PH 205 SET 7

Due Tues., Nov 13, 1990

Maximum recorded score = 70 points

1) In a scattering experiment, the differential cross-section is observed to be

\[ \frac{d\sigma}{d\omega \theta} = \frac{\pi a^2}{2} (1 + \varepsilon \cot \theta) \]

\( \varepsilon \) small

Supposing the scattering is elastic scattering off a hard object, what is the shape?

If \( \varepsilon = 0 \) it would be a sphere (Prob 8, Set 6)

The object is almost a sphere - say an ellipsoid

\[ \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1 \]

Find \( \frac{d\sigma}{d\omega \theta} \) for scattering off an ellipsoid

with arbitrary axes \( A \) and \( B \). What are \( A \) and \( B \) corresponding to the cross-section stated above, supposing \( \varepsilon \) is small?

2) A digression into optics. [A book: "Rainbows, Haloes & Glories" by Greener.]

a) Rainbow scattering. Consider the scattering of light off a water drop. When light hits a boundary between air and water, some light is transmitted and some is reflected. So many outgoing light rays are possible.

The 1st 4 outgoing rays are shown in the sketch.

Case 1: Reflection \( \Rightarrow \) Hard Scattering \( \Rightarrow \) Isotropic

Case 2 = Prob 10, Set 6

We are interested in cases 3 and 4 as they lead to rainbows!

Since \( b = a \sin \alpha \) as shown

\[ \frac{d\sigma}{d\omega \theta} = \frac{b \frac{d\sigma}{d\omega \theta}}{a \sin \theta} = \frac{\pi a^2}{2} \frac{\sin \alpha}{\sin \theta} \frac{d\alpha}{d\theta} \]

If \( \frac{d\theta}{d\alpha} = 0 \), then \( \frac{d\sigma}{d\omega \theta} \to \infty \).
That is, if many different $\alpha$'s (and hence $\beta$'s) lead to the same $\Theta$, the scattered light will get very bright -- this is the rainbow effect!

Let $m = \#$ of internal reflections before the ray emerges. Calculate \( \Theta = f(\alpha, \beta, m) \) from geometry. Use Snell's law to relate $\alpha$ and $\beta$ and the index of refraction $n$.

Show that \( \frac{d\Theta}{d\alpha} = 0 \) when \( \sin^2 \alpha = \frac{(M+1)^2 - n^2}{(M+1)^2 - 1} \)

For water $n = 4/3$. Evaluate $\alpha, \beta, \Theta$ for the first two rainbows, $m = 1 \& 2$ ($\Theta = 138^\circ, 129^\circ$)

If you are watching the rainbow, what is the angle between the light you see and the direction to the sun?

The index of refraction, $n$, varies with the wavelength of light. Long $\lambda \Rightarrow$ small $n$. What is the order of the colors in the 1st & 2nd rainbows?

The explanation of the rainbow is attributed to Descartes.

b) Glories (Strictly cultural, there is no problem assigned)

If you look at the shadow of the airplane on a cloud when you are flying, you will see a halo or 'glory' of light immediately outside the shadow. That is, there is an enhancement of scattering at 180° off water drops. This was first observed in 1735 by a Spanish mountain climber in the Andes. But a good explanation seems to have been given only in 1959.

\[ \text{This is the limit of geometrical optics for } n = 4/3 \]

Apparently some light gets trapped in a thin layer at the surface of the drop and is carried thru a few degrees of arc before refracting into the drop. The light which is carried by $74^\circ$ on the way in & also on the way out is then scattered by 180° and causes the glory.
A car is driving along a washboard road such that the axle undergoes a forced vertical oscillation:

\[ x = A \cos \omega t + x_0 \]

Mass \( M \) of the car is supported above the axle via a shock absorber of rest length \( l \) and spring constant \( k \). The damping of the shock absorber is proportional to the rate of change of its length, i.e.,

\[ F_{\text{damping}} = -b \left( \dot{x} - \dot{x}_0 \right) \]

where \( x = \) height of top of the shock absorber above the mean height of the road.

Write down and solve the differential equation for the vertical motion of mass \( M \). Show that the average height is \( \bar{x} = x_0 + l - \frac{g}{\omega_0^2} \) and the amplitude of the oscillation is

\[ A = \sqrt{\frac{\omega_0^4 + 4\beta^2\omega_0^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \]

where \( \omega_0^2 = \frac{k}{M} \) and \( 2\beta = \frac{b}{M} \).

Suppose the shock absorber is critically damped. At what frequency is the oscillation amplitude a maximum, and what is the maximum amplitude? \( \text{Ans:} \) Amplitude max = \( \frac{2\sqrt{3}}{3} A \).

4. a) Give the Fourier series expansion of the sawtooth wave function

\[ F(t) = \frac{F_0}{T} \left( -\frac{T}{2} < t < \frac{T}{2} \right) \]

\[ w = \frac{2\pi}{T} \]

\[ \text{Ans:} \quad F(t) = \frac{F_0}{T} \left( \sin \omega t - \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t - \ldots \right) \]

b) Give the Fourier series expansion of the half-wave function

\[ F(t) = \begin{cases} \sin \omega t & 0 < t < \frac{T}{2} \\ 0 & \frac{T}{2} < t < \frac{T}{2} \end{cases} \]

\[ \text{Ans:} \quad F(t) = \frac{1}{\pi} + \frac{1}{2} \sin \omega t - \frac{2}{3\pi} \cos 2\omega t - \frac{2}{15\pi} \cos 4\omega t - \ldots \]

Which of a) or b) converges faster?
5 a) A damped oscillator is driven by a step function force

\[ F(t) = \begin{cases} 
0 & t < 0 \\
F_0 & t \geq 0 
\end{cases} \]

Use Green's method to calculate the motion \( x(t) \)

\[ x(t) = \frac{F_0}{M\omega_0^2} \left(1 - e^{-\beta t} \cos \omega_1 t - \frac{\beta}{\omega_1} \sin \omega_1 t\right) \quad t > 0 \]

Sketch this for \( \beta > 0 \), and \( \beta = 2\omega_0 \).

