The Harlem Globetrotters can balance a basketball by spinning it on a finger. We see that stability might be possible if the spinning ball acts like a gyroscope and precesses, rather than rolls off.

Consider a sphere of radius \( a \), mass \( m \) which rolls without slipping on a fixed sphere of radius \( b \).

Derive and decompose into components the equation of motion.

Some milestones:

\[
\ddot{\omega} = \omega_1 \hat{\mathbf{\hat{\mathbf{0}}}} + \frac{a+b}{a} \hat{x} \dot{d}^2 \frac{dt}{dx}
\]

\[
(I + ma^2) \frac{a+b}{a} \hat{x} \dot{d}^2 \frac{dt}{dx} + I \omega_1 \dot{d} + mg a \hat{x} \dot{\dot{d}} = 0
\]

Note that \( \hat{\mathbf{\hat{\mathbf{0}}} \) rotates about \( \hat{\mathbf{0}} \) with rate \( \dot{\theta} \), and about \( \hat{x} \) at rate \( \dot{\varphi} \) (careful of signs).

After obtaining the 3 component equations of motion, first consider the steady solution, \( \dot{\theta} = 0 \), \( \dot{\varphi} = \omega_0 \). To show that the angular velocity normal to \( \hat{\mathbf{\hat{\mathbf{0}}} \) axis must obey

\[
\omega_1 > \frac{a}{I} \sqrt{mg (a+b)(I+ma^2) \omega_0 \Theta}
\]

The sphere will fall off if the radial component of the contact force vanishes. Show that this requires

\[
\Sigma^2 < \frac{g \omega_0 \Theta}{(a+b) \sin^2 \Theta}
\]

Use the relation between \( \omega_0 \) and \( \omega_1 \) to show that this indicates that too much spin is bad as well as too little.

Consider nutations about steady precession

\[
\theta = \theta_0 + \epsilon \cos \omega t
\]

\[
\phi = \Sigma t + \delta \sin \omega t
\]

To show \( \Sigma^2 = \frac{I^2 \omega_1^2 - 4 mg (a+b)(I+ma^2) \omega_0 \Theta + \Sigma^2}{[ (I+ma^2)(\frac{b+a}{a}) ]^2} \)
So indeed small oscillations are stable if $\omega_1 > \omega_{\text{min}}$ found above.

For a basketball of radius 15 cm which is a hollow sphere, $I = \frac{2}{3}ma^2$, balanced vertically on your finger, $b = 1$ cm, our result predicts a minimum angular frequency of $\approx 80$ Hz for stability. This is rather fast & we reluctantly conclude that the Harlem Globetrotters never took Phys 205.

2. **The Golfer's Nemesis**

Can a golf ball roll into the cup, roll around on the vertical wall & then pop back out?

Consider a sphere of radius $a$, mass $M$, rolling without slipping inside a vertical cylinder of radius $b$.

If $\omega = \dot{\theta} = \text{rotation of the point of contact about the vertical}$, show that the components of the equation of motion lead to

\[ \begin{align*}
\hat{r} & : \quad a\ddot{\omega}_1 = \omega \dot{z} \\
\hat{z} & : \quad (I + ma^2)\ddot{z} = -ma^2g - Ia\omega_1\dot{\omega} \\
\hat{\theta} & : \quad \dot{\omega} = 0
\end{align*} \]

Show that $z$ of C.M. executes simple harmonic motion, and if at $t = 0$, $z = 0$, $\dot{z} = 0$ and $\omega_1 = \omega_{10}$, then

\[ z = \left( \frac{ma^2g + Ia\omega_{10}}{I\omega_2^2} \right) \left( \cos \omega_2 t - 1 \right) \]

where $\omega_2 = \omega \sqrt{\frac{I}{I + ma^2}}$

Note that if the ball rolls into the cup with velocity $v_0$, then $\omega = \frac{v_0}{b-a}$ (if conditions are right for rolling without slipping...)

