

The distance between classical and quantum systems

Deanna Abernethy and John R. Klauder*

Department of Mathematics

University of Florida

Gainesville, FL 32611

Electronic mail: daber82@ufl.edu and klauder@phys.ufl.edu

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In a recent paper, a “distance” function, \mathcal{D} , was defined which measures the distance between pure classical and quantum systems. In this work, we present a new definition of a “distance”, D , which measures the distance between either pure or impure classical and quantum states. We also compare the new distance formula with the previous formula, when the latter is applicable. To illustrate these distances, we have used 2×2 matrix examples and 2-dimensional vectors for simplicity and clarity. Several specific examples are calculated.

1. INTRODUCTION

It is a property of nature that classical and quantum mechanics have quite different characteristics. Classical mechanics deals with ordinary second order differential equations leading to particle trajectories, while quantum mechanics involves the study of partial differential equations for functions such

*Also Physics Department.

as the wave function. Despite this basic difference we wish to find a common formulation of both classical and quantum mechanics. We plan to use that common formulation to define a distance between classical and quantum systems.

Earlier work in this field, Ref. [1], focused solely on pure states, which were represented by vectors. In units where Planck's constant $\hbar = 1$, this distance formula was given by

$$\mathcal{D} = \min_{\alpha} \int \|i\dot{\psi}(t) - [\dot{\alpha} + \mathcal{H}]\psi(t)\| dt. \quad (1)$$

Here, we let $L \equiv i\dot{\psi}(t) - [\dot{\alpha} + \mathcal{H}]\psi(t)$, where $\psi(t)$ is a Hilbert space vector and $\|L\|$ denotes the vector norm. If ψ satisfies Schrödinger's equation, then $\mathcal{D} = 0$. However, if ψ has a different temporal behavior, then generally $\mathcal{D} > 0$.

This formula for \mathcal{D} corresponds to so-called *pure states*, however there are other quantum states, namely *impure states*, that cannot be covered by this original definition.

The purpose of the present work is to study some simple, two-dimensional states and examine several examples of pure and impure state situations. We also verify that the results for the pure states with the new formulation agree with the results obtained by means of the previous formulation. While these examples are only for two-dimensional vectors and 2×2 matrices, they illustrate the general principles involved.

Our task is to find the distance between classical and quantum systems. At first glance, this seems comparable to finding the distance between two dissimilar objects, such as a rock and a leaf. Thus, we begin by trying to find a way to describe classical and quantum systems in a similar manner in order that the two theories can be compared.

The connection of classical and quantum mechanics used by Klauder in forming his expression for the distance has been well discussed in several papers; see Ref. [2]. The essence of this connection assumes the Schrödinger equation, but *restricts* the time dependence of the wave function so it generally is *not* a solution to this equation. When possible, this restricted time dependence reflects the time dependence associated with classical dynamics; the appropriate time dependence can be obtained by restricting ψ to evolve solely within coherent states for which the parameters evolve as classical solutions. However, for present purposes, it suffices to choose a time dependence

such that $L \neq 0$. The deviation from L being an exact solution is then turned into an expression for the distance.

Klauder has proposed an alternative formula that should cover both the pure and the impure states. This new formula uses *density matrices* ρ and defines a distance function D by

$$D = \int \left\| i\dot{\rho}(t) - [\mathcal{H}, \rho(t)] \right\| dt. \quad (2)$$

For ease of notation, we let $A \equiv i\dot{\rho}(t) - [\mathcal{H}, \rho(t)]$, where, in our examples, $\rho(t)$ is a Hermitian 2×2 matrix, $\dot{\rho}(t)$ is the time derivative of $\rho(t)$, and \mathcal{H} is the Hamiltonian 2×2 matrix. When A is a matrix, as is the case here, the symbol $\|A\|$ denotes the operator norm; for a discussion of operators and norms, see, e.g., Ref. [3]. Although the operator norm of A is often moderately difficult to evaluate, it is rather easy to calculate for the 2×2 matrices A , as we now demonstrate. It is because of this ease of calculation that we focus our attention on matrices of such a small dimension.

