

REVIEW ARTICLE

The Physics of Information

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Abstract. Information theory is a statistical theory concerned with the relative state of detectors and physical systems. As a consequence, the classical framework of Shannon needs to be extended to deal with quantum detectors, possibly moving at relativistic speeds, conceivably within curved space-time. Considerable progress toward such a theory has been achieved in the last ten years, while much is still not understood. This review recapitulates some milestones along this road, and speculates about future ones.

1. Entropy and Information: Classical Theory

Since Shannon's historical pair of papers [1], information theory has changed from an engineering discipline to a full-fledged theory within physics [2]. While a considerable part of Shannon's theory deals with communication channels and codes [3], the concepts of entropy and information he introduced are crucial to our understanding of the physics of measurement, and turn out to be more general than thermodynamical entropy. Thus, information theory represents an important part of statistical physics.

When discussing the relationship between information theory and statistical physics, it is impossible not to mention Jaynes' work on the subject [4], who realized that optimal inference (that is, making predictions based on available information) involves choosing probability distributions that maximize Shannon's entropy. In this manner, he was able to *justify* certain parts of statistical physics using probability theory. The general point of view promulgated here goes beyond that. It is argued that information theory is a unifying framework which can be used to describe those circumstances in nonequilibrium physics that involve an observer and an observed.

In the following, I present an overview of some crucial aspects of entropy and information in classical and quantum physics, with extensions to the special and general theory of relativity. While not exhaustive, the treatment is at an introductory level, with pointers to the technical literature where appropriate.

1.1. Entropy

The concepts of entropy and information quantify the ability of observers to make *predictions*, in particular how well an observer equipped with a specific measurement

apparatus can make predictions about another physical system. Shannon entropy (also known as *uncertainty*) is defined for mathematical objects called *random variables*. A random variable X is a system that can take on a finite number of discrete states x_i , where $i = 1, \dots, N$ with probabilities p_i . Now, physical systems are not mathematical objects, nor are their states necessarily discrete. However, if I want to quantify my uncertainty about the state of a physical system, then in reality I need to quantify my uncertainty about the *possible outcomes of a measurement of that system*. In other words, my maximal uncertainty about a system is not a property of the system, but rather a property of the measurement device with which I am about to examine the system. If my measurement device, for example, is simply a “presence-detector”, then the maximal uncertainty I have about the physical system under consideration is 1 bit, which is the amount of *potential information* I can obtain about that system.

Thus, the entropy of a physical system is undefined if we do not specify the device that we are going to use to reduce that entropy. A standard example for a random variable that is also a physical system is the six-sided even die. Usually, the maximal entropy attributed to this system is $\log_2(6)$ bits. Is this all there is to know about it? If it is a physical system, the die is made of molecules and these can be in different states depending on the temperature of the system. Are those knowable? What about the state of the atoms making up the molecules? They could conceivably provide labels such that the number of states is in reality much larger. What about the state of the nuclei? Or the quarks and gluons inside those?

This type of thinking makes it clear that indeed we cannot speak about the entropy of an isolated system without reference to the coarse-graining of states that is implied by the choice of detector. And even though detectors exist that record continuous variables (such as, say, a mercury thermometer), each detector has a finite resolution such that it is indeed appropriate to consider only the *discrete* version of the Shannon entropy, which is given in terms of the probabilities p_i as[‡]

$$H(X) = - \sum_i^N p_i \log p_i . \quad (1)$$

For any physical system, how are those probabilities obtained? In principle, this can be done both by experiment and by theory. Once I have defined the N possible states of my system by choosing a detector for it, the *a priori* maximal entropy is defined as

$$H_{\max} = \log N . \quad (2)$$

Experiments using my detector can now sharpen my knowledge of the system. By tabulating the frequency with which each of the N states appears, we can estimate the probabilities p_i . Note, however, that this is a biased estimate that approaches the

[‡] From now on, I shall not indicate the basis of the logarithm, which only serves to set the units of entropy and information (base 2, e.g., sets the unit to a “bit”).

true entropy Eq.(1) only in the limit of an infinite number of trials. On the other hand, some of the possible states of the system (or more precisely, possible states of my detector interacting with the system) can be eliminated by using some knowledge of the physics of the system. For example, we may have some initial data about the system. This becomes clear in particular if the degrees of freedom that we choose to characterize the system with are position, momentum, and energy, i.e., if we consider the *thermodynamical entropy* of the system (see below).

1.2. Conditional Entropy

Let us look at the basic process that reduces uncertainty: a measurement. When measuring the state of system X , I need to bring it into contact with a system Y . If Y is my measurement device, then usually I can consider it to be completely known (at least, it is completely known with respect to the degrees of freedom I *care* about). In other words, my device is in a particular state y_0 with certainty. After interacting with X , this is not the case anymore. Let us imagine an interaction between the systems X and Y that is such that

$$x_i y_0 \rightarrow x_i y_i \quad i = 1, \dots, N, \quad (3)$$

that is, the states of the measurement device y_i end up reflecting the states of X . This is a perfect measurement, since no state of X remains unresolved. More generally, let X have N states while Y has M states, and let us suppose that $M < N$. Then we can imagine that each state of Y reflects an *average* of a number of X 's states, so that the probability to find Y in state y_j is given by q_j , where $q_j = \sum_i p_{ij}$, and p_{ij} is the joint probability to find X in state x_i and Y in state y_j . The measurement process then proceeds as

$$x_i y_0 \rightarrow \langle x \rangle_j y_j \quad (4)$$

where

$$\langle x \rangle_j = \sum_i p_{i|j} x_i. \quad (5)$$

In Eq.(5) above, I introduced the *conditional probability*

$$p_{i|j} = \frac{p_{ij}}{q_j} \quad (6)$$

that X is in state i *given* that Y is in state j . In the perfect measurement above, this probability was 1 if $i = j$ and 0 otherwise (i.e., $p_{i|j} = \delta_{ij}$), but in the imperfect measurement, X is distributed across some of its states i with a probability distribution $p_{i|j}$, for each j .

We can then calculate the *conditional entropy* (or remaining entropy) of the system X given we found Y in a particular state y_j after the measurement:

$$H(X|Y = y_j) = - \sum_i^N p_{i|j} \log p_{i|j}. \quad (7)$$

This remaining entropy is guaranteed to be smaller than or equal to the unconditional entropy $H(X)$, because the worst case scenario is that Y doesn't resolve *any* states of X , in which case $p_{i|j} = p_i$. But since we didn't know anything about X to begin with, $p_i = 1/N$, and thus $H(X|Y = y_j) \leq \log N$.

Let us imagine that we did learn something from the measurement of Y , and let us imagine furthermore that this knowledge is permanent. Then we can express our new-found knowledge about X by saying that we know the probability distribution of X , p_i , and this distribution is *not* the uniform distribution $p_i = 1/N$. Of course, in principle we should say that this is a conditional probability $p_{i|j}$, but if the knowledge we have obtained is permanent, there is no need to constantly remind ourselves that the probability distribution is conditional on our knowledge of certain other variables connected with X . We simply say that X is distributed according to p_i , and the entropy of X is

$$H_{\text{actual}}(X) = - \sum_i \log p_i \log p_i . \quad (8)$$

According to this strict view, all Shannon entropies of the form (8) are conditional if they are not maximal. And we can quantify our knowledge about X simply by subtracting this uncertainty from the maximal one:

$$I = H_{\text{max}}(X) - H_{\text{actual}}(X) . \quad (9)$$

This knowledge, of course, is *information*.

1.2.1. Example: Thermodynamics We can view thermodynamics as a particular case of Shannon theory. First, if we agree that the degrees of freedom of interest are position and momentum, then the maximal entropy of any system is defined by its volume in phase space:

$$H_{\text{max}} = \log \Delta\Gamma , \quad (10)$$

where $\Delta\Gamma = \frac{\Delta p \Delta q}{k}$ is the number of states within the phase space volume $\Delta p \Delta q$. Now the normalization factor k introduced in (10) clearly serves again to coarse-grain the number of states, and should be related to the resolution of our measurement device. In quantum mechanics, of course, this factor is given by the amount of phase space volume occupied by each quantum state, $k = (2\pi\hbar)^n$ where n is the number of degrees of freedom of the system. Does this mean that in this case it is not my type of detector that sets the maximum entropy of the system? Actually, this is still true, only that here we assume a quantum mechanical perfect detector, while still averaging over certain internal states of the system inaccessible to this detector.

