

# A Simple Proof that Toffoli and Hadamard are Quantum Universal

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## Abstract

Recently Shi [15] proved that Toffoli and Hadamard are universal for quantum computation. This is perhaps the simplest universal set of gates that one can hope for, conceptually; It shows that one only needs to add the Hadamard gate to make a 'classical' set of gates quantum universal. In this note we give a few lines proof of this fact relying on Kitaev's universal set of gates [11], and discuss the meaning of the result.

## 1 Introduction

Quantum computers, believed to be computationally stronger than classical devices, are constructed from elementary quantum building blocks, namely *qubits* and *gates*. The gates are drawn from a *universal set* of gates, namely, a set which can be used to perform general quantum computation. Following the pioneering result of universality of three qubit gates by Deutsch [8], the question of universality has been studied extensively, and a wide array of sets of gates were proven to be universal (Just a few examples: [5, 10, 1, 13, 11, 2, 16, 7, 4].)

Given the existence of convenient sets such as the one consisting of all one qubit gates plus the gate controlled-NOT [5], one might wonder why it is interesting to prove universality of many more different sets of gates. The reason is that different sets of gates are suitable for different tasks. For example, to implement a certain algorithm in the laboratory, one would use a restricted set which consists of gates which are possible to implement in the particular

physical realization of a quantum computer. However, the theoretical design of the algorithm might be easier using a completely different universal set. Often, one is interested in fault tolerant implementation of quantum circuits, in which case one is interested in universal sets consisting of gates which can be implemented fault tolerantly. Thus, in quantum computation, we often translate between different universal sets; These can be viewed as different *programming languages* encoding the same algorithm. The reason we can interchange between different sets without it being too costly both of the number of gates and in terms of calculating the description of the new circuit, is a deep theorem due to Solovay and Kitaev [14] which states that translations between different universal sets is not too costly; It causes only a polylogarithmic overhead. This fact allows freedom in choosing the building blocks suitable for our purposes, be it designing algorithms, lower bounding the quantum computational power, fault tolerant constructions, or experimental implementations.

Apart from practical and convenience reasons, there is a purely philosophical reason to study different universal sets. A fundamental question in the theory of quantum computation complexity is where does the quantum computational power come from. From the physical point of view the question is what physical systems are capable of performing truly quantum evolutions. There have been several surprising results in this direction. The Gottesman-Knill theorem [14] shows that computation using CNOT and Hadamard gates can be simulated efficiently by classical computers. Valiant [18] and Terhal and DiVincenzo [17] gave classical simulations for another restricted set of gates, relating them to non interacting fermionic systems. On the other hand,

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Bacon *et. al.* [4] showed the surprising result that the exchange interaction is universal. Naturally, we would like to understand what sets of gates achieve the full quantum computational power and when is this power lost.

One of the most natural sets of gates to consider in this study is the set consisting of the Toffoli gate denoted  $T$  and the Hadamard gate denoted  $H$ . The question of universality of the set  $\{T, H\}$  was asked by various researchers, and was recently solved by Shi [15] to the affirmative. The purpose of this note is to give a few lines proof of the universality of  $\{T, H\}$  based on a simple reduction to a known universal set, due to Kitaev [11].

The fact that  $\{T, H\}$  is universal has philosophical interpretations. The Toffoli gate  $T$  can perform exactly all *classical* reversible computation. The result says that Hadamard is all that one needs to add to classical computations in order to achieve the full quantum computation power; It perhaps explains the important role that the Hadamard gate plays in quantum algorithms, and can be interpreted as saying that Fourier transform is really all there is to quantum computation on top of classical, since the Hadamard gate is the Fourier transform over the group  $Z_2$ . From a conceptual point of view, this is perhaps the simplest and most natural universal set of gates that one can hope for.

In the rest of the note we define universality more rigorously, prove the universality of  $\{T, H\}$  and conclude with a few remarks.

## 2 Preliminaries

We will use the following notation.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (1)$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, P(i) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Given a matrix  $U$  on  $k$  qubits,  $\Lambda(U)$  is the gate which applies  $U$  on the last  $k$  qubits conditioned that the first qubit is in the state  $|1\rangle$ , and does nothing otherwise. The Toffoli gate denoted by  $T$  is the gate

that applies NOT, or  $X$  on the third qubit conditioned that the first two qubits are in the state  $|1\rangle$ ; It can be written as  $\Lambda^2[X]$ .  $U[j_1, \dots, j_k]$  denotes a gate  $U$  operating on qubits  $j_1, \dots, j_k$ . We assume that all our quantum circuits use gates that operate on a constant number of qubits.

Bernstein and Vazirani showed that quantum circuits can be transformed to circuits that use only real matrices [6]. This is done by adding one extra qubit to the circuit, the state of which indicates whether the system's state is in the real or imaginary part of the Hilbert space, and replacing each complex gate  $U$  operating on  $k$  qubits by its *real version*, denoted  $\tilde{U}$ , which operates on the same  $k$  qubits plus the extra qubit.  $\tilde{U}$  is defined by:

### Definition 1

$$\begin{aligned} \tilde{U}|i\rangle|0\rangle &= [Re(U)|i\rangle]|0\rangle + [Im(U)|i\rangle]|1\rangle \\ \tilde{U}|i\rangle|1\rangle &= -[Im(U)|i\rangle]|0\rangle + [Re(U)|i\rangle]|1\rangle. \end{aligned}$$

Here  $Re(U), Im(U)$  means the real and the imaginary part of the matrix  $U$ , respectively. The new circuit computes the same function with the overhead of one qubit.