Let us note that ρ and \mathcal{H} are both Hermitian and that ρ is normalized so that $\text{Tr}(\rho) = 1$. It then follows that A is anti-Hermitian and $\text{Tr}(A) = 0$. Thus, A necessarily has the form

$$A = U^\dagger \begin{pmatrix} d & 0 \\ 0 & -d \end{pmatrix} U, \quad (3)$$

where U is a unitary 2×2 matrix.

For our calculations we need to evaluate the usual operator norm of A . The norm of A is given by its largest eigenvalue and therefore is $|d|$. In order to determine this value, it is useful for our examples to observe that

$$A^\dagger A = U^\dagger \begin{pmatrix} |d|^2 & 0 \\ 0 & |d|^2 \end{pmatrix} U. \quad (4)$$

Therefore

$$\text{Tr}(A^\dagger A) = 2 |d|^2. \quad (5)$$

Stated otherwise, it follows that

$$\|A\| = \sqrt{\frac{1}{2} \text{Tr}(A^\dagger A)}. \quad (6)$$

It is important to note that this expression holds only when A is given by a 2×2 matrix. For higher dimensional examples this formula is generally inappropriate.

2. GENERAL STRATEGY

As mentioned, the previous work on this subject dealt only with pure states. We need to verify that the new distance formula accurately measures the distance for pure states before we can move on to impure states. We can accomplish this by comparing a pure state example using our distance formula to the distance found in previous work. For convenience, we may find these examples as part of a more general analysis.

We first start with the general form of a 2×2 matrix, which will lead to a general ρ . This will allow us to apply the general form to several, more specific pure and impure cases. Let

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (7)$$

However, ρ is defined as a positive, Hermitian matrix with $\text{Tr}(\rho) = 1$. This mandates that a be real, $c = b^*$, the complex conjugate of b , and $d = 1 - a$, as well as $0 \leq a \leq 1$. Thus,

$$\rho = \begin{pmatrix} a & b \\ b^* & 1 - a \end{pmatrix}. \quad (8)$$

To differentiate between pure and impure states, we can appeal to the value of $\text{Tr}(\rho^2)$. A pure state has $\text{Tr}(\rho^2) = 1$, while an impure state has $\text{Tr}(\rho^2) < 1$. Since

$$\rho^2 = \begin{pmatrix} a^2 + |b|^2 & ab + b(1 - a) \\ ab^* + b^*(1 - a) & |b|^2 + (1 - a)^2 \end{pmatrix}, \quad (9)$$

we find that

$$\begin{aligned} \text{Tr}(\rho^2) &= a^2 + 2|b|^2 + (1 - a)^2 \\ &= 2a^2 + 2|b|^2 + 1 - 2a. \end{aligned} \quad (10)$$

To develop pure cases of ρ , we let $2a^2 + 2|b|^2 + 1 - 2a = 1$, i.e.,

$$|b| = \sqrt{a - a^2}. \quad (11)$$

Similarly, for impure cases, we set $|b| < \sqrt{a - a^2}$. Thus, for any choice of a , we can find a suitable value for b . We next apply this formula to create several examples.

General example 1:

We will be complete in our calculations for this example, but certain steps will be omitted in further calculations for simplicity.

Let us begin by choosing $a = \cos^2(t)$, then

$$\begin{aligned} |b| &= [\cos^2(t) - \cos^4(t)]^{1/2} \\ &= \frac{1}{2} |\sin(2t)|. \end{aligned} \quad (12)$$

Therefore, we may set $b = \frac{1}{2} \sin(2t)$ for pure cases, and $|b| < \frac{1}{2} |\sin(2t)|$ for impure cases. For example, if we choose $b = \frac{1}{4} \sin(2t)$, we have satisfied this condition for an impure case, for general t values.

Notice that the two cases differ by just a coefficient β . If we keep the same value for a , but allow $b = \beta \sin(2t)$ with $|\beta| \leq \frac{1}{2}$, then we can evaluate both pure ($\beta = \frac{1}{2}$) and selected impure cases ($\beta = \frac{1}{4}$ and $\beta = 0$) from this general matrix. Thus we adopt ρ and \mathcal{H} for our first set of examples as

$$\rho = \begin{pmatrix} \cos^2(t) & \beta \sin(2t) \\ \beta \sin(2t) & \sin^2(t) \end{pmatrix} \quad (13)$$

and

$$\mathcal{H} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (14)$$

To evaluate the distance D (defined in Eq. 2), we begin by finding $i\dot{\rho}(t)$ as given by

$$i\dot{\rho}(t) = i \begin{pmatrix} -2 \cos(t) \sin(t) & 2\beta \cos(2t) \\ 2\beta \cos(2t) & 2 \sin(t) \cos(t) \end{pmatrix} \quad (15)$$

and $[\mathcal{H}, \rho(t)]$ as

$$[\mathcal{H}, \rho(t)] = \lambda \begin{pmatrix} 0 & 2\beta \sin(2t) \\ -2\beta \sin(2t) & 0 \end{pmatrix}. \quad (16)$$