Suppose I am contemplating a system whose maximum entropy I have determined to be Eq. (10), but I have some additional information. For example, I know that this system has been undisturbed for a long time, and I know its total energy E , and perhaps even the temperature T . Of course, this kind of knowledge can be obtained in a number of different ways. It could be obtained by experiment, or it could be obtained

by inference, or theory. How does this knowledge reduce my uncertainty? In this case, we use our *knowledge of physics* to predict that the probabilities $\rho(p, q)$ going into our entropy

$$H(p, q) = - \sum_{\Delta p, \Delta q} \rho(p, q) \log \rho(p, q) \quad (11)$$

are given by the canonical distribution[§]

$$\rho(p, q) = \frac{1}{Z} e^{-E(p, q)/T}, \quad (12)$$

where Z is the usual normalization constant, and the sum in (11) goes over all momenta in the phase space volume $\Delta p \Delta q$. The amount of knowledge we have about the system is then just the difference between these two uncertainties:

$$I = \log \Delta \Gamma - \log Z - \frac{E}{T}. \quad (13)$$

1.3. Information

In Eq. (9), we quantified our knowledge about the states of X by the difference between the maximal and the actual entropy of the system. This was a special case because we assumed that after the measurement, Y was in state y_j with certainty, i.e, everything was known about it. In general, we can imagine that Y instead is in state y_j with probability q_j (in other words, we have some information about Y but we don't know everything, just as for X). We can then define the *average conditional entropy* of X simply as

$$H(X|Y) = \sum_j q_j H(X|Y = y_j) \quad (14)$$

and the information that Y has about X is then the difference between the unconditional entropy $H(X)$ and Eq. (14) above,

$$H(X : Y) = H(X) - H(X|Y). \quad (15)$$

The colon between X and Y in the expression for the information $H(X : Y)$ is conventional, and indicates that it stands for an entropy *shared* between X and Y . According to the strict definition given above, $H(X) = \log N$, but in the standard literature $H(X)$ refers to the actual uncertainty of X given whatever knowledge allowed me to obtain the probability distribution p_i , i.e., Eq. (8).

Eq. (15) can be rewritten to display the symmetry between the observing system and the observed:

$$H(X : Y) = H(X) + H(Y) - H(XY), \quad (16)$$

where $H(XY)$ is just the joint entropy of both X and Y combined. This joint entropy would equal the sum of each of X 's and Y 's entropy only in the case that there are *no*

[§] We set Boltzmann's constant equal to 1 throughout. This constant, of course, sets the scale of thermodynamical entropy, and would end up multiplying the Shannon entropy just like any particular choice of base for the logarithm would.

correlations between X 's and Y 's states. If that would be the case, we could not make any predictions about X just from knowing something about Y . The information (16), therefore, would vanish.

1.3.1. Measurement Example An instructive example illustrating the effect of a measurement on uncertainty has been given by Peres [5]. Suppose the random variable X represents the location of a key, and prior knowledge has established the following: the key is in my pocket with probability $p = 0.9$, but if it is not in my pocket, it can be in exactly 100 places with equal probability. The random variable is thus actually composed of two *correlated* variables: the pocket P (with two states, *yes* and *no*), and the “other” places O , that has 100 states : $X = PO$. The entropy of X is:

$$H(X) = H(O|P) + H(P) , \quad (17)$$

where naturally $H(P)$ is my uncertainty about whether the key is in my pocket, given by $H(P) = -0.1 \ln 0.1 + 0.9 \ln 0.9 \approx 0.325$, and $H(O|P)$ is the average conditional entropy of the “other” places, given I know whether or not the key is in my pocket. So:

$$\begin{aligned} H(O|P) &= p H(O|P = \text{yes}) + (1 - p) H(O|P = \text{no}) \\ &= 0.1 \times 0 + 0.9 \times \ln(100) \approx 0.4605 , \end{aligned} \quad (18)$$

since if the key is in my pocket, it is not in any of the 100 other places. Thus, my uncertainty about the key location is $H(X) \approx 0.7856$. Now, this type of example is often used to claim that a measurement can sometimes *increase* uncertainty, by nothing that, should I *not* find the key in my pocket ($P = \text{no}$), my uncertainty is now $\ln 100 \approx 4.605$, much larger than 0.7856! But it is in fact not $H(X)$ that has increased, since the new uncertainty is of course just $H(X|P = \text{no})$, a *conditional* uncertainty. The entropy of X was changed only by reducing $H(P)$ in Eq. (17) to zero (since the state of the pocket will be known with certainty after the measurement), and therefore

$$H(X) \rightarrow H(O|P) \approx 0.4605 . \quad (19)$$

Thus, conditional entropies can increase or decrease due to a measurement, but the unconditional entropy *must* decrease. This example is also instructive to illustrate that in almost all cases, the entropy of random variables in physics is going to be conditional on the state of other, measured variables. Indeed, *subjectively* you sense that your uncertainty about the key's location has increased after not finding it in your pocket, because *your* uncertainty has become a conditional one after measurement. The fact that it has decreased *on average* is irrelevant to you as an observer, because this may be the one and only time you perform the measurement.

1.4. Information and the Second Law

Thermodynamics' second law is often regarded as one of physics' most curious, because it appears to be intuitively correct while it cannot be derived from first principles. I will take the position here that this is so because the second law is usually formulated

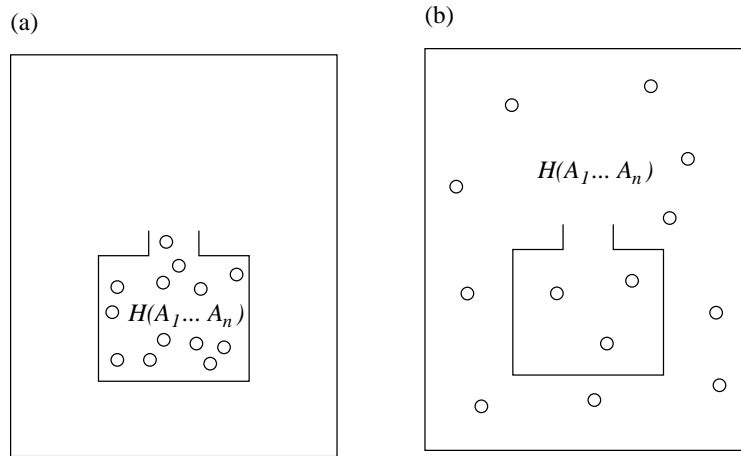


Figure 1. Diffusion of an ideal gas from a small into a larger container. (a) The molecules with entropy $H(A_1 \dots A_n)$ occupy the smaller volume, and their correlation entropy is zero. (b) The molecules have escaped into the larger container, which increases the sum of the per-particle entropies and increases the correlation entropy commensurately such that the overall entropy remains unchanged.

without giving sufficient heed to the notion that conditional and unconditional entropies are fundamentally different, both from the point of view of our intuition and from their mathematical structure. I have argued above that thermodynamics can be viewed as a special case of information theory, and I elaborate this point here.

The second law makes a prediction about the behavior of closed systems that evolve from a non-equilibrium state towards an equilibrium state. In particular, the second law predicts that the entropy of such a system will *almost always* increase. The central observation about the inconsistency of this formulation lies in recognizing that the second law describes *non-equilibrium dynamics* using an equilibrium concept (namely Boltzmann-Gibbs entropy). Above, we have seen that Shannon entropies, once we have chosen thermodynamical variables such as position and momentum as those relevant to us, turn into Shannon-Gibbs entropies if equilibrium distributions are used in Eq. (11). But while a system moves from non-equilibrium to equilibrium, we certainly cannot do this. Indeed, we know that as a system equilibrates, conditional and unconditional entropies are *not* equal. In order to correctly describe this, we have to use information theory.

