of the molecules containing Cl\textsuperscript{37} have missed the detector because of reorientations in the transition field. This amount is much greater than can be expected to occur for a nucleus whose \( g \) is as small as 0.55. We have no knowledge, however, that the molecules in our beam are LiCl rather than (LiCl)\(_n\) with \( n \approx 2\). If \( n = 2 \) then 42 percent of the molecules in the beam will contain Cl\textsuperscript{37}, and the assignment of the larger \( g \) to Cl\textsuperscript{35} would imply that 24 percent of the molecules in the beam have missed the detector due to reorientations. This fraction is also greater than can reasonably be expected, especially for a curve as broad as that in Fig. 3. It therefore seems quite certain that the \( g \) value of 0.546 is to be assigned to Cl\textsuperscript{35} and that of 0.454 to Cl\textsuperscript{37}. If we take the band spectra\textsuperscript{8} value of 5/2 for the spin of Cl\textsuperscript{35} its moment is 1.365±0.005 nuclear magnetons. It must be borne in mind, however, that determinations of spin from measurements of alternating intensities in band spectra are not decisive for spin values as large as 5/2. The correction due to diamagnetic susceptibility is less than 0.1 percent and can be neglected. Since the spin of Cl\textsuperscript{37} is not known its magnetic moment cannot be obtained from the observed \( g \).

DISCUSSION

Our values, 2.741 and 1.345, for the magnetic moments of Rb\textsuperscript{87} and Rb\textsuperscript{85}, respectively, are to be compared with 2.67 and 1.32, calculated from the h.f.s. of the ground state by the use of the G.F.S. formula.\textsuperscript{11} It is seen that the results obtained from the G.F.S. formula are in good agreement with those obtained from our direct measurements. The value which we obtain for the ratio, 2.038±0.010, is to be compared with the ratio obtained by Millman and Fox\textsuperscript{4} from atomic beam measurements, 2.026±0.004. The discrepancy of 0.6 percent lies within the limits of experimental error. Any contribution to the h.f.s. of an atomic energy state by a form of an interaction, not electromagnetic in character, between the nucleus and the electronic structure is certainly very small.

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\textsuperscript{11} The results given by Millman and Fox are slightly high because of an error in the use of the relativity correction factor, \((1−dz/dw)\), in the G.F.S. formula. The moments here given have been recalculated from their data.

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The Electron-Positron Field Theory of Nuclear Forces

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A simple, spin-independent interaction is assumed between heavy particles (neutrons and protons) and electrons. Saturation of resulting forces between heavy particles is assured by choosing a bounded interaction. The change in energy of electrons in negative levels which is caused by the presence of a heavy particle is calculated. Certain general restrictions on the choice of interactions between electrons and heavy particles are discussed.

INTRODUCTION

It has been suggested\textsuperscript{1} that nuclear forces are characterized by an exchange of electron-

\textsuperscript{1} G. Gamow and E. Teller, Phys. Rev. 51, 289 (1937).

positron pairs between the interacting particles. This type of field theory was conceived of as a possible explanation of the equality of proton-proton and neutron-proton forces which became apparent from the results of scattering of protons
by protons. An interaction between the heavy elementary particles which does not entail an exchange of charge or of spin, however, will not show the same kind of saturation which has been customarily assumed to exist between nuclear particles.

One species of saturation which may occur if neutrons and protons interact strongly with electrons and positrons in certain states has been investigated. It was assumed that a single heavy particle attracts electrons of either spin if they occupy a particular spherical state which will be designated by \( \psi_A \) and repels electrons in the spherical state \( \psi_B \). \( \psi_A \) and \( \psi_B \) are assumed to have the same space dependence, centered on the position of the heavy particle, and to differ only in the components of the four-valued spin of relativistic electron theory. The interaction is assumed to be strong enough so that it is permissible to say that in the zero-order approximation the lowest state of a single heavy particle system is that in which \( \psi_A \) states of both spins are filled and \( \psi_B \) states empty. Let the interaction energy per electron be \( \pm \eta \), so that the energy in zero-approximation of a heavy particle at rest is \(-2\eta\), neglecting the proper mass of the heavy particle. In first approximation one should add the average kinetic energy of the light particle field, \( E_{\text{kin}} \). The perturbation method used is valid if \( \eta \gg E_{\text{kin}} \). Interaction with electrons in states orthogonal to the \( \psi_A \)'s and \( \psi_B \)'s has been assumed to be zero.

Attraction between heavy particles which interact with electrons as assumed above was found to be a consequence of the change in average kinetic energy of the light particle field which was effected by bringing the heavy particles close together. In particular, disregarding the application of the exclusion principle to the heavy particles, \( N \) heavy particles located at the same point will be in the lowest state if the \( \psi_A \) states centered at that point are filled and the \( \psi_B \) states empty. The energy of this system in first approximation is \(-2N\eta + E_{\text{kin}}\) as compared with \(-2N\eta + NE_{\text{kin}}\) in case the \( N \)

heavy particles are widely separated. It has also been shown that the interaction energy per particle, \(-2\eta\), was independent of the configuration of heavy particles and could therefore be considered as a part of the rest energy of the particles. The maximum attractive potential in a nucleus of \( N \) particles is thus \((N-1)E_{\text{kin}}\). A binding energy per particle which is independent of the number of other particles in heavy nuclei may then be obtained. The range of forces which is essentially the linear dimension of states \( \psi_A \) and \( \psi_B \) is known to be of the order of \( \eta^2/mc^2 \) so that \( E_{\text{kin}} \sim c\eta/(\eta^2/mc^2) \sim 137mc^2 \) in agreement with the order of magnitude of depths of potential wells found necessary to account for the stability of light nuclei.