This allows us to calculate the value of A as

$$A = \begin{pmatrix} -2i \cos(t) \sin(t) & 2i \beta \cos(2t) - 2\lambda \beta \sin(2t) \\ 2i \beta \cos(2t) + 2\lambda \beta \sin(2t) & 2i \sin(t) \cos(t) \end{pmatrix}. \quad (17)$$

As follows from Eq. (6), $\|A\|$ is given by

$$\begin{aligned}\|A\| &= \left[\frac{1}{2} \text{Tr}(A^\dagger A)\right]^{1/2} \\ &= \left[4 \cos^2(t) \sin^2(t) + 4\beta^2 \cos^2(2t) + 4\lambda^2 \beta^2 \sin^2(2t)\right]^{1/2}.\end{aligned}\quad (18)$$

This gives us a guideline for computing the distance for this ρ and this \mathcal{H} .

Now we can confirm our use of the distance formula D by comparing our distance for a pure state to the distance using the method previously described in Ref. [1].

Example 1a (pure case, $\beta = \frac{1}{2}$):

Using this general sample case, we begin with a pure case of ρ by choosing $a = \cos^2(t)$ and $b = \frac{1}{2} \sin(2t) = b^*$. Thus

$$\rho = \begin{pmatrix} \cos^2(t) & \frac{1}{2} \sin(2t) \\ \frac{1}{2} \sin(2t) & \sin^2(t) \end{pmatrix}\quad (19)$$

and

$$\mathcal{H} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\quad (20)$$

If we apply $\beta = \frac{1}{2}$ to Eq. (18), we get

$$\begin{aligned}\|A\| &= \left[4 \cos^2(t) \sin^2(t) + \cos^2(2t) + \lambda^2 \sin^2(2t)\right]^{1/2} \\ &= \left[1 + 4\lambda^2 \cos^2(t) \sin^2(t)\right]^{1/2}.\end{aligned}\quad (21)$$

Figure 1 illustrates the equation for $\|A\|$ in this example for $\lambda = 1$.

Comparison of the Two Distance Functions

Before proceeding further, let us verify that the new distance function D agrees with the distance \mathcal{D} using Eq. (1). We begin by identifying $\psi(t)$ as follows

$$\psi(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.\quad (22)$$

by It follows that the integrand in Eq. (1) involves

$$L = i\dot{\psi}(t) - [\dot{\alpha} + \mathcal{H}]\psi(t) = \begin{pmatrix} -i \sin(t) - (\dot{\alpha} + \lambda) \cos(t) \\ i \cos(t) - (\dot{\alpha} - \lambda) \sin(t) \end{pmatrix},\quad (23)$$

