Parametrization of the Kobayashi-Maskawa Matrix

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(Received 22 August 1983)

The quark mixing matrix (Kobayashi-Maskawa matrix) is expanded in powers of a small parameter \( \lambda \) equal to \( \sin \theta_c = 0.22 \). The term of order \( \lambda^5 \) is determined from the recently measured \( B \) lifetime. Two remaining parameters, including the \( CP \)-nonconservation effects, enter only the term of order \( \lambda^3 \) and are poorly constrained. A significant reduction in the limit on \( c / f \) possible in an ongoing experiment would tightly constrain the \( CP \)-nonconservation parameter and could rule out the hypothesis that the only source of \( CP \) nonconservation is the Kobayashi-Maskawa mechanism.

PACS numbers: 11.30.Er, 12.10.Ck, 13.25.+m

The quark mixing of the weak-interaction current in the standard model is described by the \( 3 \times 3 \) Kobayashi-Maskawa (KM) matrix\(^1 \)

\[
V = \begin{pmatrix}
V_{ud} & V_{us} & V_{ub} \\
V_{cd} & V_{cs} & V_{cb} \\
V_{td} & V_{ts} & V_{tb}
\end{pmatrix}.
\]

(1)

The element \( V_{us} \) is quite well determined to be equal to 0.22. This and other information suggest that \( V \) differs from unity by a small quantity. Here we set

\[
0.22 = V_{us} = \lambda
\]

(2)

and consider an expansion of \( V \) in powers of \( \lambda \). A recent measurement of the lifetime \( \tau_B \) of \( B \) particles yields the result\(^2 \)

\[
V_{cb} = 0.06.
\]

(3)

This suggests to us that \( V_{cb} \) is of order \( \lambda^2 \) rather than \( \lambda \) so that we set

\[
V_{cb} = A \lambda^2
\]

with \( A = \frac{s_3}{s_2} \). To order \( \lambda^2 \) the KM matrix can then be written

\[
V = \begin{pmatrix}
1 - \frac{3}{2} \lambda^2 & \lambda & 0 \\
-\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\
0 & -A \lambda^2 & 1
\end{pmatrix}.
\]

(4)

We now want to go to order \( \lambda^3 \). Unitarity then prescribes the following form:

\[
V = \begin{pmatrix}
1 - \frac{1}{4} \lambda^2 & \lambda & \lambda^2 A (\rho - i \eta) \\
-\lambda & 1 - \frac{3}{4} \lambda^2 & \lambda^2 A \\
\lambda^2 A (1 - \rho - i \eta) & -\lambda^2 & 1
\end{pmatrix},
\]

(5)

where two new parameters \( \rho \) and \( \eta \) must be introduced. Equation (4) can be derived from the standard KM form by assuming that \( s_3 \) and \( s_2 \) are of order \( \lambda^2 \) and making the replacements\(^3 \)

\[
\lambda = s_1, \quad A \lambda^2 = (s_2^2 + s_3^2 + 2 s_2 s_3 \cos \delta)^{1/2}, \quad A^2 \lambda^3 = s_2 s_3 \sin \delta, \quad A \lambda^2 (\rho^2 + \eta^2)^{1/2} = s_3,
\]

(6)

or

\[
A \lambda^2 [(1 - \rho^2 + \eta^2)^{1/2} = s_3.
\]

(7)

Only three of the equations (5b)–(5e) are independent. The phase convention has been changed from the standard form so that \( CP \) nonconservation enters only terms of order \( \lambda^3 \).

Given the values of \( \lambda \) and \( A \) we look for empirical constraints on \( \rho \) and \( \eta \). If we neglect \( CP \) nonconservation for the moment, terms of the order \( \lambda^4 \) (which enter along the diagonal and in the \( \lambda^2 A \) terms) are too small to be of importance given experimental and theoretical uncertainties. Therefore the simple form (4) is adequate for present analyses. The only significant constraint now comes from the limit on the ratio of \( b \) to \( u \) to \( b \) to \( c \) transitions which yields\(^4 \)

\[
|V_{ub}/V_{cb}| < 0.2
\]

(8)

or

\[
\rho^2 + \eta^2 < 1.
\]

(9)

From Eq. (7) it follows that

\[
V_{td} < 2 A \lambda^3.
\]

(10)