2.

The present paper differs from the previous work (I) in two respects. First, the interaction between heavy particles and the light particle field has a different form. This change was introduced in order to assure that the interaction has the proper symmetry with respect to inversion. The consequences of the lack of invariance of the previously assumed interaction with respect to inversion are briefly discussed.

On the other hand, it will be assumed again that the heavy particle does not change its state as a consequence of the interaction with the electron-positron field so that we do not obtain spin-dependent forces. A modification of the theory leading to spin dependence will be discussed in a forthcoming paper.

Second, it is shown that saturation will appear for any value of the interaction constant, \( \eta \). In fact, we shall see at once that the reason of the saturation is that the assumed interaction of a heavy particle with the light particle field has a lower bound (i.e., the characteristic values of the interaction operator do not extend to \(-\infty\)).

Let the interaction be \(-\eta\) between the heavy particle and the light particles (electrons) in certain states, \( \psi_A \), and let the interaction be \(+\eta\) between the heavy particle and the light particles in certain other states, \( \psi_B \), which are orthogonal to the \( \psi_A \). Electrons in states orthogonal to both the \( \psi_A \) and the \( \psi_B \) shall not interact with the heavy particle. The lowest value of the

\footnotesize
6 C. Critchfield and E. Teller, Phys. Rev. 53, 812 (1938). This reference will be quoted throughout the remainder of the paper as I.
interaction energy will be obtained if all states \( \psi_A \) are filled and all states \( \psi_B \) empty. The total minimum of the interaction energy of a heavy particle with the light particle field will be \(-2\eta\) if there are two states \( \psi_A \) (and two states \( \psi_B \)). The total energy of the system will consist of three parts: the energy of the heavy particle, the above interaction energy and the increase of the kinetic energies of the electrons under the influence of the heavy particle. This last quantity is always positive as the sum of the kinetic energies of the electrons in the vacuum (when all levels with negative kinetic energy are filled, all levels with positive kinetic energy empty) is the absolute minimum which the value of the kinetic energies of the electrons can assume. The lowest possible value of the sum of these three energies will correspond to the observable mass \( M \) of a heavy particle.

The total energy of \( N \) heavy particles will also contain three parts. First the proper energy of the heavy particles, \( NMc^2 \), second the interaction energy between light and heavy particles and third the increase of the kinetic energy for the light particles. As this last quantity is always positive, the total energy of \( N \) particles is always greater than \( E_N > NMc^2 - 2N\eta \) as \(-2\eta\) is the absolute minimum of the interaction of a heavy particle with the electron field. Thus the total binding energy between \( N \) particles, \( NMc^2 - E_N < N(M - M_B)c^2 + 2N\eta \), will be below a constant, \( (M - M_B)c^2 + 2\eta \), times \( N \), which will assure the saturation character of the assumed interaction for sufficiently large values of \( N \) for any \( \eta \).

The assumption of a finite value for \( \eta \) appears necessary because a very large \( \eta \) would lead to a strong repulsion of heavy particles which are not at rest with respect to each other. One can see this in the following way. Naturally, the states \( \psi_A \) and \( \psi_B \) will depend on the position of the heavy particle and will be, in fact, centered around it. Let us assume now that all \( \psi_A \) are orthogonal to all \( \psi_B \), also to those centered around a different heavy particle. This condition is fulfilled if the spin parts of the \( \psi_A \) are perpendicular to the spin parts of the \( \psi_B \). If this is true, the lowest value of the interaction energy of the electron field with a system of heavy particles will be assumed if both \( \psi_A \) of each heavy particle are occupied and both \( \psi_B \) of each heavy particle unoccupied. If \( \eta \) is very large, this will be a reasonable first approximation. The total energy \( Mc^2 \) of one heavy particle will then be in first approximation

\[
Mc^2 = M\rho c^2 - 2\eta + E_1 \text{ kin},
\]

where \( E_1 \text{ kin} \) is the lowest value of the increase of kinetic energy of the electrons, compatible with the condition that the \( \psi_A \) are occupied, the \( \psi_B \) unoccupied. Similarly, for \( N \) particles, the total energy becomes in first approximation

\[
E_N = NMc^2 - 2N\eta + E_N \text{ kin},
\]

where \( E_N \text{ kin} \) is the lowest possible increase in the kinetic energy of the electrons which allows all \( 2N \) of the \( \psi_A \) to be occupied and all \( 2N \) of the \( \psi_B \) to be empty. The binding energy will be, therefore, in first approximation

\[
NMc^2 - E_N = NE_1 \text{ kin} - E_N \text{ kin}.
\]

