1 Problem

Discuss the relation of “hidden” mechanical momentum to the so-called Abraham-Minkowski debate as to the significance of the expressions \( E \times H / 4\pi c \) and \( D \times B / 4\pi c \) for “electromagnetic” momentum (in Gaussian units), where \( c \) is the speed of light in vacuum.

2 Solution

In 1903 Max Abraham noted [3] that the Poynting vector [4], which describes the flow of energy in the electromagnetic field,

\[
S = \frac{c}{4\pi} E \times H,
\]  

(1)

when divided by \( c^2 \) has the additional significance of being the density of momentum stored in the electromagnetic field,\(^2\)

\[
P_{EM}^{(A)} = \frac{E \times H}{4\pi c} \quad \text{(Abraham).}
\]  

(2)

Of course, \( D = E + 4\pi P \) and \( H = B - 4\pi M \), where \( P \) and \( M \) are the densities of electric and magnetic polarization, respectively.

In 1908 Hermann Minkowski gave an alternative derivation [10] that the electromagnetic-momentum density is\(^3\,4\)

\[
P_{EM}^{(M)} = \frac{D \times B}{4\pi c} \quad \text{(Minkowski),}
\]  

(3)

and the debate over the merits of these two expressions continues to this day. Minkowski died before adding to the debate, while Abraham published several times on it [14, 15, 16]. For recent reviews, see [17, 18, 19].

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\(^1\)This debate has been characterized by Ginzburg as a “perpetual problem” [1]. For a lengthy bibliography on this topic, see [2].

\(^2\)J.J. Thomson wrote the electromagnetic momentum as \( D \times H / 4\pi c \) in 1891 [5] and again in 1904 [6]. This form was also used Poincaré in 1900 [7], following Lorentz’ convention [8] that the force on electric charge \( q \) be written \( q(D + \mathbf{v}/c \times \mathbf{H}) \) and that the Poynting vector is \((c/4\pi)D \times H\). For discussion of these forms, see, for example, [9].

\(^3\)Heaviside gave the form (3) in 1891, p. 108 of [11], and a derivation (1902) essentially that of Minkowski on pp. 146-147 of [12].

\(^4\)See also, for example, sec. 2.1 of [13].
A general consensus has emerged that in dielectric media the Abraham momentum (2) is indeed the momentum stored in the electromagnetic field, while the Minkowski momentum (3) includes the momentum of matter that interacts with the electromagnetic fields. This suggests that the quantity,

\[ P_{\text{hidden}} = \int (\mathbf{p}(M)_{\text{EM}} - \mathbf{p}(A)_{\text{EM}}) \, d\text{Vol} = \int \frac{\mathbf{D} \times \mathbf{B} - \mathbf{E} \times \mathbf{H}}{4\pi c} \, d\text{Vol} = \int \frac{\mathbf{P} \times \mathbf{B} + \mathbf{E} \times \mathbf{M}}{c} \, d\text{Vol}, \quad (4) \]

might have the significance of mechanical momentum “hidden” within the system.

### 2.1 Shockley’s Version of “Hidden” Mechanical Momentum

The term “hidden” mechanical momentum is more commonly associated with a different context, first noted by Shockley [23], in which a system whose center of mass/energy is at rest but for which the electromagnetic field momentum, \( \mathbf{P}_{\text{EM}} \), is nonzero. The total momentum of such a system must be zero [24], so there must be an equal and opposite “hidden” mechanical momentum

\[ P_{\text{hidden, mech}} = -\mathbf{P}_{\text{EM}}. \quad (5) \]

The Abraham-Minkowski debate over the meaning of \( \mathbf{P}_{\text{EM}} \) indicates that the meaning of “hidden momentum” is also ambiguous if it is only defined by eq. (5). A more general definition of “hidden momentum” for any subsystem of a possibly larger system is given in [26],

\[ P_{\text{hidden}} \equiv \mathbf{P} - M \mathbf{v}_{\text{cm}} - \int_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})(\mathbf{p} - \rho \mathbf{v}_b) \cdot d\text{Area} = -\int f^\mu (\mathbf{x} - \mathbf{x}_{\text{cm}}) \, d\text{Vol}, \quad (6) \]