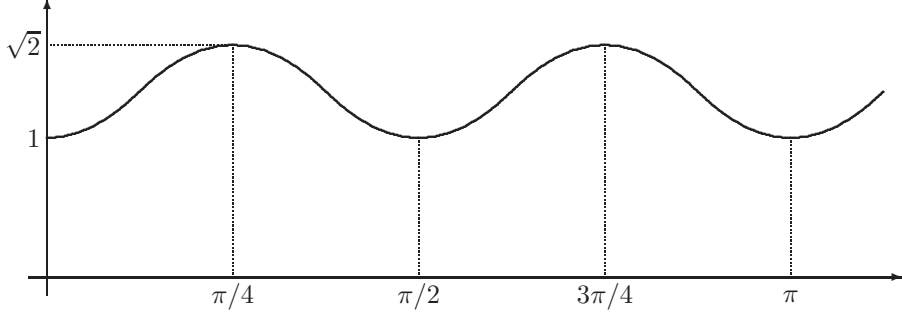


Figure 1: Graph of $\|A\|$ vs. t for Example 1a

and the vector norm of this expression is

$$\begin{aligned}
& \|i\dot{\psi}(t) - [\dot{\alpha} + \mathcal{H}]\psi(t)\| \\
&= [\sin^2(t) + (\dot{\alpha} + \lambda)^2 \cos^2(t) + \cos^2(t) + (\dot{\alpha} - \lambda)^2 \sin^2(t)]^{1/2} \\
&= [1 + \dot{\alpha}^2 + \lambda^2 + 2\dot{\alpha}\lambda \cos(2t)]^{1/2}. \tag{24}
\end{aligned}$$

We can minimize $\|L\|$ over $\dot{\alpha}$ by choosing

$$\dot{\alpha} = -\lambda \cos(2t). \tag{25}$$

Substituting this value for $\dot{\alpha}$ into Eq. (24), we obtain

$$\begin{aligned}
\|L\| &= \{1 + [-\lambda \cos(2t)]^2 + \lambda^2 + 2[-\lambda \cos(2t)]\lambda \cos(2t)\}^{1/2} \\
&= [1 + 4\lambda^2 \cos^2(t) \sin^2(t)]^{1/2}, \tag{26}
\end{aligned}$$

a result that agrees with Eq. (21). Thus our definition of the new distance function D is appropriate, and we will continue to apply this formula to further examples.

3. ADDITIONAL SPECIFIC CASES

Example 1b (impure case, $\beta = \frac{1}{4}$):

Resuming our calculations, we will evaluate an impure case with $a = \cos^2(t)$ and $b = \frac{1}{4} \sin(2t)$. This gives

$$\rho = \begin{pmatrix} \cos^2(t) & \frac{1}{4} \sin(2t) \\ \frac{1}{4} \sin(2t) & \sin^2(t) \end{pmatrix} \tag{27}$$

and

$$\mathcal{H} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (28)$$

Again using equation (18), we obtain a value for $\|A\|$ as

$$\begin{aligned} \|A\| &= [4 \cos^2(t) \sin^2(t) + \frac{1}{4} \cos^2(2t) + \frac{1}{4} \lambda^2 \sin^2(2t)]^{1/2} \\ &= [\frac{1}{4} + (3 + \lambda^2) \cos^2(t) \sin^2(t)]^{1/2}. \end{aligned} \quad (29)$$

Example 1c (impure case, $\beta = 0$):

We will now choose $b = 0$, which also gives an impure case. We have

$$\rho = \begin{pmatrix} \cos^2(t) & 0 \\ 0 & \sin^2(t) \end{pmatrix} \quad (30)$$

and \mathcal{H} is still given by

$$\mathcal{H} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (31)$$

We evaluate $\|A\|$ as

$$\begin{aligned} \|A\| &= [4 \cos^2(t) \sin^2(t)]^{1/2} \\ &= |\sin(2t)|. \end{aligned} \quad (32)$$

In this case, when we evaluate the distance, say for $T = 4\pi$, we are led to

$$\begin{aligned} D &= \int_0^{4\pi} |\sin(2t)| dt \\ &= 8 \int_0^{\pi/2} \sin(2t) dt \\ &= 8. \end{aligned} \quad (33)$$

General example 2:

With $a = \cos^2(t)$, we again have $b = \beta \sin(2t) = b^*$ for $|\beta| \leq \frac{1}{2}$ to give us

$$\rho(t) = \begin{pmatrix} \cos^2(t) & \beta \sin(2t) \\ \beta \sin(2t) & \sin^2(t) \end{pmatrix}, \quad (34)$$

but now we choose a new expression for \mathcal{H} given by

$$\mathcal{H} = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (35)$$

It follows that

$$\begin{aligned} A &= i\dot{\rho}(t) - [\rho(t), \mathcal{H}] \\ &= \begin{pmatrix} -i \sin(2t) & 2i\beta \cos(2t) + \lambda \cos(2t) \\ 2i\beta \cos(2t) - \lambda \cos(2t) & i \sin(2t) \end{pmatrix}. \end{aligned} \quad (36)$$