This can be easily obtained for a coincident position of the heavy particles: in this case all pairs of \( \psi_A \) are the same and all pairs of \( \psi_B \) also. Thus the kinetic energy \( E_N \text{ kin} = E_1 \text{ kin} \) and the total binding becomes for coincident heavy particles

\[
NMc^2 - E_N = (N - 1)E_1 \text{ kin}.
\]

The above consideration, which is a repetition of that in (I), assumes that the \( \psi_A \) are all orthogonal to all \( \psi_B \). This cannot be true, however, since the states \( \psi_A \), \( \psi_B \), etc., are defined in the system of reference in which the corresponding heavy particle is at rest. If one of them is moving with a velocity \( v \), the corresponding \( \psi_B \) states will not be orthogonal any more to the \( \psi_A \) states of a heavy particle at rest but will contain them with a coefficient of the order of magnitude \( v/c \). Hence, it will be impossible to fill all the \( \psi_A \) states of the particle at rest and leave the \( \psi_B \) states of the moving particle empty. Thus, under these conditions, the interaction energy between light particle field and heavy particles will not be the sum of the interaction energies of the light particles with the heavy particles separately but will lie higher by an amount of the order of magnitude \( \eta v^2/c^2 \).

Although other, at present unpredictable, effects may arise from the difficulty connected with the simultaneous creation of pairs at finite
distances from the heavy particle, there is good reason for assuming that \( \eta \) is less than about 100 times greater than the kinetic energy of the electron. For simplicity, we can assume that \( \eta \) is small enough so that the kinetic energy of the electron can be neglected. Otherwise it will be found that the wave function of the particle will be modified in a way which is quite different.

**Form of Interaction**

In the earlier treatment of saturation of pair-emission forces (I) it was assumed for simplicity that a heavy particle created and annihilated pairs in spherically symmetrical states. A consequence of the Dirac theory of the electron is, however, that the states of negative kinetic energy and spherical states of positive kinetic energy have opposite transformation properties under inversion of the coordinate system. According to the hole theory of positrons the creation of an electron-positron pair in a spherically symmetrical state is interpreted as a transition of an electron from a spherical state of negative energy to one of positive energy. Since no change is assumed to take place in the heavy particle state during pair-emission or reabsorption it follows that the Hamiltonian representing the interaction between heavy particle and light particle field is odd under inversion. With an odd term in the Hamiltonian it would no longer be possible to have nuclear wave functions which are purely odd or purely even and the usual argument showing that nuclei cannot have permanent electric dipoles will break down. In fact, with an even Hamiltonian nuclear wave functions will be either odd or even; in both cases the charge distribution is even and no dipoles can result.

Nuclear electric dipoles would give ortho-para hydrogen and (or) deuterium conversions in the presence of a strong dipole gas at atmospheric pressure in about \( 10^{-4} \) seconds. Such conversions fail to take place even in a month. If an odd Hamiltonian is introduced new reasons should be found for the absence of dipoles in nuclei. In addition there seems to be no reason to abandon the postulate of the symmetry of elementary particles with respect to inversion.

It is known\(^4\) that the sixteen linearly independent square matrices which can be formed as linear operators on the four-component electron spin can be represented as one scalar, four components of a polar vector, six components of an antisymmetric tensor, four components of a pseudovector and one pseudoscalar. Scalar additions to the Hamiltonian which refer to the spin-coordinates of two particles (heavy particle and electron) can therefore be constructed in five ways: viz., as products of each of the five covariant operators associated with one particle with the corresponding contravariant operator of the other. Since we are at present interested only in those operators which, in nonrelativistic approximation, leave the heavy particle in the same state the only combinations of possible use are the scalar-scalar and vector-vector products. The vector-vector product would give the same attraction for electrons in positive states as for electrons in negative states and a strong interaction could not be assumed without attracting electrons to the heavy particle and changing the charge with which the heavy particle is associated in its lowest state. The scalar-scalar interaction will therefore be adopted.

We shall write the scalar-scalar interaction in accordance with the formulism of the quantization of the wave equation. Let \( \psi_b(x) \) be the four-component operator of the electron state \( \psi_b(x) \) and \( \psi_b(x)^\dagger \) the adjoint operator. \( \psi_b(x), \psi_b(x)^\dagger \) then obey the commutation rules:

\[
\begin{align*}
\psi_b(x)\psi_b(x')^\dagger + \psi_b(x')^\dagger \psi_b(x) &= \delta_{xx'}
\end{align*}
\]