For a uniform sphere show that the ball rises again to the rim of the cup after 1.87 revolutions, and so indeed might pop out!
3) **OFF THE RIM**

A frequent occurrence in golf or basketball is that the ball rolls around the rim of the cup or basket for quite a while—then sometimes goes in, sometimes not...

Consider a sphere of radius \( a \), mass \( M \), rolling without slipping on a horizontal hoop of radius \( b \). An equilibrium of steady rolling exists with no 'spin' component, \( \omega_0 = \omega \), \( \dot{\theta} = 0 \). In this case show that the angular velocity of the point of contact about the vertical is

\[
\Omega = \sqrt{\frac{3g \tan \Theta}{5(b-a \cos \Theta)}}
\]

for a hollow sphere.

For a basketball of radius 15 cm, a hoop of radius 30 cm, this gives \( n \) rev/sec. at \( \Theta = 45^\circ \).

Show that the equilibrium is unstable. If \( \Omega > \Omega_{\text{equi}} \), then the ball rises and will leave the hoop. If \( \Omega < \Omega_{\text{equi}} \), then the ball will fall thru as desired.

4) A CIRCULAR HOOP OF RADIUS \( a \) ROTATES WITH CONSTANT ANGULAR VELOCITY \( \Omega \) IN A HORIZONTAL PLANE. THE PIVOT IS A POINT ON THE HOOP. A BEAD OF MASS \( M \) SLIDES FREELY ON THE HOOP.

a) Use \( \Theta \) as shown and Lagrange's method to show that the equation of motion is

\[
\ddot{\Theta} = -\frac{2}{5} \lambda \sin \Theta \quad \text{and} \quad \dot{\Theta} = \frac{2}{5} \lambda \sin \Theta \quad \text{lead to the result of 9).}
\]

b) Show that the Hamiltonian is

\[
H = \frac{P_\Theta^2}{2} - P_\Theta \Omega \cdot \omega - \frac{M^2 \lambda^2 \sin^2 \Theta}{2a^2}
\]

And that Hamilton's equations also lead to the result of 9). Is energy conserved?

c) Analyze the problem in a rotating frame. With care, the result of a) follows fairly quickly.
The Piano  A piano wire is struck by a hammer with a sharp blow, and a fairly pure note is produced. This is surprising—given the analysis in the notes of the effect of an impulse, Helmholtz has suggested that a better approximation to the effect of the hammer is

\[ F(t) = \begin{cases} F \delta(t-\frac{T}{2}) \sin \frac{2\pi t}{T} & \text{if } 0 < t < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} \]

i.e. The force goes thru one half-period of a harmonic oscillation.

The force is applied at a point \( b \) from the end of the wire of length \( L \). The wire is fixed at both ends and stretched so that the wave velocity is \( c \).

Use Green's method to show that the string vibrates as

\[ s(x,t) = \frac{2FT}{\pi^2 c \rho} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi b}{L} \cos \frac{n\pi c}{4L} \sin \frac{n\pi x}{L} \sin \frac{n\pi c (t-T)}{2L}}{n \left( 1 - \frac{(n\pi c)^2}{2L^2} \right)} \]

Suppose we choose \( b = \frac{L}{2} \), the midpoint, and \( T = \frac{2L}{c} \) = fundamental period.

Then \[ s(x,t) = \frac{2FT}{\pi^2 c \rho} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi x}{L} \sin \frac{n\pi c (t-\frac{L}{2c})}{2}}{n \left( 1 - \frac{n^2 \pi^2}{4} \right)} \]

so all harmonics vanish except \( n=1 \)

\[ \lim_{n \to 1} \frac{\sin \frac{n\pi}{1-n^2} \cos \frac{n\pi}{-2}}{-2n} = \frac{1}{2} \]

Even if \( T = \frac{2L}{c} \) cannot be exactly achieved in practice, the series converges quickly since the terms go like \( n^{-3} \).
A string of mass \( M \), length \( l \) is clamped at both ends and stretched with a tension \( T \).

a) A mass \( M \) is attached to the midpoint

\[ \dot{\sigma}_0 = 2 \sqrt{\frac{T}{Ml}} \]

b) Suppose the mass is attached at a distance \( b \) from one end. Let \( c = \sqrt{\frac{T}{\rho}} \) = velocity along string.