where \( \mathbf{P} \) is the total momentum of the subsystem, \( M = U/c^2 \) is its total “mass,” \( U \) is its total energy, \( \mathbf{x}_{\text{cm}} \) is its center of mass/energy, \( \mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt \), \( \mathbf{p} \) is its momentum density, \( \rho = u/c^2 \) is its “mass” density, \( u \) is its energy density, \( \mathbf{v}_b \) is the velocity (field) of its boundary, and

\[ f^\mu = \frac{\partial T^{\mu\nu}}{\partial x_\nu}, \quad (7) \]

\[ ^5 \text{However, this author considers that the field-only momentum is given by } \mathbf{P}_{\text{EM}} = \int \mathbf{E} \times \mathbf{B} \, d\text{Vol}/4\pi c. \]

\[ ^6 \text{When dealing with waves of angular frequency } \omega \text{ in a dispersive medium with index } n(\omega) \text{ it is useful to introduce the quantity } n_g = c/n_g = c \frac{dk}{d\omega} = d(\omega n)/d\omega = n + \omega \frac{dn}{d\omega}, \text{ which is sometimes called the group-velocity index. This velocity is positive in a passive medium, but can be negative in a gain medium [20]. The emerging consensus [13, 17, 18, 19] is that the Abraham momentum density corresponds to the momentum of a photon of angular frequency } \omega \text{ in a dielectric medium of group-velocity index } n_g \text{ being } \hbar \omega / c n_g, \text{ and is sometimes called the kinetic momentum density. The Minkowski momentum density (in a dielectric) corresponds to the momentum } n^2 \hbar \omega / c n_g \text{ of a photon of angular frequency } \omega, \text{ and is sometimes called the pseudomomentum or the quasimomentum. The momentum of a photon most often used in quantum theory is } \hbar \mathbf{k} = n \hbar \omega \mathbf{k}/c, \text{ which is often called the canonical momentum. In a nondispersive medium with } n > 0 \text{ the Minkowski momentum is the same as the canonical momentum. For discussion of negative-index materials, see [21].} \]

\[ ^7 \text{A similar issue arises in acoustics, where one sometimes speaks of the “pseudomomentum” of sound waves, which is analogous to the Minkowski momentum in electrodynamics. See [22] and references therein.} \]

\[ ^8 \text{Classical systems with nonzero “hidden” mechanical momentum have moving parts, as noted in [25]. If magnetic charges existed, a system of static electric and magnetic charges would have no “hidden” mechanical momentum, and its total field momentum must also be zero.} \]
is the 4-force density exerted by the subsystem on the rest of the system, with $T^{\mu \nu}$ being the stress-energy-momentum 4-tensor of the subsystem.

The definition (6) indicates that the value of the “hidden” momentum depends on the subsystem under consideration. In the classic examples considered by Shockley and others, the entire system was partitioned into two subsystems that occupied that same volume, the electromagnetic fields $E$ and $B$, and the “mechanical” components of the system; the (relative) permittivity $\epsilon$ and the (relative) permeability $\mu$ were both unity.

For an isolated, closed system with total stress-energy-momentum tensor $T^{\mu \nu}$, the 4-divergence of the latter is zero, $\partial T^{\mu \nu}/\partial x^\nu = 0$. If the system contains two subsystems $A$ and $B$ which occupy the same volume, then $f_A^\mu = \partial T_A^{\mu \nu}/\partial x^\nu = -\partial T_B^{\mu \nu}/\partial x^\nu = -f_B^\mu$, where $f_B^\mu$ is the 4-force density exerted by subsystem $A$ on $B$. Hence, according to the last form of the definition (6), subsystems $A$ and $B$ have equal and opposite “hidden” momenta. In particular, if the entire system is partitioned into “electromagnetic” and “mechanical” subsystems, we have that

$$P_{\text{hidden, EM}} = -P_{\text{hidden, mech}}. \tag{8}$$

For the electromagnetic subsystem the macroscopic electromagnetic energy-momentum-stress tensor (secs. 32-33 of [27], sec. 12.10B of [28]) is, in a linear medium\(^9\)

$$T^{\mu \nu}_{\text{EM}} = \left( \begin{array}{c|c} u_{\text{EM}} & \frac{c P_{\text{EM}}}{c p_{\text{EM}}} \\ \hline \frac{c p_{\text{EM}}}{c P_{\text{EM}}} & -T^{ij}_{\text{EM}} \end{array} \right), \tag{9}$$