In the same way \( \Psi_m(x) \) and \( \Psi_m(x)^\dagger \) are operators associated with the heavy particle state. The interaction between a heavy particle located at the origin, \( \Psi_m(0) \), and an electron in the state \( \psi_b(x) \) may then be written:

\[
J = \sum_i \eta \int dx \int dx' \rho_{mn}^i \rho_{ib}^3 \Psi_m(0)^\dagger \Psi_m(0)
\]

\[
\times \psi_b(x)^\dagger \psi_b(x') \delta(x) \delta(x')
\]

where the form of $\rho_{mn}$ and $\rho_{ij}$ is given by

$$\rho = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (2a)$$

referring to the four spin-states of the heavy particle in one case and of the light particle in the other, and $\eta$ is a constant. For simplicity we restrict our attention to operators which do not change the state of the heavy particle in nonrelativistic approximation (neglecting the last two rows) so that the operator of the heavy particle may be considered to be equivalent to unity and will not be explicitly mentioned hereafter. It is further supposed, in order to avoid divergent interaction between heavy particles, that the $\delta$-functions in (2) are replaced by spherically symmetrical functions of finite extent, $u(x)$, $u(x')$, and, in particular, of linear dimensions comparable to the electron radius, $e'/n$. Invariance under Lorentz transformation cannot be had for $J$ when these assumptions are made and, in fact, it will be seen that this failure is just the usual one found in applying relativistic methods to the theory of interactions of elementary particles with fields. The proposed addition to the energy of a single electron is then

$$J = \sum_{j} -\eta \int dx \int dx' \rho_{ij}\psi_i(x)'\psi_j(x')u(x)'^u(x'). \quad (3)$$

It will be convenient to have $J$ expressed in momentum space, so the following transformations are used:

$$u(x) = C\int e^{i\sigma_1}u(p)dp,$$

$$u(x') = C\int e^{-i\sigma_1}u(p)'dp,$$

$$\varphi_k(p) = C\int e^{i\sigma_1}\varphi_k(x)dx,$$

$$\varphi_k(p)' = C\int e^{-i\sigma_1}\varphi_k(x)dx,$$

$$C = (2\pi)^{-\frac{3}{2}}\hbar^{-1},$$

and $v(p)$ will be assumed to be normalized so that

$$\int v(p)^2 4\pi p^2 dp = 1.$$

$$J = -\eta \sum_{j} \int \int \rho_{ij}v(p)v(p)'\varphi_i(p)\varphi_j(p')dpdp'. \quad (5)$$

The whole Hamiltonian is obtained by adding to this the Hamiltonian corresponding to Dirac's wave equation of the electron in its quantized form. The whole Hamiltonian has, therefore, the form $\sum_{ij}\psi_i\rho_{ij}\psi_j$ and is therefore equivalent to the Hamiltonian of a system of electrons which do not interact with each other and the energy equation of which is in Schrödinger form

$$-\sum_{k}[\epsilon_k(p\sigma)^2 + \rho_{ij}m^2c^2]\varphi_k(p)$$

$$-\eta v(p)\sum_{j} \rho_{ij} \int v(p')\varphi_j(p')dp' = E\varphi_i(p), \quad (6)$$

where $E$ is the proper value of the energy. We shall solve now this equation and determine its characteristic values and characteristic states. In the state of the whole system for which the total energy is lowest, all electron states defined by (6) will be occupied if the corresponding $E$ is negative, and all electron states with a positive $E$ will be unoccupied.

For the further work, we drop the subscripts on the $\rho$, $\sigma$ and $\psi$ and consider them as four-dimensional matrices and vectors, respectively. For $\eta = 0$, one has the ordinary Dirac equation which can be transformed by

$$T(p) = (T(p)^{-1})! = [E_p + mc^2 + ic\rho_{ij}(\sigma, \rho)] \times [2E_p(E_p + mc^2)]^{-1}, \quad (7)$$

$$E_p = (m^2c^2 + c^2p^2)^\frac{1}{2},$$

into $-\rho E_p$. One may hope, therefore, to obtain some simplification by transforming the whole Eq. (6) with $T(p)$. This corresponds to replacing $\varphi(p)$ by

$$\varphi'(p) = T(p)\varphi(p). \quad (8)$$

The transformed Eq. (6) becomes particularly


$^6$ E. Fermi, Rev. Mod. Phys. 4, 131 (1932).
simple if we restrict ourselves to high energy values \( E_\gamma > m c^2 \). The \( T(p) \) is then, if we neglect \( m c^2 \) in comparison with \( E_\gamma \),

\[
T(p) = 2^{-i} \left( 1 + i \frac{\rho_3 (\sigma, p)}{p} \right),
\]

\[
T(p)^{-1} = 2^{-i} \left( 1 - i \frac{\rho_3 (\sigma, p)}{p} \right).
\]

and we have, instead of (6),

\[
(E - \rho_3 E_\gamma) \phi' (p) + \nu(p) \int T(p) \rho_3 T(p')^{-1} \chi \nu(p') \phi'(p') dp' = 0. \quad (10)
\]

**Solution of Proper-Value Problem**

Equation (10) will now be used to determine the proper energies of electrons. It must be remembered that in this equation only the kinetic energy of the electron for \( E_\gamma \gg m c^2 \) and the non-Coulomb interaction with the heavy particle is taken into account. One important term which has been neglected is the Coulomb interaction between electrons. Neglecting this interaction makes it possible to restrict ourselves to the one-electron problem. The effect of all the terms neglected may be expected to be of the general order of magnitude of \( m c^2 \).