I note in passing that Eqs. (6)–(8) show the consistency of the expansion in powers of \( \lambda \) because they limit the coefficients of the \( \lambda^3 \) terms. Many other experimental constraints on the KM matrix are discussed in the literature.\(^1 \) When we neglect \( CP \) nonconservation all of these are now of no significance. For example, an upper limit on
Re($V_{td} V_{ts}$) may be set based on the box-diagram contribution to $K_L - \mu^+\mu^-$. For $m_t \sim 40$ GeV this limit is about 0.02; a stronger limit comes from Eq. (8):

$$|V_{td} V_{ts}| < 2\lambda^2 A^2 \approx 2 \times 10^{-3}.$$  

All CP-nonconserving effects are proportional to $\text{Im} V_{td} = -\text{Im} V_{ub} = \lambda^2 A \eta \approx 1.5 \times 10^{-2}$ where the last inequality follows from Eq. (7). For the $K^0$ system CP-nonconserving effects depend on $V_{td}$

$$V = \begin{pmatrix} 1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^2 (\rho - i \eta + i \eta \lambda^2) \\ -\lambda & 1 - \frac{1}{2} \lambda^2 - i \eta A \lambda^4 & A \lambda^2 (1 + i \eta \lambda^2) \\ A \lambda^2 (1 - \rho - i \eta) & -A \lambda^2 & 1 \end{pmatrix},$$

where we have demanded that the imaginary part of the unitarity relation be satisfied to order $\lambda^3$ and the real part only to order $\lambda^2$. The term $i \eta A \lambda^2$ in $V_{ub}$, which could be transferred to $V_{td}$, is needed for unitarity but can be neglected in present calculations. The two new terms $i \eta A \lambda^4$ in $V_{us}$ and $V_{cb}$ represent the leading CP-nonconserving pieces of these elements and cannot be neglected in general.

As first emphasized by Gilman and Wise, as a result of penguin diagrams a nonzero value of the CP-nonconserving parameter $\epsilon'$ is expected from the KM matrix. Following Hagelin and Gilman one finds

$$|\epsilon'| \approx 0.02 \ A^3 \lambda^2 \eta \approx 0.03 \eta |\epsilon|, \quad (11)$$

where the empirical value of $\epsilon$ is used for the last equality and the coefficient (0.02 or 0.03) has an uncertainty of at least a factor of 2 associated with the hadronic matrix element. The present limit ($|\epsilon'|/\epsilon < 0.02$) provides at best only a modest constraint on $\eta$. However, if this limit is reduced by a factor of 5, as is possible in an ongoing experiment, the upper limit on $|\eta|$ would be reduced to the order of 0.1. Since the calculation involves an evaluation of $\epsilon'$ and not of the ratio $\epsilon'/\epsilon$, this constraint holds even if CP nonconservation is not entirely due to the KM mechanism; for example, $\epsilon$ might result from a combination of a superweak and a KM contribution. Of course, if the $\epsilon'$ due to the KM mechanism is approximately cancelled by some other source of CP nonconservation this constraint would not be correct.

As has been commented by many people, the measured $B$ lifetime has the consequence that if CP nonconservation is to be explained entirely by the KM mechanism the value of the CP-nonconserving parameter $(A^3 \lambda^2 = s_2 s_2 \sin \delta)$ must be close to the upper limit of Eq. (9) if $m_t$ is not too large. Thus for $m_t \approx 40$ GeV a value of $\eta \approx 0.5$ is needed. From Eq. (11) this would correspond to a value of $\epsilon'/\epsilon$ close to the present upper limit. If the present limit is reduced by a factor of 5 and if $m_t$ is found to be of the order 40 GeV or less it would be very difficult to explain the value of $\epsilon$ using the KM mechanism alone. The KM matrix may in this case still be one of the sources of CP nonconservation and as discussed above it may be a major source of a nonzero value for $\epsilon'$.

This research was carried out at the Aspen Center for Physics and was partially funded by the U. S. Department of Energy and a John Simon Guggenheim Memorial Fellowship.


$^7$ If use the value $\tau_B = 1.2 \times 10^{-12}$ see reported by the Mark II group at the storage ring PEP in Proceedings of the International Lepton-Photon Conference, Cornell University, August 1983 (to be published), together with Eq. (6a) from P. H. Ginsparg, S. L. Glashow and M. B. Wise, Phys. Rev. Lett. 50, 1415 (1983), based on the work of Cortes et al., Phys. Rev. D 25, 188 (1983). A somewhat better procedure is to use the theoretical formula for $b \rightarrow c e \nu$ together with $\tau_B$ and the experimental branching ratio; within the present uncertainties (probably 20% experimental and 20% theoretical on the number 0.06) the two procedures
give the same result.

3My notation is more closely related to that of L. Maiani, in Proceedings of the International Symposium on Lepton and Photon Interaction at High Energies, Hamburg, 1977 (DESY, Hamburg, 1977), p. 867. His parameters are given by \( s_\theta = \lambda, s_\gamma = \lambda \alpha, \) and \( s_\theta e^{i\delta} = \lambda (\alpha - i\eta). \)


7For values of \( m, \) much larger than \( m_w,\) box-diagram constraints may become significant. I am indebted to J. S. Hagelin for this point.


9B. Winstein, in Proceedings of the Symposium on Intense Medium Energy Sources of Strangeness, Santa Cruz, California, March 1983 (to be published).

10See, for example, P. H. Ginsparg et al., in Ref. 3.

11I have used Eq. (10) of Ref. 6. This has an uncertainty of at least a factor of 2.