where $u_{\text{EM}}$ is the electromagnetic field energy density, $P_{\text{EM}}$ is the electromagnetic momentum density, and $T^{ij}_{\text{EM}}$ is the 3-dimensional (symmetric) electromagnetic stress tensor. If the tensor (9) is independent of time (as, for example, in the rest frame of a medium with static charge and steady current distributions), then the quantity $f^0$ in eq. (6) is

$$f^0 = \frac{\partial T^{\nu}}{\partial x^\nu} = c \nabla \cdot P_{\text{EM}}, \tag{10}$$

for the electromagnetic subsystem, and hence,

$$P_{\text{hidden, EM}} = -\int \frac{f^0}{c} (x - x_{\text{cm}}) d\text{Vol} = -\int x (\nabla \cdot P_{\text{EM}}) d\text{Vol} + x_{\text{cm}} \int \nabla \cdot P_{\text{EM}} d\text{Vol}. \tag{11}$$

The last integral in eq. (11) transforms into a surface integral at infinity that is negligible for a system with bounded charge and current distributions. The term $-\int x (\nabla \cdot P_{\text{EM}}) d\text{Vol}$ can be integrated by parts, with the resulting surface integral at infinity also being negligible, such that

$$P_{\text{hidden, EM}} = \int P_{\text{EM}} d\text{Vol} = P_{\text{EM}}. \tag{12}$$

Then, together with eq. (8) we have that

$$P_{\text{hidden, EM}} = P_{\text{EM}} = -P_{\text{hidden, mech}}. \tag{13}$$

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\(^9\)For a nonlinear medium, Minkowski’s stress tensor [10] is not symmetric, whereas Abraham’s [14] is.
For a “static” case, the “visible” mechanical momentum is zero in the rest frame of the medium, and any mechanical momentum is “hidden.” That is,

\[ P_{\text{hidden,EM}} = P_{\text{EM}} = -P_{\text{hidden,mech}} = -P_{\text{mech}}, \]

and the total momentum of the system is zero,

\[ P_{\text{total}} = P_{\text{EM}} + P_{\text{mech}} = 0. \]

Thus, the definition (6) is consistent with concept of “hidden” momentum as discussed by Shockley and others as explaining how/why the total momentum of an electromechanical system “at rest” is zero.

The result (15) holds for any (valid) form of the electromagnetic field momentum density \( p_{\text{EM}} \) and the associated stress-energy-momentum tensor \( T^\mu_{\nu \text{EM}} \), so the present considerations of “hidden” momentum cannot resolve the Abraham-Minkowski debate. That is, if one accepts either the Abraham or the Minkowski form of the stress-energy-momentum tensor, the definition (6) leads one to a computation of the “hidden” mechanical momentum that is consistent with eqs. (14)-(15). One the other hand, one expects that mechanical momentum, “hidden” or not, is uniquely specifiable for a given system, so that two different values for the “hidden” mechanical momentum cannot both be correct. If, by some argument other than that presented here, the value of the “hidden” mechanical momentum in a medium with electric and magnetic polarization could be determined, this could provide a resolution of the Abraham-Minkowski debate, as least for “static” examples.

In any case, the definition (6) is not consistent with the conjecture (4).

### 2.2 Romer’s Example

In many static examples involving only free charges and free/conduction currents, the free charge/current distributions can be replaced by densities of (fixed) polarization \( \mathbf{P} \) and magnetization \( \mathbf{M} \) such that the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) are unchanged. Following Romer [29], we consider a spherical shell of radius \( a \) with surface charge density proportional to \( \cos \theta \) (with respect to the \( z \)-axis), such that the charge distribution has electric dipole moment \( \mathbf{p} \) and the electric field \( \mathbf{E} \) has the form

\[
E = \begin{cases} 
-\frac{\mathbf{p}}{a} & (r < a), \\
\frac{3(\mathbf{p} \cdot \hat{r})\mathbf{p} - \mathbf{p}}{r^3} & (r > a),
\end{cases}
\]

for which the tangential component of \( \mathbf{E} \) is continuous across \( r = a \). The system also includes an electrically neutral spherical shell of radius \( b \) with surface currents proportional

\[ \footnote{This conclusion appears to differ from that in [34].} \]

\[ \footnote{A precursor to Romer’s example, with \( a = b \) and a uniform surface charge density, was discussed in [30], where it is attributed to J.J. Thomson around 1904. This example has zero field momentum but the field angular momentum is \( \mathbf{L} = 2QM/2ac \), where \( Q \) is the total electric charge and \( M \) is the magnetic moment of the sphere.} \]

