The Equivalence Principle and Round-Trip Times for Light

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1 Problem

If an observer, at rest (in flat spacetime) between a pair of mirrors each at distance D but in opposite directions, emits two pulses of light simultaneously they return to the observer simultaneously after time $2D/c$, where c is the speed of light. Discuss the case that the observer and mirrors have uniform acceleration (with respect to the inertial lab frame) along their common line.

Compare with the case that the observer and mirrors are at rest in a "uniform gravitational field".

2 Solution

2.1 Accelerated Observer and Mirrors

We consider an observer, initially at $z = 0$, and two mirrors with $z = \pm D$. All three have uniform acceleration $\mathbf{a} = a\,\hat{\mathbf{z}}$ with respect to an inertial frame in flat spacetime.[1]

The observer emits pulses of light along the $\pm z$-axis at time $t = 0$, which thereafter reflect off the mirrors and return to the observer at some later time. The light pulses obey

$$z_{p^\pm} = \pm ct \tag{1}$$

until they reflect off the mirrors. The latter have equations of motion

$$z_{m^\pm} = \pm D + \frac{c^2}{a}\left(\sqrt{1 + a^2t^2/c^2} - 1\right) \approx \pm D + \frac{at^2}{2}\,, \tag{2}$$

where the approximation holds if the velocity of the mirrors is small compared to c when the light pulses reach them. Then, the times when the pulses reach the mirrors are related by

$$t_\pm = \frac{c}{a}\left(\pm 1 \mp \sqrt{1 \pm \frac{2aD}{c^2}}\right) \approx \frac{D}{c}\left(1 \pm \frac{aD}{2c^2}\right), \tag{3}$$

expanding the square root to second order, and the corresponding positions of the mirrors are

$$z_{m^\pm} \approx \pm D + \frac{aD^2}{2c^2} \tag{4}$$

[1]See Appendix A for discussion of the meaning of the term "uniform acceleration".

keeping terms only to order D^2. The reflected photons obey, to the same order,

$$z_{p^\pm} = z_{m^\pm} \mp c(t - t_\pm) \approx \pm 2D \mp ct + \frac{aD^2}{c^2}. \tag{5}$$

Of course, the observer has position $z \approx at^2/2$, so the reflected pulses reach the observer at times

$$T_{p^\pm} \approx \frac{c}{a}\left(\mp 1 \pm \sqrt{1 \pm \frac{4aD}{c^2} + \frac{2a^2D^2}{c^4}}\right) \approx \frac{2D}{c} \mp \frac{aD^2}{c^3}\,. \tag{6}$$

The difference in the round-trip times of the two light pulses is

$$\Delta T \approx \frac{2aD^2}{c^3}\,. \tag{7}$$

For example, if $a = g \approx 10$ m/s^2 and $D = 1$ m, then $\Delta T \approx 6 \times 10^{-25}$ s. The period of optical light is about 2×10^{-15} s, so this time offset is about 3×10^{-10} periods. To have the time offset be one period, which might be measurable, we would need $D \approx 200$ km.

2.2 Observer and Mirrors at Rest in a Uniform Gravitational Field

The notion of a uniform gravitational field is somewhat elusive.. If one associates gravitational fields with sources of mass/energy, then physical gravitational fields are typically associated with distortions of spacetime.[2] On the other hand, the equivalence principle implies that a uniformly accelerated reference frame in flat spacetime should be equivalent to a uniform gravitational field. Of course, a uniform field over all spacetime is a mathematical idealization, such that there is room for discussion as to the relevant physical approximation to this concept. Lengthy debate on this topic may or may not have converged, but present wisdom seems to be that reasonably physical assumptions as to the sources of a uniform gravitational field are consistent with it being associated with flat spacetime [5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Often a weak, uniform gravitational is taken to be described by the metric

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\left(1 + \frac{gz}{c^2}\right)^2 dt^2, \qquad (|z| < c^2/g), \tag{8}$$

where $g = 2\pi G\rho$, G is Newton's gravitational constant and ρ is the density of mass/energy. See, for example, [9].