Note that the oscillation makes a large overshoot of the equilibrium position \( \frac{F_0}{M\omega_0^2} \). What is the time at which the maximum of the first overshoot, and what is \( x \) then?

\[ x = \frac{F_0}{M\omega_0^2} \left(1 + e^{-\beta t/\omega_1}\right) \]

5 b) The same oscillator is now subject to a finite impulse

\[ F(t) = \begin{cases} 
0 & t < 0 \\
F_0 & 0 \leq t < T \\
0 & t \geq T 
\end{cases} \]

Now what is \( x(t) \)?

Suppose the damping is strong enough that the initial oscillations have died out before the force is turned off, i.e., \( e^{-\beta t} \to 0 \). Sketch \( x(t) \) in this case.

6 a) A damped oscillator is subject to the driving force \( F(t) = F_0 e^{-\alpha t} \). Solve for the 'steady' motion \( x(t) \) by making a suitable guess as to the form of \( x(t) \).

6 b) Now suppose \( F(t) = \begin{cases} 
0 & t < 0 \\
F_0 e^{-\alpha t} & t \geq 0 
\end{cases} \)

Use Green's method to solve for the transient response. (Which should also include the 'steady' motion of part a)!

\[ x = \frac{F_0}{M\omega_0^2 + \omega_0^2 - 2\alpha \beta} \left\{ e^{-\alpha t} + e^{-\beta t} \left(\frac{\alpha - \beta}{\omega_1} \sin \omega_1 t - \omega_0 \omega_1 t\right)\right\} \]

Sketch this for the case \( \alpha = \beta \).
PH 205 SET 7

[In my opinion you should work as many of the last 4 problems as you can.]

7) A hoop of mass M and radius R is attached to a massless rod of length L to form a pendulum. The hoop pivots freely about the connection to the rod. All the motion is in a vertical plane.

Find the frequencies of the normal modes.

Hint: Make the small-angle approximation before deriving the equations of motion—but remember you must keep terms to some order in T and V if you use Lagrange’s method.

As a special case, setting R = 0, I got \( \omega = \sqrt{\frac{g}{L}} (4 \pm 2\sqrt{2}) \)

8) An equilateral triangle of mass M (a thin plate) is suspended from 3 springs at the 3 corners. The equilibrium position of the plate is horizontal (all 3 corners at the same vertical height).

All 3 springs have the same force constant \( k \), and the same rest length.

What are the frequencies of the normal modes?

(Ignore rotation about a vertical axis, and consider only motion in which the c.m. moves vertically—no pendulum motion)

We are left with 3 degrees of freedom \( \Rightarrow \) 3 modes. You may wish to write down the general equation of motion in 3 coordinates. However, it is sufficient to guess the form of the motion of each of the 3 modes. Then derive an oscillatory equation for one motion at a time. Each mode should be described by only 1 coordinate.

Sketch the 3 modes, and show that the frequencies are

\[ \omega_1 = \sqrt{\frac{3k}{M}}, \quad \omega_2 = \omega_3 = 2\omega_1 \]

The fact that \( \omega_2 = \omega_3 \) means you could have made other definitions of the motion of these 2 modes. In pictures, what does an arbitrary mode with \( \omega = 2\omega_1 \) look like?
\[ \text{Problem 205 Set 7} \]

9) A uniform disk of mass \( M \), radius \( a \), rests on a smooth table. It is connected via 3 springs of constant \( k \), rest length \( l_0 \), to 3 fixed points 120° apart. At equilibrium the springs have length \( l > l_0 \).

**What are the frequencies of the 3 normal modes (including rotation)?**

You might guess the modes and solve them one by one, or try Lagrange's method.

**Ans:** \( \omega_1 = \omega_2 = \sqrt{\frac{3k}{2m}} \frac{2l-l_0}{l} \), \( \omega_3 = \sqrt{\frac{6k}{m}} \frac{(l-l_0)(a+l)}{a} \).

The problems on p. 60, 61, 63 may help with the geometry.

10) a) Consider the linear triatomic molecule of Problem 1 p. 72, 61, 63. Then solve it by guessing the modes, using the constancy of the C.M. to reduce the problem to 2 degrees of freedom. (We will ignore the bending mode here.) Work this problem by deriving the 3 coupled equations of motion of coordinates \( x_1, x_2, x_3 \).

Assume oscillatory solutions to derive the characteristic equation for \( \omega^2 \). **Ans** \( \omega^2 = 0, \frac{k}{m_a}, \frac{k}{m_b} \frac{2m_a + m_b}{m_2m_b} \).

The solution \( \omega = 0 \) means there is a non-oscillatory motion possible in this system — which we know is just translation of the C.M.

b) Suppose the middle atom \( m_b \) is tied to the origin by a spring, also of constant \( k \) (rest length zero).

Now what are the normal frequencies?

I got: \( \omega^2 = \frac{k}{m_a}, \frac{k}{m_b} \left\{ \frac{3m_a + m_b}{2m_a + 2m_b m_b + m_b^2} \right\} \).