\[ \footnote{Another example of this type has been considered in [31].} \]
to \sin \theta' \text{ (with respect to the } z'\text{-axis), such that the current distribution has magnetic dipole moment } \mathbf{m}, \text{ and the magnetic field } \mathbf{B} \text{ has the form}

\begin{equation}
\mathbf{B} = \begin{cases} \frac{2\mathbf{m}}{b^3} & (r < b), \\ \frac{3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m}}{r^4} & (r > b), \end{cases}
\end{equation}

for which the normal component of \mathbf{B} \text{ is continuous across } r = b. \text{ The system is in vacuum.}

We consider the case that } a > b. \text{ The usual argument in vacuum is that the electromagnetic-field momentum } \mathbf{P}_{\text{EM}} \text{ can be computed as}

\begin{align}
\mathbf{P}_{\text{EM}} &= \mathbf{P}_{\text{EM}}^{(A)} = \mathbf{P}_{\text{EM}}^{(M)} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \, d\text{Vol} \\
&= \int_{r < b} \frac{-\mathbf{p} \times 2\mathbf{m}}{4\pi a^3 b^3 c} \, d\text{Vol} + \int_{b < r < a} \frac{-\mathbf{p} \times [3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m}]}{4\pi a^3 r^3 c} \, d\text{Vol} \\
&\quad + \int_{r > a} \frac{[3(\mathbf{p} \cdot \hat{r})\hat{r} - \mathbf{p}] \times [3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m}]}{4\pi b^3 c} \, d\text{Vol} \\
&= -\frac{2\mathbf{p} \times \mathbf{m}}{3a^3 c} - \frac{\mathbf{p} \times \mathbf{m}}{a^3 c} \ln \frac{a}{b} + \frac{\mathbf{p} \times \mathbf{m}}{a^3 c} \ln \frac{a}{b} - \frac{2\mathbf{p} \times \mathbf{m}}{3a^3 c} + \frac{\mathbf{p} \times \mathbf{m}}{3a^3 c} \\
&= \frac{\mathbf{m} \times \mathbf{p}}{a^3 c}.
\end{align} \tag{18}

If } a < b \text{ the result is } \mathbf{P}_{\text{EM}} = \mathbf{m} \times \mathbf{p}/b^3 c.

This system is at rest and must have zero total momentum \cite{24}, which leads to eq. (5). Hence, we infer that the system also contains “hidden” mechanical momentum,

\begin{equation}
\mathbf{P}_{\text{hidden,mech}} = -\mathbf{P}_{\text{EM}} = \frac{\mathbf{p} \times \mathbf{m}}{a^3 c}, \tag{19}
\end{equation}

when } a > b.

Rather than supposing the fields to be due to free charges and currents, we can consider the cases that there exist uniform electric polarization density } \mathbf{P} = 3\mathbf{p}/4\pi a^3 \text{ for } r < a, \text{ and/or uniform magnetic polarization density } \mathbf{M} = 3\mathbf{m}/4\pi b^3 \text{ for } r < b. \text{ For nonzero } \mathbf{P} \text{ the sphere of radius } a \text{ is now an electret, while for nonzero } \mathbf{M} \text{ the sphere of radius } b \text{ is a permanent magnet. Then, the fields } \mathbf{E} \text{ and } \mathbf{B} \text{ are identical to those of eqs. (16)-(17), so we suppose that the field-only momentum is still given by eq. (18). Hence, for } a > b,}