1.4.1. Equilibration Let us analyze the quintessential irreversible dynamics, the notorious “perfume bottle” experiment, in which a diffusive substance (let’s say, an ideal gas) is allowed to escape from a small container into a larger one (see Fig. 1a). Both the initial and the final state of the system can be described by equilibrium equations; common wisdom however states that the entropy of the gas is *increasing* during the process, reflecting the non-equilibrium dynamics. I shall now show that this is not the case, by describing the gas in the smaller container by a set of variables A_1, \dots, A_n , one

for each molecule. (What I will show is that it is instead a *conditional* entropy that is increasing.) The entropy $H(A_i)$ thus represents the entropy per molecule. The entire gas, on the other hand, is described by the *joint entropy*

$$H_{\text{gas}} = H(A_1 \cdots A_n) , \quad (20)$$

which can be much smaller than the sum of per-particle entropies because there are strong *correlations* between the variables A_i . In information theory, such correlations are described by information terms such as (15). And as we discussed above, they must vanish at equilibrium. The sum of per-particle entropies, because it ignores correlations between subsystems, is just the standard thermodynamical entropy S_{eq}

$$H(A_1 \cdots A_n) \ll \sum_{i=1}^n H(A_i) = S_{eq} . \quad (21)$$

The difference between S_{eq} and (20) is given by the n -body correlation entropy

$$H_{\text{corr}} = \sum_{i=1}^n H(A_i) - H(A_1 \cdots A_n) , \quad (22)$$

which can be calculated in principle, but becomes cumbersome already for more than three particles.

We see that in this description, the molecules after occupying the larger volume cannot be independent of each other, as their locations are *in principle* correlated (as they all used to occupy a smaller volume, see Fig. 1a). It is true that once the molecules occupy the larger volume (Fig. 1b) the observer has lost track of these correlations, and the second law characterizes just how much information was lost. This statement, however, has nothing to do with physics, but rather concerns an observer's capacity to make predictions. Indeed, these correlations are not manifest in two- or even three-body correlations, but are complicated n -body correlations which imply that their positions are not independent, but linked by the fact that they share initial conditions. This state of affairs can be summarized by rewriting Eq. (22):

$$H(A_1 \cdots A_n) = \sum_{i=1}^n H(A_i) - H_{\text{corr}} . \quad (23)$$

We assume that before the molecules are allowed to escape, they are uncorrelated with respect to each other: $H_{\text{corr}} = 0$, and the entire entropy is given by the extensive sum of the per-molecule entropies. After expansion into the larger volume, the standard entropy increases because of the increase in available phase space, but this increase is balanced by an increase in the correlation entropy H_{corr} in such a manner that the actual joint entropy of the gas, H_{gas} , remains unchanged.

Note that this description is not, strictly speaking, a redefinition of thermodynamical entropy. While in the standard theory, entropy is an *extensive* (i.e., additive) quantity for uncorrelated systems, the concept of a thermodynamical entropy in the absence of equilibrium distributions has been formulated as the number of ways to realize a given set of occupation numbers of states of the joint system (which gives rise to (1) by use

of Stirling’s approximation, see, e.g., [6]) and is thus fundamentally *non-extensive*. Assuming the A_i are uncorrelated reduces $H(A_1 \cdots A_n)$ to the extensive sum $\sum_{i=1}^n H(A_i)$, and thus to an entropy proportional to the volume they inhabit. From a calculational point of view the present formalism does not represent a great advantage in this case, as the correlation entropy H_{corr} can only be obtained in special situations, when only few-body correlations are important.

The examples of non-equilibrium processes treated here (measurement and equilibration) suggest the following information-theoretical reformulation of the second law:

In a thermodynamical equilibrium or non-equilibrium process, the unconditional (joint) entropy of a closed system remains a constant.

Nothing can be said in principle about the conditional entropies involved (namely the conditional entropy of the system given the state of the observer, or the conditional entropy of the observer, given the state of the system), because they can be increasing or decreasing. In a measurement, the conditional entropy decreases (but the conditional entropy given a particular outcome can increase), while during equilibration, the conditional entropy usually increases. That it can sometimes *decrease* is acknowledged in the standard formulation of the second law by the words “almost always”. We recognize this as just one of these rare fluctuations encountered in Section 1.3.1, where the entropy conditional on a particular outcome behaves counter-intuitively, while on average everything is as it should be.

The formulation of the second law given above directly reflects probability conservation (in the sense of the Liouville theorem), and allows a quantitative description of the amount by which either the conditional entropy is decreased in a measurement, or the amount of per-particle entropy is increased in an equilibration process.

2. Quantum Theory

In quantum mechanics, the concept of entropy translates very easily, but the concept of information is thorny. John von Neumann introduced his eponymous quantum mechanical entropy as early as 1927 [7], a full 21 years before Shannon introduced its classical limit! In fact, it was von Neumann who suggested to Shannon to call his formula entropy, simply because “your uncertainty function has been used in statistical mechanics under that name” [8].

2.1. Measurement

In quantum mechanics, measurement plays a very prominent role, and is still considered somewhat mysterious in many respects. The proper theory to describe measurement dynamics in quantum physics, not surprisingly, is quantum information theory. As in the classical theory, the uncertainty about a quantum system can only be defined in terms of the detector states, which in quantum mechanics are a discrete set of *eigenstates* of

a measurement operator. The quantum system itself is described by a wave function, given in terms of the quantum system's eigenbasis, which may or may not be the same as the measurement device's basis.

For example, say we would like to “measure an electron”. In this case, we may mean that we would like to measure the position of an electron, whose wave function is given by $\Psi(q)$, where q is the coordinate of the electron. Further, let the measurement device be characterized initially by its eigenfunction $\phi_0(\xi)$, where ξ may summarize the coordinates of the device. Before measurement, i.e., before the electron interacts with the measurement device, the system is described by the wave function

$$\Psi(q)\phi_0(\xi) . \quad (24)$$

After the interaction, the wave function is a superposition of the eigenfunctions of electron and measurement device

$$\sum_n \psi_n(q)\phi_n(\xi) . \quad (25)$$

Following orthodox measurement theory, the classical nature of the measurement apparatus implies that after measurement the “pointer” variable ξ takes on a well-defined value at each point in time; the wave function, as it turns out, is thus *not* given by the entire sum in (25) but rather by the single term

$$\psi_n(q)\phi_n(\xi) . \quad (26)$$

The wave function (25) is said to have collapsed to (26).

Let us now study what actually happens in such a measurement in detail. For ease of notation, let us recast this problem into the language of state vectors instead. The first stage of the measurement involves the interaction of the quantum system Q with the measurement device (or “ancilla”) A . Both the quantum system and the ancilla are fully determined by their state vector, yet, let us assume that the state of Q (described by state vector $|x\rangle$) is unknown whereas the state of the ancilla is prepared in a special state $|0\rangle$, say. The state vector of the combined system $|QA\rangle$ before measurement then is

$$|\Psi_{t=0}\rangle = |x\rangle|0\rangle \equiv |x, 0\rangle . \quad (27)$$

The von Neumann measurement is described by the unitary evolution of QA via the interaction Hamiltonian

$$\hat{H} = -\hat{X}_Q\hat{P}_A , \quad (28)$$

operating on the product space of Q and A . Here, \hat{X}_Q is the observable to be measured, and \hat{P}_A the operator *conjugate* to the degree of freedom of A that will reflect the result of the measurement. We now obtain for the state vector $|QA\rangle$ after measurement (e.g., at $t = 1$, putting $\hbar = 1$)

$$|\Psi_{t=1}\rangle = e^{i\hat{X}_Q\hat{P}_A}|x, 0\rangle = e^{ix\hat{P}_A}|x, 0\rangle = |x, x\rangle . \quad (29)$$

Thus, the pointer A that previously pointed to zero now also points to the position x that Q is in. This operation appears to be very much like the classical measurement

process Eq. (3), but it turns out to be quite different. In general, this unitary operation introduces quantum *entanglement*, which is beyond the classical concept of correlations.