For spacetime described by the static metric (8), electrodynamics obey Maxwell's equation with the alterations that the vacuum has relative permittivity and permeability given by

$$\epsilon = \mu = \frac{1}{1 + gz/c^2}\,, \tag{9}$$

[2]These distortions are often called "curvature", but the case of hypothetical "cosmic strings" and "domain walls" [3, 4] spacetime is flat with topological defects. Vacuum "domain walls" are not physically viable, but remain an interesting theoretical construct.

as discussed, for example, in sec. 90 of [1]. A consequence is that the speed, u, of light emitted at $z = 0$ is a function of z according to[3,4]

$$u(z) = c\,(1 + gz/c^2), \tag{10}$$

The round-trip times $T_\pm$ for light emitted upwards and downwards at $z = 0$ and reflecting off mirrors at $z = \pm D$ (where $gD \ll c^2$) are

$$T_\pm = 2\int_0^{\pm D} \frac{\pm dz}{u} \approx \frac{2}{c}\int_0^{\pm D} \pm dz\,(1 - gz/c^2) = \frac{2D}{c} \mp \frac{gD^2}{c^3}\,. \tag{11}$$

The time difference between the two round-trips is

$$\Delta T = \frac{2gD^2}{c^3}\,, \tag{12}$$

which is the same as eq. (7) for observer and mirrors accelerated in flat spacetime with $a = g$. This is consistent with the popular understanding that a uniformally accelerated frame in flat spacetime is equivalent to the "uniform gravitational field" described by the metric (8).[5]

If we approximate a uniform gravitational field by that at the surface of the Earth, then the symbol g in eq. (10) becomes, approximately, $g_0(1 - z^2/2R_E^2)$ where $g_0 = GM_E/R_E^2$, G is Newton's gravitational constant, M_E and R_E are the mass and radius of the Earth, respectively. This results in a very small correction to eq. (12), such that in principle an observer in a box with mirror walls at the Earth's surface could determine that (s)he is not in flat spacetime by performing the present experiment (for several different distances D; a single result could always be interpreted as due to some value of uniform acceleration).

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[3]Equation (10) appears near the end of Einstein's 1907 paper [5].

[4]Our brief discussion avoids the issue of variation with z of the rate of clocks in a uniform gravitational field. However, the metric (8) indicates that a clock (that reads time t) at position z has proper time interval $d\tau = (1 + gz/c^2)dt$, such clock at $z > 0$ runs slower compared to proper time than a clock at $z = 0$. Hence, reporting the speed of light at position $z > 0$ as $u(z) = dz/dt = (dz/d\tau)(d\tau/dt) = c\,(1 + gz/c^2)$ gives a value larger than c. If light is emitted in the $+z$-direction at $z = -c^2/g$ its initial speed is zero according to eq. (10), such that it takes an infinite time interval Δt to reach $z = 0$, and we speak of $z = -c^2/g$ as the "event horizon" for the observer at $z = 0$. However, an observer at $z = -c^2/g$ could consider that the light has local speed c, and the metric to be eq. (8) with z replaced by $z + c^2/g$, such that the speed of light varies with z according to $u(z) = c\,(1 + g(z + c^2/g)/c^2) = c\,(2 + gz/c^2)$, and the event horizon for this observer is $z = -2c^2/g$. Similarly, an observer at $z = c^2/g$ who considers the local speed of light to be c concludes that light emitted at $z = 0$ takes an infinite time to reach him, so that in effect an observer at $z = 0$ cannot communicate with one at $z = c^2/g$. Hence, we say that the metric (8) is valid only for $|z| < c^2/g$.

Another way to see this is to note that the gravitational redshift brings the energy of any photon emitted at $z = 0$ to zero at $z = c^2/g$ [6], so there is no meaningful physical interaction possible between an observer at $z = 0$ and one at $z > c^2/g$.

A universe with a uniform gravitational field is effectively partitioned into regions of extent $\Delta z = \pm c^2/g$ around any observer. Each observer cannot know about the rest of the universe outside this domain. That is, early cosmological visions that assumed a flat Earth and "turtles all the way down" were actually consistent with general relativity.

[5]However, as discussed in Appendices B and C, the set of the accelerated observer plus two accelerated mirrors with constant separation in the lab frame is not an accelerated frame (in which the separation of objects at rest in that frame would be the same at all times), except for small times as considered here.