\begin{align}
\mathbf{P}_{\text{EM}}^{(A)}(\mathbf{M} = 0) &= \int \frac{\mathbf{E} \times \mathbf{H}}{4\pi c} \, d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \, d\text{Vol} = \mathbf{P}_{\text{EM}}, \tag{20} \\
\mathbf{P}_{\text{EM}}^{(A)}(\mathbf{M} \neq 0) &= \mathbf{P}_{\text{EM}} - \int_{r < b} \frac{\mathbf{E} \times \mathbf{M}}{c} \, d\text{Vol} = \frac{\mathbf{m} \times \mathbf{p}}{a^3 c} - \int_{r < b} \frac{-\mathbf{p} \times 3\mathbf{m}}{4\pi a^3 b^3 c} \, d\text{Vol} = 0, \tag{21} \\
\mathbf{P}_{\text{EM}}^{(M)}(\mathbf{P} = 0) &= \int \frac{\mathbf{D} \times \mathbf{B}}{4\pi c} \, d\text{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} \, d\text{Vol} = \mathbf{P}_{\text{EM}}, \tag{22} \\
\mathbf{P}_{\text{EM}}^{(M)}(\mathbf{P} \neq 0) &= \mathbf{P}_{\text{EM}} + \int_{r < a} \frac{\mathbf{P} \times \mathbf{B}}{c} \, d\text{Vol} \\
&= \frac{\mathbf{m} \times \mathbf{p}}{a^3 c} + \int_{r < b} \frac{3\mathbf{p} \times 2\mathbf{m}}{4\pi a^3 b^3 c} \, d\text{Vol} + \int_{b < r < a} \frac{3\mathbf{p} \times [3(\mathbf{m} \cdot \hat{r})\hat{r} - \mathbf{m}]}{4\pi a^3 r^3 c} \, d\text{Vol} \\
&= -\frac{\mathbf{m} \times \mathbf{p}}{a^3 c} = -\mathbf{P}_{\text{EM}}. \tag{23}
\end{align}
A reasonable view seems to be that “hidden” mechanical momentum is associated with the moving charges of the electrical currents in the system. If the magnetization $M$ is zero, the currents are the same whether $P$ is zero or not, so it seems reasonable to conclude that the “hidden” mechanical momentum in Romer’s example is independent of the polarization $P$, and hence

$$P_{\text{hidden, mech}}(M = 0) = -P_{\text{EM}} = \int \frac{E \times B}{4\pi c} d\text{Vol} = -P_{\text{EM}}^{(A)}, \quad (24)$$

since $H = B$ in this case. That is, an argument based on “hidden” mechanical momentum leads to the conclusion that the Abraham momentum (and stress-energy-momentum tensor) is the “correct” one for examples like Romer’s if the magnetization is zero.

If the magnetization $M$ is nonzero it is less clear that the “hidden” mechanical momentum still equals $-P_{\text{EM}}$, or that it equals the negative of any of eqs. (21)-(23). In this context, it may be useful to consider another example.

### 2.3 Hnizdo’s Example: Uniformly Magnetized Toroid

V. Hnizdo notes that a toroid with uniform azimuthal magnetization $M = M \hat{\phi}$ (in cylindrical coordinates $(\varrho, \phi, z)$) has $H = B - 4\pi M = 0$ everywhere and $B = 4\pi M$ inside the toroid. If this toroid is subject to a radial electric field $E$, then $E \times B$ is nonzero inside the toroid and parallel to $z$-axis while $E \times H$ is zero everywhere. If we accept that this example contains “hidden” mechanical momentum, then the Abraham momentum cannot be the “correct” one (in the sense of being equal and opposite to the “hidden” mechanical momentum such that the total momentum of the system, which is “at rest”, is zero).

A body with uniform magnetization is a collection of identical Ampèrean magnetic dipoles. As noted in [36], an Ampèrean magnetic dipole $m$ in an external electric field $E$ is a system “at rest” with nonzero field momentum $E \times m/c$ (computed from $\int E \times B d\text{Vol}/4\pi c$), and so this system must contain “hidden” mechanical momentum in the direction of $m \times E$. We infer that a collection such as Hnizdo’s example of identical magnetic dipoles, all in an external electric field such that $m \times E$ is always in the same direction, also contains nonzero “hidden” mechanical momentum.

Hence, it appears that neither the Abraham nor the Minkowski momenta are the “correct” field momenta in the sense of being equal and opposite to the “hidden” mechanical momentum of such systems “at rest” that possess this.

It seems to this author that only the form $\int E \times B d\text{Vol}/4\pi c$ should be called the electromagnetic field momentum, and that both the Abraham and Minkowski momenta represent combinations of electromagnetic field momentum with mechanical momentum associated with electric and/or magnetic polarization.