Hence

$$A^\dagger A = \begin{pmatrix} j_1 & j_2 \\ j_3 & j_4 \end{pmatrix}, \quad (37)$$

where

$$j_1 = \sin^2(t) + [2i\beta \cos(2t) - \lambda \cos(2t)] [-2i\beta \cos(2t) - \lambda \cos(2t)] \quad (38)$$

$$\begin{aligned} j_2 &= [i \sin(2t)] [2i\beta \cos(2t) + \lambda \cos(2t)] \\ &\quad + [i \sin(2t)] [-2i\beta \cos(2t) - \lambda \cos(2t)] \end{aligned} \quad (39)$$

$$\begin{aligned} j_3 &= [-i \sin(2t)] [-2i\beta \cos(2t) + \lambda \cos(2t)] \\ &\quad - [i \sin(2t)] [2i\beta \cos(2t) - \lambda \cos(2t)] \end{aligned} \quad (40)$$

$$j_4 = [-2i\beta \cos(2t) + \lambda \cos(2t)] [2i\beta \cos(2t) - \lambda \cos(2t)] + \sin^2(2t), \quad (41)$$

which leads to

$$\begin{aligned} \|A\| &= \sqrt{\frac{1}{2} \text{Tr}(A^\dagger A)} \\ &= \frac{1}{\sqrt{2}} [2 \sin^2(2t) + 4\beta^2 \cos^2(2t) + \lambda^2 \cos^2(2t) \\ &\quad + 4\beta^2 \cos^2(2t) + \lambda^2 \cos^2(2t)]^{1/2} \\ &= [\sin^2(2t) + \cos^2(2t)(4\beta^2 + \lambda^2)]^{1/2}. \end{aligned} \quad (42)$$

Just as in Example 1, we can choose values of β to create both pure and impure cases.

Example 2a (pure case, $\beta = \frac{1}{2}$):

Our pure choice of $\beta = \frac{1}{2}$ gives $b = \frac{1}{2} \sin(2t) = b^*$ and $\rho(t)$ as follows

$$\rho(t) = \begin{pmatrix} \cos^2(t) & \frac{1}{2} \sin(2t) \\ \frac{1}{2} \sin(2t) & \sin^2(t) \end{pmatrix} \quad (43)$$

and

$$\mathcal{H} = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (44)$$

By following Eq. (42), we get the value for $\|A\|$ when $\beta = \frac{1}{2}$ given by

$$\|A\| = [\sin^2(2t) + \cos^2(2t)(1 + \lambda^2)]^{1/2}. \quad (45)$$

Example 2b (impure case, $\beta = \frac{1}{4}$):

In this case

$$\rho(t) = \begin{pmatrix} \cos^2(t) & \frac{1}{4} \sin(2t) \\ \frac{1}{4} \sin(2t) & \sin^2(t) \end{pmatrix} \quad (46)$$

and

$$\mathcal{H} = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (47)$$

With the aid of Eq. (42), we get

$$\|A\| = [\sin^2(2t) + \cos^2(2t)(\frac{1}{4} + \lambda^2)]^{1/2}. \quad (48)$$

Example 2c (impure case, $\beta = 0$):

Our final impure case is $\beta = 0$. This gives us

$$\rho(t) = \begin{pmatrix} \cos^2(t) & 0 \\ 0 & \sin^2(t) \end{pmatrix} \quad (49)$$

and

$$\mathcal{H} = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (50)$$

Following Eq. (42) once more, we get

$$\|A\| = [\sin^2(2t) + \lambda^2 \cos^2(2t)]^{1/2}. \quad (51)$$

We have investigated several impure and pure cases with two different choices of \mathcal{H} . Next, we will explore new general examples of $\rho(t)$ with the same two choices of \mathcal{H} .