## 3 Universality

There have been several notions of Universality used in the literature. The strongest sense of universality of a set of gates  $S$  is as follows:

**Definition 2 (Strict Universality)** *A set of quantum gates  $S$  is said to be strictly universal if there exists a constant  $n_0$  such that for any  $n \geq n_0$ , the subgroup generated by  $S$  is dense in  $SU(2^n)$ , the group of unitary matrices with determinant 1 operating on  $n$  qubits.*

This means that any unitary matrix on  $n$  qubits can be approximated (in the standard operator norm induced by the  $l_2$  norm) to within arbitrary accuracy by applying a sequence of gates from  $S$ . The determinant is taken to be 1 since an overall phase can be added to the gates without changing anything. Typically  $n_0$  is a small number; 2 or 3.

Note that the above definition does not require anything regarding the *rate* of approximation, which in principle can be arbitrarily slow; Fortunately, the Solovay-Kitaev theorem [14, 11] guarantees that fast approximation is implied by the definition. The theorem states that for any fixed  $k \geq n_0$ , the number of gates from a universal set required to approximate a matrix  $U$  on  $k$  qubits to within  $\epsilon$  grows only like  $\text{polylog}(1/\epsilon)$ . Moreover, the calculation of the description of the sequence of gates approximating  $U$  can also be done efficiently.

Most of the universal sets of gates that have appeared in the literature are of the strictly universal type. Such is for example Kitaev's set of gates:

**Theorem 1** (Kitaev [11]) *The set  $\{\Lambda(P(i)), H\}$  is strictly universal with  $n_0 = 2$ .*

In fact, such a strong notion of universality is unnecessary, and often, weaker notions of universality are used. For example, universality which allows using ancilla states, as is done in Shi [15]; In this case, instead of approximating  $U$  on the input state  $|\xi\rangle$  one attempts at approximating  $U \otimes I$  on the state  $|\xi\rangle \otimes |\phi\rangle$  where  $|\phi\rangle$  is an ancilla state that can be generated using the same set of gates. Another relaxation of the definition is used in the context of certain fault tolerant constructions [4], where one is interested in *encoded universality*, namely generating all unitary matrices on some part of the Hilbert space. The Solovay-Kitaev theorem does not hold automatically for these weaker definitions of universality, and one should be careful to check on a case by case basis that fast approximation indeed holds.

We generalize these relaxations in a definition of *computational universality*. This definition captures essentially the meaning of the notion of universality, namely, that the set of gates can be used to perform general quantum computation, without too much overhead.

**Definition 3 (Computational Universality)** *A set of quantum gates  $C$  is said to be Computationally Universal if it can be used to simulate to within  $\epsilon$  error any quantum circuit which uses  $n$  qubits and  $t$  gates from a strictly universal set with only polylogarithmic overhead in  $(n, t, 1/\epsilon)$ .*

Strict universality obviously implies computational universality. Translations between universal sets satisfying either one of the definitions are polylogarithmic, and so for all purposes, it is sufficient to prove computational universality for a set of gates.

## 4 Proof

Here we are interested in the set  $\{T, H\}$ . We cannot hope to prove strict universality since the set consists only of real gates and cannot approximate complex unitary matrices. We show

**Theorem 2** *The set  $\{T, H\}$  is computationally universal.*

**Proof:** Let  $Q$  be a circuit that uses  $t$  gates from  $S = \{\Lambda(P(i)), H\}$ .  $S$  is a strictly universal set of gates by theorem 1. We replace gates from  $S$  by gates from our set  $\{T, H\}$ .  $H$  is already in our set, so we only need to deal with  $\Lambda(P(i))$ , which we convert to its real version using definition 1. We find that  $\widetilde{\Lambda(P(i))} = \Lambda^2[XZ]$  by checking its operation on the 8 basis states. Since  $XZ = XHXXH$ , this matrix is exactly a product of four gates from our set:  $T(1, 2, 3)H(3)T(1, 2, 3)H(3)$ . We have simulated  $Q$  with at most  $4t$  gates, and one additional qubit.  $\square$

## 5 Concluding Remarks

We have given a simple proof that the set  $\{T, H\}$  is computationally universal. For stronger results regarding this set, showing that it generates a dense subgroup in the group of orthogonal matrices, see [15].

We remark that an inherent disadvantage of using real matrices as the universal set of gates is that the additional extra qubit is used in many gates, which prevents full parallelization of the computation. One might worry that for this reason this set cannot be useful for fault tolerance against decoherence in the wires, which requires parallelism [3]. However this is not true; Error correction can still be performed in parallel, since the gates used in error corrections, namely Hadamard and classical gates, are real and

do not require the additional qubit, and so  $\{T, H\}$  can be used for fault tolerant purposes.

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