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13. This view originated in [32]. See also [33].
14. A version of this argument is given in [34], which also considers momentum balance when the magnetic field is reduced to zero.
15. For uniform magnetization $M$, the volume density $\rho_m = -\nabla \cdot M$ of “fictitious” magnetic charges is zero. For a toroid with azimuthal magnetization, the surface density $\sigma_m = M \cdot \hat{n}$ of “fictitious” magnetic charges is also zero (where $\hat{n}$ is normal to the surface). Then, there are no sources for the $H$-field, which is hence zero.
16. Experimental evidence that magnetization is due to Ampèrean magnetic dipoles is reviewed in [35].
2.3.1 The Toroid is Also an Electret

We record two examples of radial electric fields as discussed above. First, we consider the toroid also to be an electret with uniform radial electric polarization \( \mathbf{P} = P \hat{\rho} \).

We suppose the toroid has a rectangular cross section \( a < \rho < b, |z| < l \) and that the toroid is long, \( l \gg b \). The surface density of bound electric charge is then

\[
\sigma_e(\rho = a) = -P, \quad \sigma_e(\rho = b) = P.
\]  

(25)

The electric field inside the long toroid, at locations not close to its ends, is approximately radial, and follows from Gauss’ law as

\[
E_r(a < \rho < b) = -4\pi P \frac{a}{\rho},
\]  

(26)

and hence the interior \( \mathbf{D} \)-field is

\[
D_r(a < \rho < b) = E_r + 4\pi P_r = 4\pi P \left( 1 - \frac{a}{\rho} \right).
\]  

(27)

2.3.2 The Toroid is a Cylindrical Capacitor

Alternatively, the surfaces \( \rho = a \) and \( b \) could be conductors, forming a cylindrical capacitor, and the medium of the toroid could be a (linear) dielectric with (relative) permittivity \( \epsilon \) such that \( \mathbf{D} = \epsilon \mathbf{E} \). These conductors can support densities of free charge given by eq. (25) such that the electric field inside the toroid is again given by eq. (26). In this case, the \( \mathbf{D} \)-field inside the toroid is given by

\[
D_r(a < \rho < b) = \epsilon E_r = -4\pi P \frac{a}{\rho}.
\]  

(28)

Forces and Momenta If the Magnetization Goes to Zero

This variant permits a noteworthy phenomenon if the electrically charged conductors are physically isolated from the magnetized toroid, and that magnetization goes to zero at some time.

When the magnetization is \( \mathbf{M} = M \hat{\phi} \), the (field only) electromagnetic momentum per unit length in \( z \) associated with the (long) system is

\[
\mathbf{P}_{\text{EM}} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} d\text{Area} \approx \int_a^b \left( -\frac{(4\pi Pa/\rho)}{4\pi c} \hat{r} \times 4\pi M \hat{\phi} \right) \frac{2\pi \rho}{c} d\rho = -\frac{8\pi^2 MPa(b - a)}{c} \hat{z},
\]  

(29)

where \( \mp P \) is the surface density of electric charge on the conductors (at \( \rho = a^- \) and \( b^+ \)). At this time, there is an equal and opposite “hidden” mechanical momentum per unit length, \( \mathbf{P}_{\text{hidden, mech}} = -\mathbf{P}_{\text{EM}} = \int \frac{\mathbf{B} \times \mathbf{E}}{4\pi c} d\text{Area} = \int \frac{\mathbf{M} \times \mathbf{E}}{c} d\text{Area}, \) (30)

such that the total momentum of the system is zero. Equation (30) is consistent with the earlier comment that the “hidden” mechanical momentum of a magnetic dipole \( \mathbf{m} \) in
an electric field \( \mathbf{E} \) is \( \mathbf{m} \times \mathbf{E}/c \) [36], such that the volume density of “hidden” mechanical momentum is \( \mathbf{M} \times \mathbf{E}/c \) inside the magnetization density \( \mathbf{M} \).

If the magnetization drops to zero at some later time, a transient electric field is induced when the \( \mathbf{B} \)-field is changing, which electric field is conveniently deduced from the changing vector potential,

\[
\mathbf{E}_{\text{ind}} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}.
\]

(31)

We ignore the small additional magnetic field and vector potential associated with the transient currents, and approximate the vector potential as the quasistatic value related to the instantaneous magnetization \( \mathbf{M}(t) \), still supposed to be spatially uniform as it drops to zero. This vector potential has only a \( z \)-component, related by

\[
\mathbf{B} = \nabla \times \mathbf{A}, \quad B_\phi = -\frac{\partial A_z}{\partial \varrho} = \begin{cases} 
0 & (\varrho < a), \\
4\pi M & (a < \varrho < b), \\
0 & (\varrho > b).
\end{cases}
\]