2.3 Does a Uniform Gravitational Field Have a Source?

Using coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, the metric tensors g_{ij} and g^{ij} corresponding to eq. (8) have nonzero components[6]

$$g_{00} = \frac{1}{g^{00}} = f^2(z) = \left(1 + \frac{gz}{c^2}\right)^2, \qquad g_{11} = g_{22} = g_{33} = g^{11} = g^{22} = g^{33} = -1, \tag{13}$$

such that $g_{ik}\, g^{jk} = \delta_i^j$. The nonzero Christoffel symbols are

$$\Gamma_{i,jk} = \Gamma_{i,kj} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right), \qquad \Gamma_{0,03} = \Gamma_{0,30} = -\Gamma_{3,00} = f\frac{df}{dz} \equiv ff'. \tag{14}$$

The Riemann curvature tensor has nonzero components

$$R_{ijkl} = \frac{\partial \Gamma_{i,jl}}{\partial x^k} - \frac{\partial \Gamma_{i,jk}}{\partial x^l} + g^{mn}\Gamma_{i,mk}\Gamma_{n,jl} - g^{mn}\Gamma_{i,ml}\Gamma_{n,jk}, \tag{15}$$

$$R_{0330} = R_{3003} = -R_{0303} = -R_{3030} = ff''. \tag{16}$$

The Ricci tensor has nonzero components

$$R_{ij} = g^{kl}R_{kilj}, \qquad R_{00} = ff'', \qquad R_{33} = -\frac{f''}{f}. \tag{17}$$

The Ricci curvature scalar is[7]

$$R = g^{ij}R_{ij} = \frac{2f''}{f}. \tag{18}$$

Einstein's gravitational equations are

$$\frac{8\pi G}{c^4}T_{ij} = R_{ij} - g_{ij}R, \tag{19}$$

$$T_{00} = -\frac{c^4}{8\pi G}ff'', \qquad T_{11} = \frac{c^4}{4\pi G}\frac{f''}{f} \qquad T_{22} = \frac{c^4}{4\pi G}\frac{f''}{f} \qquad T_{33} = \frac{c^4}{8\pi G}\frac{f''}{f}. \tag{20}$$

Hence, the choice $f(z) = 1 + gz/c^2$, for which $f'' = 0$, implies that the stress-energy tensor T_{ij} is everywhere zero. The "uniform gravitational field" corresponding to the metric (8) has no source, and is only a kind of "coordinate force" akin to the centrifugal force and the Coriolis force.

Requiring a uniform gravitational field to have an infinite planar source and flat spacetime apparently leads to metrics with spatial anisotropy. See, for example, [3, 8, 14, 15, 16, 17, 18, 19].

[6] For the general case of symmetric metric tensors, see prob. 2, sec. 92 of [1].

[7] Probably, $R = f''/f$, such that $T_{00} = T_{33} = 0$, and I have errors somewhere.

A Appendix: Uniformly Accelerated Motion

"Uniform acceleration" cannot mean constant acceleration in the (inertial) lab frame, as this would eventually lead to faster-than-light motion. Rather, we suppose (following Born [20]) that the acceleration is uniform with respect to the instantaneous rest frame of the accelerated object. Quantities in this frame will be designated with the superscript *. From sec. 10 of Einstein's first paper on relativity [21] we have that for acceleration parallel to the velocity $\mathbf{v}$ of an object, the acceleration in the lab frame is related to that in the instantaneous rest frame according to

$$\frac{dv}{dt} = (1 - v^2/c^2)^{3/2} \frac{dv^\star}{dt^\star} \,. \tag{21}$$

In this, two powers of $\sqrt{1 - v^2/c^2}$ come from the transformation of relative velocity, and another comes from time dilation.

For uniform acceleration $a = dv^\star/dt^\star$, eq. (21) can be integrated to find the velocity v. Thus, the acceleration in the lab frame is related to that in the instantaneous rest frame according to

$$\frac{v}{\sqrt{1 - v^2/c^2}} = at, \qquad \text{and} \qquad \frac{dz}{dt} = v = \frac{at}{\sqrt{1 + a^2t^2/c^2}} \,. \tag{22}$$

supposing that $v = 0$ when $t = 0$. Integrating eq. (22) we obtain

$$z = Z + \frac{c^2}{a} \left(\sqrt{1 + a^2t^2/c^2} - 1 \right), \tag{23}$$

where Z is the z-coordinate of the object at time $t = 0$. The (proper) time $t^\star$ on a clock carried by the accelerating object is related by

$$dt^\star = dt\sqrt{1 - v^2/c^2} = \frac{dt}{\sqrt{1 + a^2t^2/c^2}} \,, \tag{24}$$

and hence,

$$t^\star = \frac{c}{a} \sinh^{-1} \frac{at}{c} \,, \qquad t = \frac{c}{a} \sinh \frac{at^\star}{c} \,. \tag{25}$$