The dependence of \( \nu(p') \) on \( p' \) is spherically symmetrical. Only \( \phi'(p) \) functions which contain a spherically symmetrical part will give a nonvanishing integral in (10). Since angular momentum in the electron state will be conserved we need only to consider functions with \( j = \frac{1}{2} \) which one can write in the form \( S(p) + \frac{[(\sigma, p)/p] P(p)}{P(p)} \). \( S(p) \) and \( P(p) \) are functions of the absolute value of \( p \), i.e., spherically symmetrical also. Higher \( j \) values will vary with the angle as higher spherical harmonics. The nature of the space dependence of \( \phi'(p) \) is not changed by the transformation \( T(p) \). Hence even after the transformation with \( T \) only the wave functions of the form \( S(p) + \frac{[(\sigma, p)/p] P(p)}{P(p)} \) need be considered. \( S \) and \( P \) have four components each, i.e., are four-dimensional vectors, and \( [((\sigma, p)/p] P(p) \) P(p) has the angular dependence of a first spherical harmonic, transforming like a \( P_1 \) function. Integration over \( dp' \) is understood to mean integration over \( dp \), \( dp' \), or in spherical coordinates over \( p'^2 dp' dp \). The integral over an odd power of \( (\sigma, p)/p \) will obviously be zero, whereas every even power of \( (\sigma, p)/p \) is equivalent to unity.

Substituting for \( T \) and \( \phi'(p) \), the integral in (10) may be written in the form:

\[
I = \frac{1}{2} \int \rho_3 \left[ 1 - i \frac{\rho_3 (\sigma, p)}{p} \right] \nu(p) \left[ S(p') + \frac{(\sigma, p')}{p} P(p') \right] p'^2 \sin \theta \theta d \phi dp'
\]

\[
= \frac{1}{2} \rho_3 \left[ 1 + \rho_1 \rho_3 \frac{(\sigma, p)}{p} \right] \xi_S + \left[ \rho_1 \rho_3 \frac{(\sigma, p)}{p} \right] \xi_F,
\]

where

\[
\xi_S = 4 \pi \int S(p') \nu(p') dp', \quad \xi_F = 4 \pi \int P(p') \nu(p') dp'.
\]

and Eq. (10) becomes

\[
0 = (E - \rho_3 E_\gamma) S(p) + \frac{(\sigma, p)}{p} P(p) + \frac{1}{2} \nu(p) \rho_3 \left[ 1 + \rho_1 \rho_3 \frac{(\sigma, p)}{p} \right] \xi_S + \left[ \rho_1 \rho_3 \frac{(\sigma, p)}{p} \right] \xi_F. \quad (12)
\]

Multiplying by \( (E + \rho_3 E_\gamma)/(E^2 - E_\gamma) \)

\[
S(p) + \frac{(\sigma, p)}{p} P(p) = - \frac{1}{2} \frac{E_\gamma + E_\gamma}{E^2 - E_\gamma} \nu(p) \rho_3 \left[ 1 + \rho_1 \rho_3 \frac{(\sigma, p)}{p} \right] \xi_S + \left[ \rho_1 \rho_3 \frac{(\sigma, p)}{p} \right] \xi_F. \quad (13)
\]

Multiply both sides by \( 4 \pi \nu(p)^* p^2 dp \) and integrate

\[
\xi_S = -2 \pi \eta \int_{0}^{\infty} \frac{E_\gamma + E_\gamma}{E^2 - E_\gamma} |\nu(p)|^2 p^2 dp \left[ \xi_S + \rho_1 \rho_3 \xi_F \right].
\]
and by \(4\pi v(p)\delta(p, p')\delta^4(p, p')\)
\[
\xi_p = -2\pi \int_0^\infty \frac{E_{p+} + E_p}{E^2 - E_{p'}^2} |v(p)|^2 p^5 dp \left[ \rho_1 \rho_3 \xi_s - \xi_p \right].
\]

Or, defining \(\alpha\) and \(\beta\) as
\[
\alpha = 2\pi \int_0^\infty \frac{E |v(p)|^2 p^5 dp}{E^2 - E_{p'}^2}, \quad \beta = 2\pi \int_0^\infty \frac{E_p' |v(p)|^2 p^5 dp}{E^2 - E_{p'}^2},
\]
\[
\xi_s = -(\alpha \rho_3 + \beta)(\xi_s + \rho_1 \rho_3 \xi_p), \quad \xi_p = - (\alpha \rho_3 + \beta)(\rho_1 \rho_3 \xi_s - \xi_p).
\] (14)