(32)

We take the vector potential to be zero at \( \varrho = \infty \), such that

\[
A_z = 4\pi M \begin{cases} 
 b - a & (\varrho < a), \\
 b - \varrho & (a < \varrho < b), \\
 0 & (\varrho > b).
\end{cases}
\]

(33)

When the magnetization drops at rate \( \dot{M} < 0 \), the induced electric field is

\[
E_{\text{ind},z} = -\frac{1}{c} \frac{\partial A_z}{\partial t} = -\frac{4\pi \dot{M}}{c} \begin{cases} 
 b - a & (\varrho < a), \\
 b - \varrho & (a < \varrho < b), \\
 0 & (\varrho > b).
\end{cases}
\]

(34)

There is no induced electric field at the outer conductor (\( \varrho = b \)), while the field on the inner conductor (\( \varrho = a \)) is in the +\( z \) direction when \( \dot{M} < 0 \). If the inner and outer conductors are free to move separately, only the inner conductor will move, and in the −\( z \) direction as its surface charge density \( -P \) is negative. The force per unit length on the inner conductor is

\[
F_z = 2\pi a (-P) E_{\text{ind},z}(\varrho = a^-) = \frac{8\pi^2 \dot{M} Pa(b-a)}{c},
\]

(35)

so the (negatively charged) inner conductor takes on mechanical momentum per unit length

\[
P_{\text{inner conductor}, z} = \int F_z \, dt = -\frac{8\pi^2 MPa(b-a)}{c},
\]

(36)

when the magnetization drops from \( M \) to zero. If the conductors are free to move, the inner conductor has final velocity in the −\( z \) direction, while the outer conductor remains at rest.
Meanwhile, both the initial electromagnetic field momentum and the “hidden” mechanical momentum have dropped to zero, so if only the inner conductor has final momentum, there would be a violation of momentum conservation.

It remains to consider forces on the magnetized toroid while the magnetization changes. The volume-force density $f_M$ on the (Ampèreian) magnetization $M$ is given, for example, in eq. (18) of [35],

$$f_M = (M \cdot \nabla)B + M \times \frac{1}{c} \frac{\partial E}{\partial t} + cM \times (\nabla \times M). \tag{37}$$

This force density acts to change the mechanical momentum density of the toroid, which consists of the “overt” momentum density $p_{\text{overt}} = \rho_{\text{mass}}v$ as well as the density $p_{\text{hidden, mech}} = M \times E/c$ of “hidden” mechanical momentum.\footnote{This argument was made implicitly by Shockley [23], and explicitly on p. 53 of [37].} In the present example, the first and last terms in eq. (36) vanish, so the mechanical momentum of the toroid varies according to

$$f_M = M \times \frac{1}{c} \frac{\partial E}{\partial t} = \frac{d}{dt} \left( p_{\text{toroid, overt}} + M \times \frac{E}{c} \right), \quad \frac{dp_{\text{toroid, overt}}}{dt} = -\frac{\partial M}{\partial t} \times \frac{E}{c}. \tag{38}$$

As the magnetization $M$ drops to zero, the overt mechanical momentum of the toroid changes, until finally the overt mechanical momentum per unit length of the toroid is

$$P_{\text{toroid, overt}} = \int d\text{Area} \int \frac{dp_{\text{overt}}}{dt} \, dt = \int M_{\text{initial}} \times \frac{E}{c} \, d\text{Area} = \int_{b}^{a} M \hat{\phi} \times -\frac{4\pi P a \hat{r}}{c \rho} 2\pi \rho \, d\rho = \frac{8\pi^2 M P a}{c} (b - a) \hat{z} = -P_{\text{inner conductor}}. \tag{39}$$

Hence, the final, total momentum, $P_{\text{inner conductor}} + P_{\text{toroid, overt}}$, of the system is zero, as expected.

The force density (37) is radial in the present example, so its volume integral vanishes, with the implication that the mechanical momentum of the toroid remains constant as the magnetization drops to zero. If the toroid is free to move, its final velocity is in the $+z$ direction. Hence, the appearance of the final, overt mechanical momentum of the toroid can be regarded as evidence of the initial, “hidden” mechanical momentum. This suggests that a laboratory demonstration of the present example would be useful in convincing skeptics of the existence of “hidden” mechanical momentum.