Using this, eqs. (22) and (23) can be rewritten as

$$v = c \tanh \frac{at^\star}{c} \,, \qquad \text{and} \qquad z = Z + \frac{c^2}{a} \left(\cosh \frac{at^\star}{c} - 1 \right) . \tag{26}$$

As such, uniformly accelerated motion is often called "hyperbolic motion".[8]

An object that extends from Z_1 to Z_2 when at rest at time $t = 0$ has extent $|Z_2 - Z_1|$ at all other times when all points in the object are subject to the same, uniform acceleration; there is no Lorentz contraction according to lab-frame observers for this type of of uniform acceleration of an extended object.

Finally, we note that for times such that $|at| \ll c$, the position is well approximated by the Newtonian form

$$z \approx Z + \frac{at^2}{2} \qquad (|at| \ll c). \tag{27}$$

[8]Hyperbolic motion appears to have been first discussed briefly by Minkowski [22], and then more fully by Born [20] and Sommerfeld [23].

B Appendix: Uniformly Accelerated Reference Frame

A set of uniformly accelerated observers can be used to define a uniformly accelerated reference frame. However, the distance between observers in a "rigid" frame must be independent of time in that frame. If we use the set of observers with equal spacing in the lab frame at all times during their accelerated motion according to eq. (26), the distance between observers would vary with time in the accelerated frame.

An appropriate coordinate system (x', y', z', t') for a "rigid" frame whose origin has acceleration g with the respect to the z-axis of the inertial lab frame, and which obeys the metric (8), is defined by [9]

$$x = x', \tag{28}$$

$$y = y', \tag{29}$$

$$z = \frac{gz' + c^2}{g} \cosh \frac{gt'}{c} - \frac{c^2}{g}, \tag{30}$$

$$t = \frac{gz' + c^2}{cg} \sinh \frac{gt'}{c}. \tag{31}$$

It is useful to note that according to eqs. (30)-(31),

$$\cosh \frac{gt'}{c} = \sqrt{1 + \left(\frac{gct}{gz' + c^2} \right)^2}, \tag{32}$$

$$z = \frac{gz' + c^2}{g} \sqrt{1 + \left(\frac{gct}{gz' + c^2} \right)^2} - \frac{c^2}{g}, \tag{33}$$

from which we obtain the velocity v in the lab frame of a point at constant z' in the accelerated frame as

$$v = \frac{dz}{dt} = \frac{gc^2 t}{(gz' + c^2)\sqrt{1 + \left(\frac{gct}{gz'+c^2} \right)^2}} = c \tanh \frac{gt'}{c}. \tag{34}$$

Note that

$$\sqrt{1 - v^2/c^2} = \frac{1}{\cosh gt'/c}, \tag{35}$$

The lab frame acceleration of a point at constant z' is

$$a = \frac{dv}{dt} = \frac{g}{1 + gz'/c^2} \frac{1}{\left[1 + \left(\frac{gct}{gz'+c^2} \right)^2 \right]^{3/2}} = \frac{g}{1 + gz'/c^2} \frac{1}{\cosh^3 \frac{gt'}{c}} = \frac{g(1 - v^2/c^2)^{3/2}}{1 + gz'/c^2}. \tag{36}$$

Recalling eq. (21) we see that the acceleration of point z' in its instantaneous inertial rest frame is

$$a^\star = \frac{g}{1 + gz'/c^2}, \tag{37}$$

which depends on the position z' in the accelerated frame. This further emphasizes the difference between a "rigid" accelerated frame and a collection of observers whose acceleration is the same in the lab frame.

The distance between nearby points in the accelerated frame, as measured at a fixed time t in the lab frame, follows from eq. (33),

$$dz = \frac{dz'}{\cosh gt'/c} = \sqrt{1 - v^2/c^2}\, dz' \qquad \text{(constant } t\text{)}. \tag{38}$$

Lab-frame observers find that, at time t, lengths in the "rigid" accelerated frame are Lorentz contracted, as expected, according to their instantaneous lab-frame velocity v, when the measurements are made at constant t.