The \(\xi\)'s are, of course, four-component spinors in general but since the forces in the present theory are indifferent to spin orientation we shall consider only two components of \(\xi_s\), \(\xi_s^1\) and \(\xi_s^2\). \(\xi_s^1\) and \(\xi_s^2\) are the first and third elements of the four-rowed form if the spin has one direction and they are the second and fourth elements for the opposite spin direction. Thus the formulae to be given will represent both spin orientations. Similarly we shall consider \(\xi_p^1\) and \(\xi_p^2\). The operator
\[
\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
will operate on these two-dimensional vectors. There are then two sets of equations:
\[
\begin{align*}
\xi_s^1 &= -(\alpha + \beta)(\xi_s^1 - \xi_p^2), \\
\xi_s^2 &= (\alpha + \beta)(\xi_s^1 - \xi_p^2),
\end{align*}
\xi_p^1 = (\alpha + \beta)(\xi_s^1 + \xi_p^1),
\xi_s^2 = (\alpha - \beta)(\xi_s^1 + \xi_p^1),
\begin{bmatrix} 1 + \alpha + \beta & -\alpha - \beta \\ -\alpha + \beta & 1 + \alpha - \beta \end{bmatrix} = 0,
\begin{bmatrix} 1 - \alpha - \beta & -\alpha - \beta \\ -\alpha + \beta & 1 - \alpha + \beta \end{bmatrix} = 0,
\begin{align*}
1 + 2\alpha &= 0, \\
\alpha &= -\frac{1}{2},
\end{align*}
\begin{align*}
1 + 2\beta &= 0, \\
\beta &= -\frac{1}{2}
\end{align*}
\begin{align*}
1 + 2\beta &= 0, \\
\beta &= -\frac{1}{2}
\end{align*}
\begin{align*}
\xi_p^1 &= \frac{1 + 2\beta}{1 - 2\beta} \xi_s^1, \\
\xi_p^2 &= \frac{1 - 2\beta}{1 - 2\beta} \xi_s^1.
\end{align*}
(15)

The equation \(\alpha = -\frac{1}{2}\) determines proper values, \(E\), of the problem with proper functions which are composed of \(S(p)\) functions of the type \(S^n\) and \(\prod (|p, p|/|p|)P(p)\) functions of the type \(P^n\) (where the superscripts have the same meaning as for the \(\xi\)'s in (15)). For \(\alpha = \frac{1}{2}\) the proper functions are combinations of \(S^n\) and \(P^n\) functions.

It has been assumed that electron states of kinetic energy which is large compared with \(mc^2\) are the only ones of interest in considering nuclear forces. In this approximation \(E_s = c\rho\). We evaluate the integral for \(\alpha\) by the following device. Let \(p\) be quantized in a region of length \(L\). Then \(p = \pi \hbar n/L\) and \(E_s = \pi \hbar c/L\). Under this assumption the integral is represented by a sum which, in case \(\alpha = \frac{1}{2}\), is
\[
\begin{align*}
1 &= \sum_0^n \frac{2\pi E v(p_\alpha) \rho_\alpha^2}{c(E^2 - c^2 p_\alpha^2)} + \sum_0^n \frac{4\pi E a}{c^3(E^2 - a^2 n^2)} + \sum_0^n \frac{4\pi E a}{c^3(E^2 - a^2 n^2)} [c^2 p_\alpha^2 v(p_\alpha) - E^2 v(E/c^2)],
\end{align*}
\] (16)
The second sum may be replaced by an integral,
\[
f(E) = \int_0^\infty \frac{4\pi p^5 v(p)^2 - (E^2/c^2) v(E/c^2)^2}{E^2 - c^2 p^2} Edp,
\]
\[
\begin{align*}
1 &= \sum_0^n \frac{4\pi E v(E/c)^2}{c^3 a} \left[ \frac{1}{E^2 / a^2 - n^2} \right] + f(E), \\
1 &= \frac{4\pi E v(E/c)^2}{c^3 a} \left[ \frac{\pi a}{2E \cotg a} - \frac{\pi a}{a^2} \right] f(E) = \frac{4\pi E v(E/c)^2}{c^3 a} \left[ \frac{\pi a}{2E \cotg a} - \frac{\pi a}{a^2} \right] f(E).
\end{align*}
(17')
whence as we increase \( L \) and diminish “\( a \)” toward zero we get:

\[
\left[ \frac{1}{-f(E)} \right] \frac{1}{2\pi^2 E^4 v(E/c)^2} = \cotg \frac{\pi E}{a}.
\]  

(17)

Equation (17) can be considered as the secular equation for \( E \). If \( v(E/c) \) does not vanish for \( E = na \) and \( E = (n+1)a \) and does not change its sign in this region the right-hand side will go from \( +\infty \) to \( -\infty \) when \( E \) changes from \( na \) to \( (n+1)a \). Thus there will be in general a root between \( na \) and \( a(n+1) \). We write, therefore,

\[
E = E_n + ax = a(n+x), \quad 0 \leq x \leq 1, \quad \cotg \frac{\pi E}{a} = \cotg (n+x)\pi = \cotg \pi x,
\]  

(18)

\[
x = -\arctg \left( \frac{1}{\pi} \frac{2\pi^2 E^4 v(E/c)^2/c^3}{1/\eta - f(E)} \right).
\]

If \( L \) has been chosen large enough the difference between \( E_n \) and \( E \) tends toward zero and we may use \( E_n \) instead of \( E \) in (18), thus obtaining an explicit expression for \( x \). In the following treatment we shall, however, drop the index, \( n \), as has been done in (18).