This becomes evident if we apply the unitary operation described above to an initial quantum state which is in a quantum *superposition* of two states:

$$|\Psi_{t=0}\rangle = |x + y, 0\rangle . \quad (30)$$

Then, the linearity of quantum mechanics implies that

$$|\Psi_{t=1}\rangle = e^{i\hat{X}_Q\hat{P}_M} \left(|x, 0\rangle + |y, 0\rangle \right) = |x, x\rangle + |y, y\rangle . \quad (31)$$

This state is very different from what we would expect in classical physics, because Q and A are not just correlated (like, e.g., the state $|x + y, x + y\rangle$ would be) but rather they are *quantum entangled*. They now form *one* system that cannot be thought of as composite.

This nonseparability of a quantum system and the device measuring it is at the heart of all quantum mysteries. Indeed, it is at the heart of *quantum randomness*, the puzzling emergence of unpredictability in a theory that is unitary, i.e., where all probabilities are conserved. What is being asked here of the measurement device, namely to describe the system Q , is logically impossible because after entanglement the system has grown to QA . Thus, the detector is being asked to describe a system that is *larger* (as measured by the possible number of states) than the detector, and that includes the detector *itself*. This is precisely the same predicament that befalls a computer program that is asked to determine its own halting probability, in Turing's famous *Halting Problem* analogue [9] of Gödel's Incompleteness Theorem. Chaitin [10] showed that the self-referential nature of the question that is posed to the program gives rise to randomness in pure Mathematics. A quantum measurement is self-referential in the same manner, since the detector is asked to describe its own state, which is logically impossible^{||}. Thus we see that quantum randomness has mathematical (or rather logical) randomness at its very heart.

2.2. von Neumann Entropy

Because of this inherent uncertainty, measurements of a quantum system A are then described as *expectation values*, which are averages of an observable over the system's *density matrix*, so that

$$\langle \hat{O} \rangle = \text{Tr}(\rho_A \hat{O}) , \quad (32)$$

where \hat{O} is an operator associated with the observable we would like to measure, and ρ_A is the density matrix of system A . The latter is obtained from the quantum wave function Ψ_{QA} (for the combined system QA , since neither Q nor A separately have a wave function after the entanglement occurred) by *tracing out* the quantum system:

$$\rho_A = \text{Tr}_Q |\Psi_{QA}\rangle \langle \Psi_{QA}| . \quad (33)$$

^{||} The logical impossibility of describing one's own state is intrinsically the same as that posed by the Cretan Paradox (Epimenides the Cretan says "All Cretans are liars.")

The partial trace represents an averaging over the states of the quantum system, which after all is *not* being observed: we are looking at the measurement device only. The uncertainty about quantum system A can then be calculated simply by von Neumann's entropy:

$$S(\rho_A) = -\text{Tr} \rho_A \log \rho_A . \quad (34)$$

If Q has been measured in A 's eigenbasis, then the density matrix ρ_A is diagonal, and von Neumann entropy turns into Shannon entropy, as we expect. Indeed, this is precisely the classical limit, because entanglement does not happen under these conditions.

Quantum Information Theory needs concepts such as conditional entropies and mutual entropies. They can be defined in a straightforward manner [11], but their interpretation needs care. For example, we can define a conditional entropy in analogy to Shannon theory as

$$\begin{aligned} S(A|B) &= S(AB) - S(B) \\ &= -\text{Tr}_{AB}(\rho_{AB} \log \rho_{AB}) + \text{Tr}_B(\rho_B \log \rho_B) , \end{aligned} \quad (35)$$

where $S(AB)$ is the joint entropy of two systems A and B . But can we write this entropy in terms of a conditional density matrix, just as we were able to write the conditional Shannon entropy in terms of a conditional probability? The answer is yes and no: a definition in terms of a conditional density *operator* $\rho_{A|B}$ exists [11, 12], but it is technically not a density matrix (its trace is not equal to one), and the eigenvalues of this matrix are very peculiar: they can *exceed* one (this is of course not possible for probabilities). Indeed, they can exceed one only when the system is entangled. As a consequence, quantum conditional entropies can be *negative* [11].

Even thornier is quantum mutual entropy. We can again define it simply in analogy to (16) as

$$S(A : B) = S(A) + S(B) - S(AB) , \quad (36)$$

but what does it mean? For starters, this quantum mutual entropy can be twice as large as the entropy of any of the subsystems, so A and B can share more quantum entropy than they even have by themselves! Of course, this is due to the fact, again, that “selves” do not exist anymore after entanglement. Also, in the classical theory, information, that is, shared entropy, could be used to make predictions, and therefore to reduce the uncertainty we have about the system that we share entropy with. But that's not possible in quantum mechanics. If, for example, I measure the spin of a quantum particle that is in an even superposition of its spin-up and spin-down state, my measurement device will show me spin-up half the time, and spin-down half the time, that is, my measurement device has an entropy of one bit. It can also be shown that the shared entropy is *two bits* [11]. But this shared entropy cannot be used to make predictions about the actual spin. Indeed, I still do not know anything about it! On the other hand, it is possible, armed with my measurement result, to make predictions about the state of *other* detectors measuring the same spin. And even though all these detectors will agree about their result, technically they agree about a random variable,

not the actual state of the spin they believe their measurement device to reflect [13]. Indeed, what else could they agree on, since the spin does not have a state? Only the combined system with all the measurement devices that have ever interacted with it, does.

Information, it turns out, is a concept that is altogether classical. *Quantum* information, in hindsight, is therefore really a contradiction in terms. But that does not mean that the entire field of quantum information theory is absurd. Rather, what we mean by “quantum information theory” is the study of storage, transmission, and manipulation of *qubits* (the quantum analogues of the usual bit), which are quantum particles that can exist in superpositions of zero and one. Indeed, the capacity of quantum channels to transmit classical information is higher than any classical channel [14, 15], for example, and quantum bits can be used for super-fast computation [16].

The extension of Shannon’s theory into the quantum regime not only throws new light on the measurement problem, but it also helps in navigating the boundary between classical and quantum physics. According to standard lore, quantum systems (meaning systems described by a quantum wave function) “become” classical in the macroscopic limit, that is, if the action unit associated with that system is much larger than \hbar . Quantum information theory has thoroughly refuted this notion, since we now know that macroscopic bodies can be entangled just as microscopic ones can [17]. Instead, we realize that quantum systems appear to follow the rules of classical mechanics if parts of their wave function are averaged over [such as in Eq. (33)], that is, if the experimenter is not in total control of all the degrees of freedom that make up the quantum system. Because entanglement, once achieved, is not undone by the distance between entangled parts, almost all systems will seem classical unless expressly prepared, and then protected from interaction with uncontrollable quantum systems. Unprotected quantum systems spread their state over many variables very quickly: a process known as *decoherence* of the quantum state.

3. Relativistic Theory

Once convinced that information theory is a statistical theory about the relative states of detectors in a physical world, it is clear that we must worry not only about quantum detectors, but about moving ones as well. Einstein’s special relativity established an upper limit for the speed at which information can be transmitted without the need to cast this problem in an information-theoretic language. But in hindsight, it is clear that the impossibility of superluminal signaling could just as well have been the result of an analysis of the information transmission capacity of a communication channel involving detectors moving at constant speed with respect to each other. As a matter of fact, the capacity of an additive white noise Gaussian (AWNG) channel for information transmission for the case of moving observers just turns out to be [18]

$$C = W \log(1 + \alpha SNR) , \tag{37}$$

where W is the bandwidth of the channel, SNR is the signal-to-noise ratio, and $\alpha = v'/v$ is the Doppler shift. As the relative velocity $\beta \rightarrow 1$, $\alpha \rightarrow 0$ and the communication capacity vanishes.