General example 3:

We will now obtain a new general $\rho(t)$ by choosing $a = 1/(1+t^2)$. Then the limiting b is found in the following manner

$$\begin{aligned} |b| &= \sqrt{a - a^2} \\ &= \left\{ \frac{1}{1+t^2} - \left[\frac{1}{1+t^2} \right]^2 \right\}^{1/2} \\ &= \frac{|t|}{1+t^2}. \end{aligned} \tag{52}$$

Thus, for a pure case, we can pick $b = t/(1+t^2)$, and $|b| < |t|/(1+t^2)$ for impure cases. As was previously the case we let the pure and impure choices of b differ only by a real constant β . We can evaluate a general case of $b = \beta t/(1+t^2)$ with $|\beta| \leq 1$ to enable us to consider several specific cases. Thus we are led to consider

$$\rho(t) = \frac{1}{1+t^2} \begin{pmatrix} 1 & \beta t \\ \beta t & t^2 \end{pmatrix}, \tag{53}$$

while initially we also use our original \mathcal{H} , namely,

$$\mathcal{H} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{54}$$

We find $A = i\dot{\rho}(t) - [\mathcal{H}, \rho(t)]$ is given by

$$A = \frac{1}{[1+t^2]^2} \begin{pmatrix} -2it & i\beta(1-t^2) - 2\lambda\beta t(1+t^2) \\ i\beta(1-t^2) + 2\lambda\beta t(1+t^2) & 2it \end{pmatrix}, \tag{55}$$

while our calculation of $\|A\| = \sqrt{\frac{1}{2} \text{Tr}(A^\dagger A)}$ gives

$$\|A\| (1 + \frac{\{4t^2 + \beta^2(1-t^2)^2 + 4\lambda^2\beta^2 t^2(1+t^2)^2\}^{1/2}}{(1+t^2)^2}). \tag{56}$$

Example 3a (pure case, $\beta = 1$):

Let us choose $\beta = 1$, which means that

$$\rho(t) = \frac{1}{1+t^2} \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix} \tag{57}$$

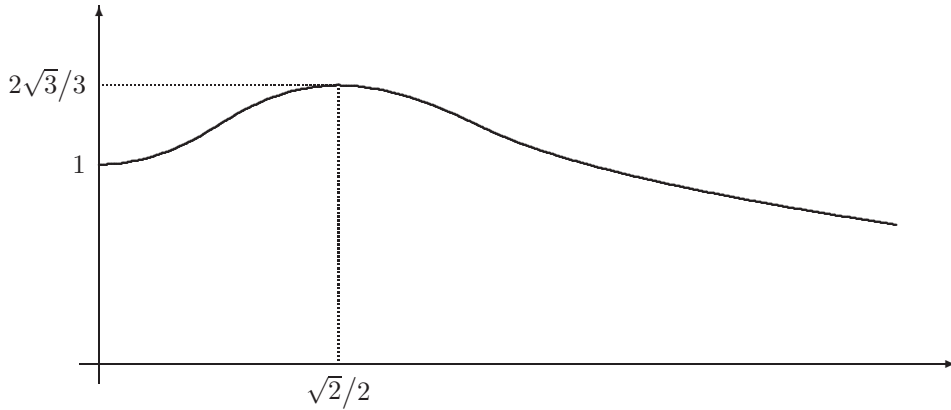


Figure 2: Graph of $\|A\|$ vs. t for Example 3a

with the same \mathcal{H} , i.e.,

$$\mathcal{H} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (58)$$

Applying Eq. (56), we find that

$$\|A\| = \frac{[1 + 4\lambda^2 t^2]^{1/2}}{1 + t^2}. \quad (59)$$

Figure 2 is an illustration of the equation for $\|A\|$ in this example for $\lambda = 1$.

4. ANOTHER COMPARISON OF THE TWO DISTANCE FUNCTIONS

Let us again verify that our definition of D is suitable by evaluating \mathcal{D} for the information in Example 3a and comparing it to Eq. (59).

We begin by identifying

$$\psi(t) = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad (60)$$

which leads directly to

$$L = i\dot{\psi}(t) - [\dot{\alpha} + \mathcal{H}]\psi(t) = \frac{1}{(1+t^2)^{3/2}} \begin{pmatrix} -it - (\dot{\alpha} + \lambda)(1+t^2) \\ i - t(\dot{\alpha} - \lambda)(1+t^2) \end{pmatrix}. \quad (61)$$

Finally, $\|L\| = \|\dot{\psi}(t) - [\dot{\alpha} + \mathcal{H}]\psi(t)\|$ is

$$\|L\| = \frac{[\dot{\alpha}^2(t^2 + 1)^2 - 2\lambda\dot{\alpha}(t^2 - 1)(t^2 + 1) + \lambda^2(t^2 + 1)^2 + 1]^{1/2}}{1 + t^2}. \quad (62)$$

