec. 2.1, the fields of the two charges add. For a distant observer on the positive x-axis, the two charges head directly towards him/her at the same time, so only a single large pulse of radiation would be detected each cycle, which pulse would have four times the intensity of the pulse in case of only a single charge. For an observer in the x-y plane at angle θ to the x-axis, two large pulses would be detected each cycle, separated by time θ/ω;[3] these pulses would have the same intensity as that in case of a single charge.

References

[1] H. Hertz, *Die Kräfte electrischer Schwimgungen, behandelt nach der Maxwell'schen Theorie*, Ann. d. Phys. **36**, 1 (1889),
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[2] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).

[2] https://en.wikipedia.org/wiki/Synchrotron_radiation

[3] Two pulses would be clearly distinguished only at angles $\theta > 1/\gamma$, *i.e.*, at angles larger than the angular spread of the "searchlight" beam.

[3] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
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[4] O. Heaviside, *Electromagneetic Theory*, Vol. III (Electrician Printing, London, 1912),
http://physics.princeton.edu/~mcdonald/examples/EM/heaviside_electromagnetic_theory_3.pdf

[5] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. 1 (Addison Wesley, 1964), http://www.feynmanlectures.caltech.edu/I_34.html