An identical procedure applied in the case \( a = -\frac{1}{2} \) leads to \( E = E_n + ax' \) and

\[
x' = -\arctg \left( \frac{1}{\pi} \frac{2\pi^2 E^4 v(E/c)^2/c^3}{1/\eta - f(E)} \right).
\]  

(19)

The sum of \( a(x+x') \) over all states of negative \( E \) of both spins will be the total change effected by the introduction of \( J \) in the energy of the lowest state of the system:

\[
\Delta E = 2 \sum_{E<0} a(x+x'), \quad 0 \leq x \leq 1
\]  

(20)

\[
\Delta E = \frac{2}{\pi} \int_{-\infty}^{0} \arctg \left( \frac{1}{\eta^2 - f(E)^2} + \frac{4\pi^2/c^3 E^4 v(E/c)^2 f(E)}{(4\pi^2/c^3 E^4 v(E/c)^2)^3} \right) dE.
\]  

(21)

\[
\Delta E = \frac{1}{\eta^2 - f(E)^2} + \frac{4\pi^2/c^3 E^4 v(E/c)^2 f(E)}{(4\pi^2/c^3 E^4 v(E/c)^2)^3} \frac{dE}{\pi},
\]

DISCUSSION

Since \( 4\pi \int \rho\varphi(p)^2 dp = 1 \) and therefore \( \rho\varphi(p)^2 \) decreases more strongly than \( p^{-1} \) for large \( p \) one finds that \( f(E) \) decreases to zero as \( E \rightarrow -\infty \) for large \( E \). Neither \( f(E) \) nor \( E\varphi(E/c)^2 \) are infinite anywhere. If, therefore, \( \eta \) is very small the denominator of (21) is always positive and essentially equal to \( \eta^2 \). Furthermore, \( \eta^2 \) times the numerator remains small and one finds \( \Delta E \sim \eta^2 \) in accordance with expectation for small perturbations.

On the other hand if \( \eta \) is large compared with the argument for maximum \( v(E/c) \) the denominator will change sign as \( E \) varies from \( -\infty \) to 0. It is required that perturbations of levels as \( E \rightarrow -\infty \) become zero. Since \( f(E) \) and with it the argument of the arc tangent will be negative

\[\text{O. K. Rice, Phys. Rev. 33, 748 (1929).}\]
in the region of the maximum. This latter contribution will correspond to the main part of the change in average kinetic energy due to the presence of the strong interaction with the heavy particle. For large \( \eta \) it will become independent of \( \eta \) provided that the integral

\[
\int_0^\infty \frac{E^2 \nu(E/c)^2}{f(E)} dE \cong \int_0^\infty \frac{E^2 \nu(E/c)^2}{E} dE
\]

converges. This is also the condition for a finite kinetic energy in the state \( \nu(E/c) \). We can thus write \( \Delta E = -2(\eta - E_{\text{kin}}) \), the factor 2 being a consequence of summing over the spins. If \( N \) heavy particles are located at the same point the total change in energy is found from (21) by replacing \( \eta \) by \( N\eta \). For sufficiently large values of \( \eta \), therefore, \( \Delta E(N) = -2N\eta + 2E_{\text{kin}} \) and the total attractive potential for the \( N \) particles is \( 2(N-1)E_{\text{kin}} \). Thus the type of saturation obtained in this case is essentially the same as that obtained in (1). That the terms depending on \( \eta \) alone are additive for any configuration of the heavy particles follows from our general arguments since proper states of the term proportional to \( \eta \) are also proper states of \( \rho_1 \) and thus the states belonging to positive and negative proper values are orthogonal regardless of the positions of the heavy particles. The arguments given in the beginning of this paper also show that saturation would be obtained for any value of \( \eta \).

It is not necessary in the theory presented here to require the average kinetic energy in the state \( \nu(p) \) to be finite, i.e.,

\[
\int_0^\infty p^2 \nu(p)^2 dp
\]

may diverge. As an example of the above calculation we take such a particular function. We assume \( u(r) \sim (1/r)e^{-E_0r/2} \).

\[
\nu(p) = \frac{1}{\pi} \left( \frac{c}{E_0} \right)^4 \frac{1}{1 + c^2 p^2 / E_0^2},
\]

\[
\int E^2 \nu(E/c)^2 dE = E_0 \int E^2 dE \left( E_0^2 + E^2 \right)^2
\]
diverges but the evaluation of (21) becomes particularly simple. We obtain

\[
f(E) = E(E^2 - E_0^2)/(E^2 + E_0^2)^2,
\]

\[
\Delta E = -E_0 \left[ G \arccos (1 - 2F^2) / \pi \right] - F \arccosh (1 + 2G^2),
\]

\[
F^2 = [(\eta^4 + 16E_0^2\eta^2) - \eta^2]/8E_0^2,
\]

\[
G^2 = [(\eta^4 + 16E_0^2\eta^2) + \eta^2]/8E_0^2,
\]

and the arccos in \( \Delta E \) is restricted by the boundary conditions to lie in the first and second quadrants.