Similarly, observers in the accelerated frame at time t' of a small length dz' find that corresponding length dz in the lab frame is related according to eq. (30) by

$$dz = dz' \cosh \frac{gt'}{c} = \frac{dz'}{\sqrt{1 - v^2/c^2}} \qquad \text{(constant } t'\text{)}. \tag{39}$$

That is, the lengths of objects in the lab frame are all also Lorentz contracted, when observed from the "rigid" accelerated frame at constant t'.

The relation between time intervals in the lab and accelerated frames for clocks at fixed z' follows from eq. (31) as

$$dt(z') = dt' \left(1 + \frac{gz'}{c^2}\right) \cosh \frac{gt'}{c} = \frac{dt'(z')}{\sqrt{1 - v^2/c^2}} \left(1 + \frac{gz'}{c^2}\right) \qquad \text{(constant } z'\text{)}, \tag{40}$$

In particular a clock at $z' = 0$ is related by the time dilation

$$dt_0 = \frac{dt'_0}{\sqrt{1 - v^2/c^2}} \,. \tag{41}$$

As all clocks in the inertial lab frame run at the same rate, we can take $dt(z') = dt_0$ to find

$$dt'(z') = dt'_0 \left(1 + \frac{gz'}{c^2}\right). \tag{42}$$

That is, clocks at larger z' in the accelerated frame run faster than clocks at smaller z', relative to clocks in the inertial lab frame, as noted by Einstein in 1907 [5].

Likewise, using eq. (30) to eliminate z' from eq. (31) in favor of z, we find

$$t = \frac{gz + c^2}{cg} \tanh \frac{gt'}{c} \,, \tag{43}$$

and hence,

$$dt = dt' \left(1 + \frac{gz}{c^2}\right) \frac{1}{\cosh^2 gt'/c} = dt' \left(1 - \frac{v^2}{c^2}\right) \left(1 + \frac{gz}{c^2}\right) \qquad \text{(constant } z\text{)}, \tag{44}$$

Clocks at fixed z appear to observers in the accelerated from to run slow (time dilation), but by a factor $1 - v^2/c^2$ rather than $\sqrt{1 - v^2/c^2}$. In addition, this time-dilation factor varies with the coordinate of the clock in the lab frame.

For completeness we note that eqs. (30) and (43) can be combined to give

$$\frac{1}{\cosh gt'/c} = \sqrt{1 - \left(\frac{gct}{gz + c^2}\right)^2}, \tag{45}$$

$$z' = \frac{z + c^2/g}{\cosh gt'/c} - \frac{c^2}{g} = \frac{gz + c^2}{g}\sqrt{1 - \left(\frac{gct}{gz + c^2}\right)^2} - \frac{c^2}{g}, \tag{46}$$

C Appendix: Bell's Spaceship Paradox

An interesting example of the difference between a "rigid" accelerated frame and a collection of observers with the same lab-frame accelerations was given by Dewan and Beran [24, 25], and popularized by Bell [26].

Here, two spaceships move, with a rope connecting them, along the z-axis with identical accelerations and constant separation dz for any time t in the inertial lab frame. Then, according to eq. (38), the separation of the spaceships in the accelerated frame of, say, the left spaceship is $dz' = dz/\sqrt{1 - v^2/c^2} > dz$. In the frame of either of the spaceships the rope appears to be stretched, and eventually breaks.

This result is very disconcerting to those who think that the spaceships define a "rigid" accelerated frame, in which the distance between two points would be independent of time. But, as discussed around eq. (39), the distance between the spaceships is increasing in the "rigid" accelerated frame associated with either of the spaceships, so it should be no surprise that the rope eventually breaks.

According to the equivalence principle, a uniformly accelerated frame is equivalent to a frame at rest in a uniform gravitational field. An object at rest in a uniform gravitational field has a constant length, as does an object in a uniformly accelerated frame (according to observers in that frame). However, many people seem to suppose that the two spaceships in Bell's paradox define a uniformly accelerated frame, in which case the rope should not be expected to break. Or, if one accepts that the rope breaks, but one supposes that the two spaceships define a uniformly accelerated frame, then according to the equivalence principle, a rope suspended at rest in a uniform gravitational field would be expected to break after a while.

These paradoxes reinforce the insight of Appendix B that a uniformly accelerated frame is not a collection of observers with the same acceleration in the inertial lab frame.

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