Historically, however, no-one seems to have worried about an “information theory of moving bodies”, not the least because such a theory had, or indeed has, little immediate relevance. (The above-mentioned reference [18] is essentially unknown in the literature.) A standard scenario of relativistic information theory would involve two random variables moving with respect to each other. The question we may ask is whether and how relative motion is going to affect any shared entropy between the variables. First, it is important to point out that Shannon entropy is a *scalar*, and we therefore do not expect it to transform under Lorentz transformations. This is also intuitively clear if we adopt the “strict” interpretation of entropy as being *unconditional* (and therefore just equal to the logarithm of the number of degrees of freedom). On the other hand, probability distributions (and the associated conditional entropies) could conceivably change under Lorentz transformations. How is this possible given the earlier statement that entropy is a scalar?

We can investigate this with a *gedankenexperiment* where the system under consideration is an ideal gas, with particle velocities distributed according to an arbitrary distribution. In order to define entropies, we have to agree on which degrees of freedom we are interested in. Let us say that we only care about the two components of the velocity of particles confined in the x – y -plane. Even at rest, the mutual entropy between the particle velocity components $H(v_x : v_y)$ is non-vanishing, due to the finiteness of the magnitude of v . A detailed calculation [19] using continuous variable entropies of a uniform distribution shows that, at rest

$$H(v_x : v_y) = \log(\pi/e) . \quad (38)$$

The velocity distribution, on the other hand, will surely change under Lorentz transformations in, say, the x -direction, because the components are affected differently by the boost. In particular, it can be shown that the mutual entropy between v_x and v_y will *rise* monotonically from $\log(\pi/e)$, and tend to a constant value as the boost-velocity $\beta \rightarrow 1$ [19]. But of course, β is just another variable characterizing the moving system, and if this is known precisely, then we ought to be able to recover Eq. (38), and the apparent change in information is due entirely to a reduction in the uncertainty $H(v_x)$. Similar conclusions can be reached if the Maxwell distribution is substituted for the uniform one. This example shows that in information theory, even if the entire system’s entropy does not change under Lorentz transformations, the entropies of subsystems, and therefore also information, can.

While a full theory of relativistic information does not exist, pieces of such a theory can be found when digging through the literature, For example, relativistic thermodynamics is a limiting case of relativistic information theory, simply because as we have seen above, thermodynamical entropy is a limiting case of Shannon entropy. But unlike in the case constructed above, we do not have the freedom to choose our variables

in thermodynamics. Hence, the invariance of entropy under Lorentz transformations is assured via Liouville's theorem, because the latter guarantees that the phase-space volume occupied by a system is invariant. Yet, relativistic thermodynamics is an odd theory, not the least because it is intrinsically inconsistent: the concept of equilibrium becomes dubious. In thermodynamics, equilibrium is defined as a state where all relative motion between the subsystems of an ensemble has ceased. Therefore, a joint system where one part moves with a constant velocity with respect to the other cannot be at equilibrium, and relativistic information theory has to be used instead.

One of the few questions of immediate relevance that relativistic thermodynamics has been able to answer is how the temperature of an isolated system will appear from a moving observer. Of course, temperature itself is an equilibrium concept and therefore care must be taken in framing this question [20]. Indeed, both Einstein and Planck [21] tackled the question of how to Lorentz-transform temperature, with different results. The controversy [22] can be resolved by realizing that no such transformation law can in fact exist [23], as the usual temperature (the parameter associated with the Planckian blackbody spectrum) becomes direction-dependent if measured with a detector moving with velocity $\beta = v/c$ and oriented at an angle θ' with respect to the radiation [24, 25]

$$T' = \frac{T\sqrt{1-\beta^2}}{1-\beta\cos\theta'} . \quad (39)$$

In other words, an ensemble that is thermal in the rest frame is non-thermal in a moving frame, and in particular cannot represent a standard heat bath because it will be *non-isotropic*.

4. Relativistic Quantum Theory

While macroscopic quantities like temperature lose their meaning in relativity, microscopic descriptions in terms of probability distributions clearly still make sense. But in a quantum theory, these probability distributions are obtained from quantum measurements specified by local operators, and the space-time relationship between the detectors implementing these operators becomes important. For example, certain measurements on a joint (i.e., composite) system may require communication between parties, while certain others are impossible *even though* they do not require communication [26]. In general, a relativistic theory of quantum information needs to pay close attention to the behavior of the von Neumann entropy under Lorentz transformation, and how such entropies are being reduced by measurement.

4.1. Boosting Quantum Entropy

The entropy of a qubit (which we take here for simplicity to be a spin-1/2 particle) with wave function

$$|\Psi\rangle = \frac{1}{\sqrt{|a|^2 + |b|^2}} \left(a|\uparrow\rangle + b|\downarrow\rangle \right) , \quad (40)$$

(a and b are complex numbers), can be written in terms of its density matrix $\rho = |\Psi\rangle\langle\Psi|$ as

$$S(\rho) = -\text{Tr}(\rho \log \rho) . \quad (41)$$

A wave function is by definition a completely known state (called a “pure state”), because the wave function is a complete description of a quantum system. As a consequence, (41) vanishes: we have no uncertainty about this quantum system. As we have seen earlier, it is when that wave function interacts with uncontrolled degrees of freedom that mixed states arise. And indeed, just by boosting a qubit, such mixing will arise [27]. The reason is not difficult to understand. The wave function (40), even though I have just stated that it completely describes the system, in fact only completely describes the *spin* degree of freedom! Just as we saw in the earlier discussion about the classical theory, there may always be other degrees of freedom that our measurement device (here, a spin-polarization detector) cannot resolve. Because we are dealing with particles, ultimately we have to consider their *momenta*. A more complete description of the qubit state then is

$$|\Psi\rangle = |\sigma\rangle \times |\vec{p}\rangle , \quad (42)$$

where σ stands for the spin-variable, and \vec{p} is the particle’s momentum. Note that the momentum wave function $|\vec{p}\rangle$ is in a product state with the spin wave function $|\sigma\rangle$. This means that both spin and momentum have their own state, they are *unmixed*. But as is taught in every first-year quantum mechanics course, such momentum wave functions (plane waves with perfectly well-defined momentum \vec{p}) do not actually exist; in reality, they are wave packets with a momentum *distribution* $f(\vec{p})$, which we may take to be Gaussian. If the system is at rest, the momentum wave function does not affect the entropy of (42), because it is a product.

What happens if the particle is boosted? The spin and momentum degrees *do* mix, which we should have expected because Lorentz transformations always imply frame rotations as well as changes in linear velocity. The product wave function (42) then turns into

$$|\Psi\rangle \longrightarrow \sum_{\sigma} \int f(\vec{p}) |\sigma, \vec{p}\rangle d\vec{p} , \quad (43)$$

which is a state where spin-degrees of freedom and momentum degrees of freedom are entangled. But our spin-polarization detector is insensitive to momentum! Then we have no choice but to average over the momentum, which gives rise to a spin density matrix that is mixed,

$$\rho_{\sigma} = \text{Tr}_{\vec{p}}(\rho_{\sigma\vec{p}}) , \quad (44)$$

and which consequently has positive entropy. Note, however, that the entropy of the joint spin-momentum density matrix remains unchanged, at zero. Note also that if the momentum of the particle was truly perfectly known from the outset, i.e., a plane wave $|\vec{p}\rangle$, mixing would also not take place [28].

While the preceding analysis clearly shows what happens to the quantum entropy of a spin-1/2 particle under Lorentz transformations (a similar analysis can be done for photons [29]), what is most interesting in quantum information theory is the entanglement *between* systems. While some aspects of entanglement are captured by quantum entropies [30] and the spectrum of the conditional density operator [12], quantifying entanglement is a surprisingly hard problem, currently without a perfect solution. However, some good measures exist, in particular for the entanglement between two-level systems (qubits) and three-level systems.

4.2. Boosting Quantum Entanglement

If we wish to understand what happens to the entanglement between two spin-1/2 particles, say, we have to keep track of four variables: the spin states $|\sigma\rangle$ and $|\lambda\rangle$ and the momentum states $|\vec{p}\rangle$ and $|\vec{q}\rangle$. A Lorentz transformation on the joint state of this two-particle system will mix spins and momenta just as in the previous example. In fact, it is known that this type of mixing will affect the entanglement between pairs of particles that are used to test the violation of Bell inequalities, for example [31]. In order to investigate this effect from the point of view of quantum information theory, we need to study the behavior of an entanglement measure under Lorentz boosts.