So, we consider some numbers for a possible demonstration experiment. We take $a \approx b \approx 1$ cm. A practical voltage across the 1-cm cylindrical capacitor might be around 1000 V = 3.3 statvolt. This voltage is also given by $V = \int_{a}^{b} E \, d\rho \approx 4\pi P \ln 2 \approx 3P$, so the surface charge density is $P \approx 1$ statCoulomb/cm². The magnetic field inside a strong permanent magnet is about $B \approx 1$ T = 10,000 G = $4\pi M$, so $M \approx 1000$ in Gaussian units. Then, the final, overt momentum would be $\approx 8\pi^2 MP/c \approx 10^{-7}$ g-cm/s, and for a toroid with mass of a few grams, its final velocity would be $\approx 10^{-7}$ cm/s, too small to be observable in a simple demonstration.
Forces and Momenta If the Electric Field Goes to Zero

If the electric field of the cylindrical capacitor drops to zero but the magnetization of the toroid remains constant, then according to eq. (38) there is no change in the “overt” mechanical momentum of the toroid, which therefore remains at rest as its “hidden” mechanical momentum drops to zero.

Meanwhile, the charges on the conductors of the cylindrical capacitor experience no axial electric field as the radial electric field drops to zero, so the conductors remain at rest.

In the final state, with zero electric field and nonzero $B$ and $M$ inside the toroid, there is no mechanical momentum anywhere, “hidden” or “overt,” and the electromagnetic field momentum is also zero.\textsuperscript{18}

2.3.3 Comments

The Minkowski momenta, $\int D \times B \, d\text{Vol}/4\pi c$, for these two cases have the opposite signs, yet the microscopic electromagnetic field momenta of the magnetic dipoles is the same in both cases. This reinforces the conclusion of sec. 2.2 that the Minkowski momentum is not the “correct” one to be equal and opposite to the “hidden” mechanical momentum (which is the same in both cases).

Appendix: Toroid with Radial Magnetization and Azimuthal Polarization

For possible amusement we consider a toroid with geometry as in sec. 2.3.1 but with radial magnetization, $M = M \hat{\varrho}$, and azimuthal polarization, $P = P \hat{\phi}$. In this case there is no free or bound electric charge densities, either in the bulk or on the surface of the toroid. Hence, there are no sources of the electric field and $E = 0$ everywhere. Inside the toroid the displacement field is nonzero, $D_{\text{interior}} = E + 4\pi P = 4\pi P \hat{\phi}$.

There is no bulk density of “fictitious” magnetic charge associated with the radial magnetization, but there are “fictitious” surface magnetic charge densities given by

$$\sigma_m(\varrho = a) = -M, \quad \sigma_m(\varrho = b) = M. \quad (40)$$

The $H$-field inside the long toroid, at locations not close to its ends, is approximately radial, and follows from Gauss’ law (here $\nabla \cdot H = 4\pi \rho_m$) as

$$H_r(a < \varrho < b) = -4\pi M \frac{a}{\varrho}, \quad (41)$$

and hence the interior $B$-field is

$$B_r(a < \varrho < b) = H_r + 4\pi M r \left( 1 - \frac{a}{\varrho} \right). \quad (42)$$

\textsuperscript{18}An example of this type was considered by J.J. Thomson on p. 348 of [6]; see also [38]. For examples with “hidden” mechanical momentum in systems with an electric dipole in a magnetic fields due to current loops, all “at rest,” such that various equal and opposite “overt” mechanical momenta arise as the electromagnetic fields are brought to zero in various ways, see [39], especially secs. IV and V.
This case is the dual of that described in sec. 2.3.1, with the duality relations \( \mathbf{M} \leftrightarrow \mathbf{P} \), \( \mathbf{E} \leftrightarrow \mathbf{H} \) and \( \mathbf{D} \leftrightarrow \mathbf{B} \).

If we accept that “hidden” mechanical momentum is due to the “external” electric field \( \mathbf{E} \) on the Ampèreian currents associated with the magnetization \( \mathbf{M} \), then there is no “hidden” mechanical momentum in the example of this Appendix. This is consistent with the “field only” momentum \( \int \mathbf{E} \times \mathbf{B} \, d\text{vol} / 4\pi c \) being zero. Of course, the Abraham momentum is also zero in the case, but the Minkowski momentum is nonzero and in the \( -z \) direction.

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