We next minimize $\|L\|$ by choosing

$$\dot{\alpha} = \lambda \left(\frac{t^2 - 1}{t^2 + 1} \right). \quad (63)$$

When we substitute this choice back into Eq. (62) we find that

$$\frac{[\dot{\alpha}^2(t^2 + 1)^2 - 2\lambda\dot{\alpha}(t^2 - 1)(t^2 + 1) + \lambda^2(t^2 + 1)^2 + 1]^{1/2}}{1 + t^2} = \frac{[1 + 4\lambda^2 t^2]^{1/2}}{1 + t^2}, \quad (64)$$

a result that coincides with Eq. (59).

5. CONTINUATION OF SPECIFIC CASES

Example 3b (impure case, $\beta = \frac{1}{2}$):

By referring to Eq. (56), we can also evaluate $\|A\|$ when $\beta = \frac{1}{2}$ as

$$\|A\| = \frac{[(t^2 + 1)^2(\frac{1}{4} + \lambda^2 t^2) + 3t^2]^{1/2}}{(1 + t^2)^2}. \quad (65)$$

Example 3c (impure case, $\beta = 0$):

Similarly, we can substitute an impure case of $\beta = 0$ into Eq. (56) to find

$$\|A\| = \frac{2|t|}{(1 + t^2)^2}. \quad (66)$$

General example 4:

We will use the same value $a = 1/(1 + t^2)$ with $b = (\beta t)/(1 + t^2)$ for $|\beta| \leq 1$. As before, this gives

$$\rho(t) = \frac{1}{1 + t^2} \begin{pmatrix} 1 & \beta t \\ \beta t & t^2 \end{pmatrix}, \quad (67)$$

but now we use a new expression of \mathcal{H} given by

$$\mathcal{H} = \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (68)$$

It follows that $A = i\dot{\rho}(t) - [\mathcal{H}, \rho(t)]$ is given by

$$A = \frac{1}{(1+t^2)^2} \begin{pmatrix} -2it & i\beta(1-t^2) - \lambda(t^4-1) \\ i\beta(1-t^2) + \lambda(t^4-1) & 2it \end{pmatrix}, \quad (69)$$

and so $\|A\| = \sqrt{\frac{1}{2} \text{Tr}(A^\dagger A)}$ is

$$\|A\| t^2 \quad (70)$$

$$= \frac{[4t^2 + \beta^2(1-t^2)^2 + \lambda^2(t^4-1)^2]^{1/2}}{(1+t^2)^2}. \quad (71)$$

Example 4a (pure case, $\beta = 1$):

We first choose $\beta = 1$ to create a pure state case, namely

$$\rho(t) = \frac{1}{1+t^2} \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix}. \quad (72)$$

With $\beta = 1$, it follows from Eq. (71) that

$$\|A\| = \frac{[1 + \lambda^2(t^2-1)^2]^{1/2}}{1+t^2}. \quad (73)$$

Example 4b (impure case, $\beta = \frac{1}{2}$):

We can investigate an impure case if we choose $\beta = \frac{1}{2}$ and substitute it into Eq. (71) as given by

$$\|A\| = \frac{[\frac{1}{4}(t^2+1)^2 + 3t^2 + \lambda^2(t^4-1)^2]^{1/2}}{(1+t^2)^2}. \quad (74)$$

Example 4c (impure case, $\beta = 0$):

Finally, when we choose an impure case with $\beta = 0$, we find that

$$\|A\| = \frac{[4t^2 + \lambda^2(t^4-1)^2]^{1/2}}{(1+t^2)^2}. \quad (75)$$

6. SUMMARY

In this article we have introduced a “distance function”, defined by Eq. (2), which measures the distance between classical and quantum states, either pure or impure. We have verified that the previous method, defined in Eq. (1), and the new technique for calculating distance are equivalent in the case of pure states. We then found the distance for several combinations of density matrices, $\rho(t)$, and Hamiltonians, \mathcal{H} .

To clearly illustrate the basic ideas, we have limited our interest to 2×2 density matrices. Additional studies for larger density matrices, either finite or infinite dimensional, would also be of interest. This is especially true for examples based on coherent states which are appropriate to discuss traditional classical situations; see Ref. [1] for such an analysis for the pure state case.

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