A good measure for the entanglement of mixed states, i.e., states that are not pure such as (42), is the so-called *concurrence*, introduced by Wootters [32]. This concurrence $C(\rho_{AB})$ can be calculated for a density matrix ρ_{AB} that describes two subsystems A and B of a larger system, and quantifies the entanglement *between* A and B . For our purposes, we will be interested in the entanglement between the spins σ and λ of our pair. The concurrence is unity if two degrees of freedom are perfectly entangled, and vanishes if no entanglement is present.

In order to do this calculation we first have to specify our initial state. We take this to be a state with spin and momentum wave function in a product, but where the spin-degrees of freedom are perfectly entangled in a so-called Bell state:

$$|\sigma, \lambda\rangle = \frac{1}{\sqrt{2}} (|\uparrow, \downarrow\rangle - |\downarrow, \uparrow\rangle) . \quad (45)$$

Of course, such states have concurrence $C(\rho_{\sigma\lambda}) = 1$. We now apply a Lorentz boost to this joint state, i.e., we move our spin-polarization detector with speed $\beta = v/c$ with respect to this pair (or, equivalently, we move the pair with respect to the detector). If the momentum degrees of freedom of the particles at the outset are Gaussian distributions unentangled with each other and the spins, the Lorentz boost will entangle them, and the concurrence *between the spins* will drop [33]. How much it drops depends on the ratio between the spread of the momentum distribution σ_r (not to be confused with the spin σ) and the particle's mass m . In Fig. 2 below, the concurrence is displayed for two different such ratios, as a function of the *rapidity* ξ . The rapidity ξ is just a transformed velocity: $\xi = \sinh \beta$, such that $\xi \rightarrow \infty$ as $\beta \rightarrow 1$. We can see that if the ratio is not too large, the concurrence will drop but not disappear altogether. But

if the momentum spread is large compared to the mass, all entanglement can be lost.

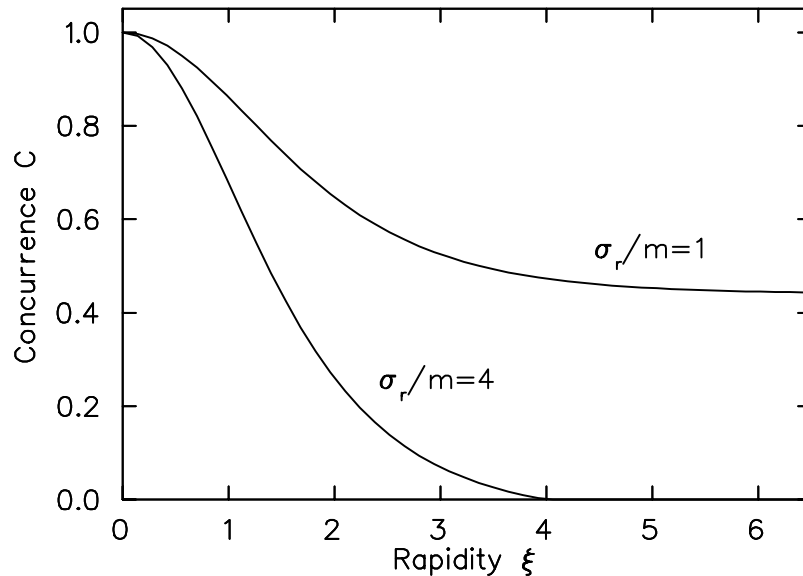


Figure 2. Spin-concurrence as a function of rapidity, for an initial Bell state with momenta in a product Gaussian. Data is shown for $\sigma_r/m = 1$ and $\sigma_r/m = 4$ (from Ref.[33]).

Let us consider instead a state that is unentangled in spins, but fully entangled in momenta. I depict such a wave function in Fig. 3, where a pair is in a superposition of two states, one moving in opposite directions with momentum \vec{p}_\perp in a relative spin state Φ^- (this is one of the four Bell spin-entangled states, Eq. (45)), and one moving in a plane in opposite orthogonal directions with momentum p , in a relative spin-state Φ^+ . It can be shown that if observed at rest, the spins are actually unentangled. But when boosted to rapidity ξ , the concurrence *increases* [33], as for this state (choosing $m = 1$)

$$C(\rho_{AB}) = \frac{p^2(\cosh^2(\xi) - 1)}{(\sqrt{1 + p^2} \cosh(\xi) + 1)^2} . \quad (46)$$

A similar analysis can be performed for pairs of entangled photons, even though the physics is quite different [34]. First of all, photons are massless and their quantum degree of freedom is the photon-polarization. The masslessness of the photons makes the analysis a bit tricky, because issues of gauge invariance enter into the picture, and as they all move with constant velocity (light speed), there cannot be a spread in momentum as in the massive case. Nevertheless, Lorentz transformation laws acting on polarization vectors can be identified, and an analysis similar to the one described above can be carried through. The difference is that the entangling effect of the Lorentz boost is now entirely due to the spread in momentum *direction* between the two entangled photon beams. This implies first of all that fully-entangled photon polarizations cannot exist, even for detectors at rest, and second that existing entanglement can either be decreased

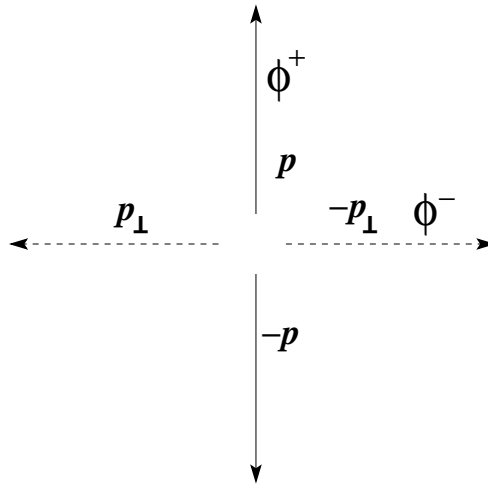


Figure 3. Superposition of Bell-states Φ^+ and Φ^- at right angles, with the particle pair moving in opposite directions.

or increased, depending on the angle with which the pair is boosted (with respect to the angle set by the entangled pair), and the rapidity [34].

5. Information in Accelerated Frames and Curved Space Time

Relativistic quantum information theory is a growing field [35] that has naturally engendered questions about the relative state of detectors in *non-inertial* frames. Accelerated detectors introduce a new twist to information theory: whether or not a detector registers depends on its state of motion, that is, even presence or absence can become relative! To some extent this is not a completely unfamiliar situation. We are used to radiation being emitted from accelerated charges, and from detectors moving through a medium with changing index of refraction [36, 37]. In such cases, the state of the detector depends on *which vacuum* it perceives. For all *inertial* measurement devices, all vacua are equivalent because they are invariant under the Poincaré group. Yet this invariance doesn't hold for accelerated observers or detectors in strong fields [38], and no particular vacuum state is singled out. Indeed, different vacua can be defined using Bogoliubov transformations to transform one set of creation/annihilation operators into another, and in principle none would be preferred. How do we then make sense of the physical world, which requires that all observers agree about the result of measurements? Usually, the agreed-upon vacuum is the one where absence of particles is perceived by *all inertial observers*, for example in the remote past or future. This is also the approach taken below when we consider black hole formation and evaporation, where quantum states $|k\rangle_{\text{in}}$ and $|k\rangle_{\text{out}}$ refer to in- and out-states in past and future infinity. In the meantime, let us discuss briefly the relative state of non-inertial detectors.