For large values of \( \eta \), one finds in first approximation:

\[
\Delta E = -2\eta + E_0 \log \frac{\eta + 5}{E_0} \frac{5}{2\pi},
\]

\[
N\Delta E - \Delta E(N) = (N-1)E_0 \left[ 8 \log \frac{\eta + 5}{E_0} \frac{5}{2\pi} \right] - \frac{8}{\pi} E_0 \log N.
\]

The binding potential per particle is therefore roughly constant but it depends upon the value of \( \eta \) and also on \( \log N \) in contrast to the case when the average kinetic energy in \( u(r) \) converges. For small \( \eta \) (or \( N\eta \))

\[
\Delta E = -2\eta^2 / 3\pi E_0.
\]

Certain general features of the theory here presented and of the simpler theory formerly discussed have been seen to be the same. One significant difference lies in the fact that the average value of the kinetic energy in the special electron states, \( \psi_A \) and \( \psi_B \), is not sufficient in the case here discussed to determine the average kinetic energy of the electron-positron field in first approximation. This is true even for the case of large \( \eta \). Then the proper states of the Hamiltonian are in first approximation the proper states of \( J \). The latter are spherically symmetrical functions \( u(x) \) and are also proper states of \( \rho_0 \). This means that they are linear superpositions of \( S_1 \) waves of positive kinetic energy and \( P_1 \) waves of negative kinetic energy for one proper value, \( -\eta \), and superpositions of \( S_1 \), negative, with \( P_1 \) positive for \( +\eta \). In fact
the absolute values of the coefficients of the normalized $S_1$ and $P_1$ waves of high kinetic energy must be equal in order to cancel out angular dependence in the result. Orthogonal linear combinations of the same $S_1$ and $P_1$ waves (for which the spherically symmetrical part of the wave is cancelled out) belong to the value zero. In taking into account the kinetic energy of the light particle field it would be necessary to calculate the effect of coupling between these orthogonal states and higher kinetic energy levels which would introduce corrections of the order of $E_{\text{kin}}$ itself. Formerly, under the simplified treatment of spherically symmetrical functions alone, such couplings led to corrections of the order of $E_{\text{kin}}^2/\eta$ and were negligible.

In this paragraph it will be understood that all wave-functions concerned are doublets and the subscript $\frac{1}{2}$ will not be used. Subscripts will rather be used to designate the sign of the kinetic energy: 1 for positive kinetic energy, 2 for negative. $S_1$ is, for example, an $S_1$ function of positive kinetic energy, $S_2$ of negative kinetic energy. With this notation and disregarding spin in the lowest state of the operator $J$ the state $S_1 - P_2$ must be filled and $S_2 + P_1$ empty. If at the same time $S_1 + P_2$ is empty the physical picture of the state is that in which pairs are produced and reabsorbed, electrons in $S$ states and positrons in $P$ states. On the other hand, if $S_1 + P_2$ is also filled, the picture is simply that of an electron in an $S$ state. Similarly, if $S_2 - P_1$ is filled there will be pair-production in the lowest state but if it is empty there is simply a positron present in an $S$ state. If $\eta$ be replaced by $-\eta$ the physical description is changed so that in the lowest state there may be either pair-emission and reabsorption or electrons and positrons present in $P$ states.

According to the mathematical solution given above the pair-emission mechanism will be still more involved. In fact, if we assume for the moment that $S_1 + P_2$ is empty, then the kinetic energy will cause transitions of electrons in the negative sea into the $S_1 + P_2$ state, thus causing a new kind of pair-production. Subsequently the kinetic energy will cause a transition from the $S_1 + P_2$ state into a state of positive kinetic energy. In this way one heavy particle may be associated with pairs the number of which has no definite upper limit.

This pair-production in the neighborhood of a heavy particle might lead to an accumulation of charge near the position of the heavy particle. It will be shown, however, that no charge is to be expected in the present theory. The proposed Hamiltonian for one electron may be written

$$H = -\rho_1(\sigma, \psi) - \rho_2[mc^2 + H'(q, q')],$$

where $H'$ is the potential due to a heavy particle located at the origin of coordinates. Disregarding the mutual interaction of electrons, (22) is also the complete Hamiltonian which operates on the antisymmetrical functional of all electrons. In the lowest state all negative levels of $H$ are filled. If the proper functions, $\psi$, are now transformed by the matrix $\rho_1$ to $\psi' = \rho_1\psi$ all states of negative energy become states of positive energy because $H$ anticommutes with $\rho_2$: $\rho_2 H \rho_1 = -H$. On the other hand the charge density operator commutes with $\rho_2$ and thus remains unchanged. The charge density when all negative levels of $H$ are filled is the same as that when all positive levels of $H$ are filled. Thus the charge density when all levels (positive and negative) are filled is double that when all negative levels of (22) are filled. Filling all positive and negative levels, however, leads to a uniform charge distribution from which it follows that there is no accumulation of charge about the heavy particle under the assumed interaction. This shows, in fact, that any addition to the Hamiltonian which anticommutes with $\rho_2$ will not lead to difficulties with a surplus of electric charge in the vicinity of a heavy particle. Since no accumulation of charge near the heavy particle seems to be a desirable feature of the theory it seems that operators anticommuting with $\rho_2$ should be first considered if an interaction of heavy particles with the light particle field is introduced.

Although many arbitrary features remain in the theory of nuclear forces proposed in the above discussion this arbitrariness is reduced to some extent by the requirements which have been pointed out. Among these probably the arguments on symmetry and on the absence of charge accumulation are most important.