Use the method of dividing the problem into two strings over intervals \([0, b]\) and \([b, l]\). Show that the normal frequencies obey the transcendental equation

\[ \sigma \sin \frac{\sigma l}{c} \sin \sigma \frac{l-b}{c} = T \sin \frac{\sigma b}{c} \]

c) Consider again the case \( b = l/2 \).

Show that there are 2 classes of solutions:

1) \( \sigma_1 = \frac{2\pi c}{l} \) in which \( M \) doesn't move at all

d) \( M \) moves and \( \sigma_1 \left( \tan \frac{\sigma_1 b}{2c} \right) = \frac{bl}{M} = \frac{m}{M} \)

d) If \( M \ll M \) show that the lowest frequency is

\[ \sigma \sim \sigma_1 \left(1 - \frac{M}{M} \right) \quad \text{where} \quad \sigma_1 = \frac{\pi c}{l} = \text{frequency if} \quad M = 0 \]

e) If \( \frac{m}{M} \ll 1 \), keep enough higher order terms to show that the lowest frequency is

\[ \sigma \sim \sigma_0 \left(1 - \frac{M}{6M} \right) \quad \Rightarrow \quad \sigma_2 = 2 \sqrt{\frac{T}{l(M + m/3)}} \]

So that the mass of the string appears as a correction \( m/3 \) to the heavy mass \( M \).
The Violin

By means of experiment, Helmholtz deduces that the action of a violin bow on a string is to force the point of contact of the string into a motion which is periodic with the period of the 1st harmonic $t_1 = \frac{2\pi}{c}$.

If the point of application of the bow is $x_0$, the rising motion occupies a time to related by $\frac{x_0}{L} = \frac{t_0}{t_1}$.

First make a Fourier analysis in time of the motion of the point of contact to show

$$S(x_0, t) = \frac{2S_0}{\pi^2 t_0 (t_1 - t_0)} \sum \frac{1}{n^2} \sin \frac{n\pi x_0}{L} \sin \frac{n\pi t}{t_1}$$

In general, we expect the motion of the entire string to be analyzable as

$$S(x, t) = \sum \frac{1}{n} \sin \frac{n\pi x}{L} \left( A_n \cos \frac{n\pi t}{t_1} + B_n \sin \frac{n\pi t}{t_1} \right)$$

Hence $A_n = 0$ at once, and

$$S(x, t) = \frac{2S_0}{\pi^2 t_0 (t_1 - t_0)} \sum \frac{1}{n^2} \sin \frac{n\pi x}{L} \sin \frac{n\pi t}{t_1}$$

From the notes, we saw that at $t = 0$, a plucked string has the analysis

$$\frac{2S_0}{\pi^2 b (L-b)} \sum \frac{1}{n^2} \sin \frac{n\pi x}{L} \sin \frac{n\pi b}{L}$$

Hence at any moment, the violin string looks like a plucked string where $\lambda/l = 2t/t_1$.

The crest of the motion moves along the string with velocity $c = 2L/t_1$. The 'vibration' is more like a travelling wave than a standing wave!
8. A spring of rest length $l_0$ has mass $m$. One end is fixed and the other has a mass $M$ attached to it.

Set up the boundary conditions, and solve the wave equation for the normal frequencies of longitudinal oscillation. (Ignore gravity) Take the spring as a uniform bar...

Show \[ \cot (\omega l_0) = \frac{M}{M} (\omega l_0) \]

where \[ \omega = \sqrt{\frac{M \omega^2}{k l_0^2}} \]

$k =$ spring constant

$\omega =$ oscillation frequency

By suitable approximation, show that the frequency of the lowest mode is

\[ \omega \sim \sqrt{\frac{k}{M + M/3}} \]

As derived on problem set 1.