That something interesting must happen to entropies in non-inertial frames is

immediately clear from the Unruh effect [39, 40, 41]. The Unruh effect is perhaps the most important clue to our understanding of quantum field theory in curved space time, which is still quite incomplete. Accelerated observers perceive a vacuum quite different from that apparent to a non-accelerated observer: they find themselves surrounded by thermal photons of temperature $T_U = \frac{\hbar a}{2\pi c}$ (the Davies-Unruh temperature), where a is the observer's acceleration, and c is the speed of light. If we were to calculate the entropy of a particle in the inertial vs. the non-inertial frame, the absence or presence of the Unruh radiation implies that they would be different. In other words, standard thermodynamic or von Neumann entropies do not transform covariantly under general co-ordinate transformations, that is, they are not scalars. Again this should not be surprising, because positive von Neumann entropies only occur if part of an entangled state is averaged over. Only the entropy of the pure state is invariant (it vanishes). There are immediate consequences for standard quantum information protocols such as quantum teleportation: while the required resource (and entangled pair) can be used to teleport one qubit perfectly in an inertial frame, the appearance of Unruh radiation dilutes the entanglement such that the fidelity of teleportation is reduced [42]. There are also consequences when positive entropies are forced upon us simply because certain parts of spacetime are inaccessible to measurements. Such is the case beyond black-hole event horizons.

5.1. Black Hole Information Paradox

Ever since the discovery of Hawking radiation [43] we are faced with what is known as the *black hole information paradox* [44]. The paradox can be summarized as follows. According to standard theory, a black hole (without charge or angular momentum) can be described by an entropy [45] that is determined entirely in terms of its mass M (in units where $\hbar = c = G = k = 1$):

$$S_{BH} = 4\pi M^2 . \tag{47}$$

From the point of view adopted in this review, this formula implies that black holes are very peculiar objects. They have an entropy given by one quarter of the surface area of the black hole event horizon ($r = 2M$ in these units), which we have to associate with an *uncertainty* in classical information theory. Yet, it seems we cannot learn anything about the black hole, because we cannot measure its states. Fortunately, quantum mechanics comes to the rescue here. Classically, a black hole has zero temperature (because it does not radiate), but a quantum treatment shows that vacuum fluctuations near the event horizon cause the black hole to radiate like a black body at a temperature $T_H = (8\pi M)^{-1}$, the Hawking temperature. In a sense, the black hole polarizes the vacuum around it, causing spontaneous emission of radiation, and a *reduction* of entropy. The alert reader should make a mental note at this point, because I have argued earlier that, strictly speaking, only *conditional* entropies can decrease. Could it be that the black hole entropy is in fact a conditional entropy? Before we enter this discussion, I should summarize an apparently very alarming state of affairs in black hole physics.

If a state that is fully known is absorbed by the black hole, i.e, if it disappears behind the event horizon, it appears as if the information about the identity of the state is lost. Even worse, after a long time, the black hole will have evaporated entirely into thermal (Hawking) radiation, and the information appears not only to be irretrievable, but *destroyed*. Indeed, it would appear that black holes have the capability to turn pure states into mixed states *without* disregarding parts of the wave function. Such a state of affairs is not only paradoxical, it is in fact completely out of the question within the standard model of physics.

It has been argued that this paradox stems from our incomplete understanding of quantum gravity. For example, in the semiclassical framework (in which Hawking's calculation was carried out [43]), the space time metric remains unquantized, and is treated instead as a classical background field. A consistent treatment instead would allow particle degrees of freedom to be entangled with the metric, creating quantum mechanical uncertainty (see [46] for a calculation of the decoherence of a qubit in an orbit around a Schwarzschild black hole due to entanglement with the metric). But it is difficult to conceive that this effect has a significant impact on black hole dynamics. Indeed, gravitational fields are weak up until the black hole is of the order of the Planck mass, when backreaction effects on the space time metric presumably become important [47]. But at this point, all the "damage" has already been done, because it is difficult to imagine that lost information can be recovered from a Planck-sized object. (An enormous amount of entropy would have to be emitted instantaneously by a black hole of size $L_{\text{Planck}} \approx 10^{-33}$ cm, see Ref. [47], p. 184). Instead, we should look at the treatment of black holes within classical *equilibrium* thermodynamics as the culprit for the paradox.

Black holes have negative heat capacity (they become hotter the more energy they radiate away), and therefore can *never* be at equilibrium with a surrounding (infinite) heat bath. As we have seen, the concept of information *itself* is a non-equilibrium one (because information implies correlations). Moreover, quantum correlations (in the form of entanglement) can exist over arbitrary distances *and* across event horizons, because entanglement does not imply signaling. Thus, a black hole can be entangled with surrounding radiation even though the two are causally disconnected. These considerations make it likely that a quantum information-theoretic treatment of black hole dynamics could resolve the paradox without appealing to a consistent theory of quantum gravity. In the following, I outline just such a scenario.

First, we must recognize that a positive entropy in quantum mechanics implies that the black hole is described by a density matrix of the form

$$\rho_{\text{BH}} = \sum_i p_i \rho_i \tag{48}$$

where $\rho_i = |i\rangle\langle i|$ are obtained from black hole eigenvectors. Such a mixed state can always be written in terms of a pure wavefunction via a Schmidt decomposition:

$$\rho_{\text{BH}} = \sum_i \sqrt{p_i} |\psi_i\rangle_{\text{BH}} |i\rangle_{\text{R}} , \tag{49}$$

where the $|i\rangle_{\text{R}}$ are the eigenstates of a “reference” system. While such a Schmidt decomposition is always possible mathematically, what physics does it correspond to? If there is a physical mechanism that associates states $|i\rangle_{\text{R}}$ *outside* of the event horizon with states $|\psi_i\rangle_{\text{BH}}$ within it, then perhaps information entering a black hole will leave some signature outside it. As it turns out, there is indeed such a physical mechanism, as I now show.

Black hole evaporation is, as mentioned earlier, due to a quantum effect that has no analogue in classical physics: the spontaneous decay of the vacuum. Quite literally, the gravitational field surrounding the black hole polarizes the vacuum so as to create virtual particle-antiparticle states. If one member of such a pair enters the event horizon while the other goes off to infinity, the black hole itself will have to provide the energy to convert the virtual particles to real ones, i.e., to put them on their mass shell. This process reduces the black hole’s mass, and thus, via Eq. (47), its entropy. Within quantum information theory, we write the black hole entropy in terms of the von Neumann entropy

$$S(\rho_{\text{BH}}) = -\text{tr} \rho_{\text{BH}} \log \rho_{\text{BH}} , \quad (50)$$

and the reduction in entropy can be understood in terms of the *removal* of positive energy modes due to the absorption of negative energy modes. As pointed out earlier, such a description only makes sense if we consider *asymptotic* modes. Then, a flux of particles of positive energy at $t \rightarrow \infty$ must correspond to a flux of negative energy of equal magnitude into the black hole. In a sense, it is our detectors in the future that are allowing us to make predictions in the past.

In this picture then, we can describe particle absorption and emission from a black hole with a standard interaction Hamiltonian, so that for an incoming particle in mode k incident on a black hole with wavefunction $|\psi\rangle_{\text{Q}}$ (the index Q will be used to label the black-hole Hilbert space throughout this section) we have the unitary evolution

$$\begin{aligned} U|k\rangle_{\text{M}}|\psi\rangle_{\text{Q}}|0\rangle_{\text{R}} &= \lim_{t \rightarrow \infty} e^{i \int_{-\infty}^{\infty} H_{\text{int}} dt} |k\rangle_{\text{M}}|\psi\rangle_{\text{Q}}|0\rangle_{\text{R}} \\ &= |k\rangle_{\text{M}}|\psi\rangle_{\text{Q}}|0\rangle_{\text{R}} + \alpha_k |0\rangle_{\text{M}}|\psi_k\rangle_{\text{Q}}|0\rangle_{\text{R}} + \beta_k |k\rangle_{\text{M}}|\psi_{-k}\rangle_{\text{Q}}|k\rangle_{\text{R}} . \end{aligned} \quad (51)$$

Here, we defined black hole wavefunctions that have either absorbed or emitted a mode $|k\rangle$ using the “ladder operators” σ_k^+ and σ_k^-

$$|\psi_{\pm k}\rangle = \sigma_k^{\pm} |\psi\rangle , \quad (52)$$

and introduced a Fock space for *stimulated* radiation, with ground state $|0\rangle_{\text{R}}$. The coefficients α_k and β_k are related to the Einstein coefficients for emission and absorption [48], so that $\beta_k = \alpha_k e^{-i\omega_k/T_{\text{H}}}$. Thus, we see that beyond spontaneous emission and absorption, stimulated emission plays a crucial role in black hole dynamics [49, 50, 51, 52]. Indeed, it is the key to the purification of the black hole density matrix. If we ignore the elastic scattering term in Eq. (51) (it does not affect the entanglement), we see that the accretion of n modes onto a pure black hole state

$|\psi\rangle$ gives rise to the entangled wavefunction

$$|\Psi\rangle_{\text{QMR}} = \sum_{i=1}^{2^n} \sqrt{p_i} |\psi_i\rangle_{\text{Q}} |i\rangle_{\text{MR}}, \quad (53)$$

as promised. Information about the identity of the particle modes, i.e., the basis states of M given by the labels attached to each incoming mode $|k_i\rangle_{\text{M}}$, are encoded in the stimulated radiation states R in such a manner that they are perfectly correlated: $I = S(M : R)$. Because M and R share the same basis system, this von Neumann mutual entropy actually reduces to a Shannon information, and we can say that M plays the role of a *preparer* of the quantum states $|k_i\rangle_{\text{R}}$. The problem of understanding the fate of information in black hole evaporation now just reduces to a problem in quantum channel theory (namely the transmission of classical information through an entanglement-assisted channel [14, 15, 53]), because spontaneous emission of particles from state (53) creates a noisy quantum channel for the classical information I .

Because of the initial entanglement between the black hole and the stimulated radiation, the final state after spontaneous emission will be an entangled *pure* state between Q, the joint system MR, and the Hawking radiation. As the states MR are not being measured, we need to trace them out to consider the joint state of black hole and Hawking radiation. In particular, the Hawking radiation will appear completely thermal with temperature T_{H} . And as long as no additional particles accrete, the wavefunction (53) ensures that the entropy of the MR system (note that $S(MR) = S(M) = S(R)$) is *always equal* to the joint entropy of black hole and Hawking radiation. This implies that we ought to be able to reformulate the second law of black hole thermodynamics in a manner very similar to the modification introduced in section 1.4. This second law [45] (see also [54] for a derivation in terms of quantum entanglement) states that the sum of black hole entropy and surrounding matter/radiation (thermodynamical) entropy can never decrease:

$$dS_{\text{tot}} = d(S_{\text{BH}} + S_{\text{therm}}) \geq 0. \quad (54)$$

But in information theory we can write *equalities* because we do not have to ignore correlations, i.e., information entropies. According to the scenario above, we can thus state that

In black hole evaporation, the joint entropy of black hole and Hawking radiation, as well as the joint entropy of black hole, radiation, and the infalling matter distribution, remain a constant.

As a corollary, we note that the only entropy that can decrease in such a process is indeed the quantum entropy of the black hole given the outgoing radiation, as we suspected earlier. A detailed description of black hole formation and evaporation in a quantum information-theoretic setting is beyond the scope of this review, and will appear elsewhere [55].

5.2. Entropies in Curved Space Time

The treatment of black hole dynamics outlined above, while suggestive, did not fully use the formalism of quantum field theory in curved space time. For example, entropies were calculated in terms of quantum mechanical wavefunctions, and in general our information degrees of freedom were particles with particular quantum numbers. But as we saw earlier, a description of detectors in non-inertial frames is inconsistent because whether or not a detector fires depends on its acceleration. Indeed, the particle concept itself is suspect in this context, and instead we should use an approach based entirely on quantum fields and their fluctuations. Such a formalism is preferred also because quantum field theory guarantees that observables interact in a manner compatible with the causal structure of space-time. Thus, in order to consistently define quantum entropies in curved space-time, we must define them within quantum field theory. To close this review, I briefly speculate about such an approach.

The first steps toward such a theory involve defining quantum fields over a manifold separated into an accessible and an inaccessible region. This division will occur along a world-line, and we shall say that the “inside” variables are accessible to me as an observer, while the outside ones are not. Note that the inaccessibility can be due either to causality, or due to an event horizon. Both cases can be treated within the same formalism (and indeed the derivation of the Unruh and Hawking effect are very similar for this reason). States in the inaccessible region have to be averaged over, since states that differ only in the outside region are unresolvable. Let me denote the inside region by R , while the entire state is defined on E . We can now define a set of commuting variables X that can be divided into X_{in} and X_{out} . By taking matrix elements of the density matrix of the entire system

$$\rho = |E\rangle\langle E| \tag{55}$$

with the complete set of variables $(X_{\text{in}}, X_{\text{out}})$, we can construct the inside density matrix (defined on R) as

$$\rho_{\text{in}} = \text{Tr}_{X_{\text{out}}}(\rho_{X_{\text{in}}, X_{\text{out}}}) . \tag{56}$$

This allows me to define the *geometric entropy* [56, 57, 58] of a state on E for an observer restricted to R

$$S_{\text{geom}} = - \text{Tr}(\rho_{\text{in}} \log \rho_{\text{in}}) , \tag{57}$$

where the trace is performed using the inside variables only. For quantum fields with equiprobable modes, we can see this expression as giving the logarithm of the number of states in the inaccessible (i.e., “out”) region that are *consistent* with measurements restricted to the “in” region [58]. Writing down such an expression, however, is just the beginning.

As with most quantities in quantum field theory, the geometric entropy (57) is divergent and needs to be renormalized. Rather than being an inconvenience, this is precisely what we should have expected: after all, we began this review by insisting

that entropies only make sense when discussed in terms of the possible measurements that can be made of the system. This is, of course, precisely the role of renormalization in quantum field theory. Quantum entropies can be renormalized via a number of methods, either using Hawking’s zeta function regularization procedure [59] or by the “replica trick”, writing

$$S_{\text{geom}} = - \left[\frac{d}{dn} \text{Tr}(\rho_{\text{in}}^n) \right]_{n=1} , \quad (58)$$

and then writing $dS(n)$ in terms of the expectation value of the stress tensor. A thorough application of this program should reveal components of the geometric entropy due entirely to the curvature of space-time, components that vanish in the flat-space limit. Furthermore, the geometric entropy can be used to write equations relating the entropy of the inside and the outside space-time regions, as

$$S(E) = S(\rho_{\text{in,out}}) = S(\rho_{\text{in}}) + S(\rho_{\text{out}}|\rho_{\text{in}}) . \quad (59)$$

A thorough application of this program, with appropriate renormalization of both ultraviolet and infrared divergencies, should finally yield an origin of the mysterious Bekenstein entropy fully in accord with information theory. First steps in this direction have indeed been taken very recently by Terno [60], who studied the transformation properties of geometric entropy, and found that S_{geom} is not a scalar under Lorentz transformation, while the Bekenstein-Hawking entropy is. Clearly, we are still not close to a full quantum field-theoretic description of information in arbitrary space-times, but it would appear that the necessary tools are available.

6. Summary

Entropy and information are statistical quantities describing an observer’s capability to predict the outcome of the measurement of a physical system. Once couched in those terms, information theory can be examined in all physically relevant limits, such as quantum, relativistic, and gravitational. Information theory is a non-equilibrium theory of statistical processes, and should be used under circumstances (such as measurement, non-equilibrium phase transitions, etc.) where an equilibrium approach is inappropriate. Because an observer’s capability to make predictions (quantified by entropy) is not a characteristic of the object the predictions apply to, it does not have to follow the same physical laws (such as reversibility) as that befitting the objects. Thus, the arrow of time implied by the loss of information under standard time-evolution is even less mysterious than the second law of thermodynamics, which is just a consequence of the former.

In time, a fully relativistic theory of quantum information, defined for quantum fields on curved space-time, should allow us to tackle a number of problems in cosmology and other areas that have as yet resisted a consistent treatment. These developments, I have no doubt, would have made Shannon proud.

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