Current Algebra and Gauge Theories. II. Non-Abelian Gluons*

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The considerations of the first paper in this series are extended to non-Abelian gauge models of the strong, weak, and electromagnetic interactions. It is shown that for a large class of such theories, the strong interactions naturally conserve parity and strangeness, and possibly isospin and other quantum numbers as well. The corrections of second order in gauge couplings to such natural symmetries are convergent. In addition to the ordinary photon-exchange term, these corrections include other terms of order α, which take the form of shifts in the effective quark mass matrix, and which automatically conserve parity and strangeness. In theories with free-field asymptotic behavior, these order-α mass shifts may be correctly calculated ignoring all effects of the strong interactions. It is suggested that in such theories, the strong gauge group is not broken, and that the infrared divergences associated with the massless vector gluons prevent the production of quarks or gluons in collisions of ordinary hadrons.

I. INTRODUCTION

This is the second paper of a series, which aims at the incorporation of strong interactions into unified renormalizable gauge theories of the weak and electromagnetic interactions, through the use of current algebra.

In the first paper1 of this series, we studied how the observed symmetries of the strong interactions can arise in a natural way, and how they are broken by second-order effects of the weak and electromagnetic interactions. There was no commitment in this work to any particular renormalizable gauge model of the weak and electromagnetic interactions, but the strong interactions were specifically chosen to be transmitted by a massive neutral vector gluon coupled to the conserved baryon-number current, with no strongly interacting scalar fields. The chief results of Ref. 1 were

(a) The strong interactions exhibit a number of "natural" symmetries,2 valid for all values of the parameters in the original Lagrangian (or at least all values in a finite range). These natural symmetries always include parity and strangeness conservation, and may also include isospin conservation, chiral SU(2)⊗SU(2), etc., depending on the gauge group and field content of the theory.

(b) The commutation and conservation relations satisfied by the weak and electromagnetic currents always guarantee that the second-order corrections to natural symmetries of the strong interactions are finite.3

(c) The second-order corrections include contributions of order α as well as truly weak contributions of order $\alpha/\mu_w^2$. The order-α terms consist of the usual one-photon-exchange contribution (with a natural ultraviolet cutoff) plus corrections to the quark mass matrix. Consequently parity and strangeness are automatically conserved in order α. It seems likely that the weak corrections of order α to the quark mass matrix provide the nonelectromagnetic isospin breaking needed in isovector mass differences and $\eta$ decay.4

Since the preparation of Ref. 1, there has been something of a revolution in our understanding of the strong interactions, brought about through the observation5 that non-Abelian gauge theories can exhibit free-field asymptotic behavior. It is extremely attractive to suppose that the strong interactions are described by such an asymptotically free gauge theory, because this would provide a means of understanding Bjorken scaling in inelastic electron scattering, and because it would open up the possibility of carrying out reliable strong-interaction calculations by perturbation theory.

These developments seem to me to be so important that it is necessary to interrupt what was to have been the order of papers in this series, and immediately to adapt the formalism of Ref. 1 to deal with non-Abelian gauge theories of the strong interactions. A short report of our conclusions has already been published6; the present paper presents the details.

One difficulty encountered in extending the work of Ref. 1 to non-Abelian theories has to do with the natural symmetries of strong interactions. In non-Abelian theories of the "Berkeley" type,7 parity, strangeness, etc. are generally not natural symmetries,8 although of course it is possible to adjust the parameters in the Lagrangian so that they are conserved to any desired degree of ac-
Second-order weak and electromagnetic effects generally produce divergent corrections in these theories, although, again, one can cancel as much as one likes of these corrections with suitably chosen counterterms. Whether or not one is content with this is a matter of personal judgment; to me, it seems that although theories of this type may play a useful phenomenological role, a fundamental theory ought to allow calculations at least in principle of symmetry-breaking effects, and in particular, ought to explain why parity and strangeness violations are not of order \( \alpha \), but much weaker.

Fortunately, it turns out that there are non-Abelian gauge theories in which strong-interaction symmetries can be understood in a natural manner. These theories are based on the direct product of a strong gauge group \( G_s \) and a weak and electromagnetic gauge group \( G_w \), so that the strong gauge bosons are neutral with respect to the weak and electromagnetic interactions, and the weak gauge bosons have no strong interactions. The only special conditions that have to be imposed are that \( G_s \) be nonchiral, and that there be no strongly interacting scalar fields (or at least no scalar fields having strong Yukawa interactions with the fermions). It is shown here, using almost precisely the same methods as in Ref. 1, that this class of non-Abelian theories enjoys all of the properties a, b, and c listed above, and therefore provides as satisfactory an understanding of the known strong-interaction symmetries as the Abelian gluon model of Ref. 1.

However, another problem associated with non-Abelian gauge theories of the strong interactions arises in the course of this work. It is difficult to understand how the strong gauge group \( G_s \) can be broken, without introducing so many scalar fields that we lose the free-field asymptotic behavior, which, after all, was our chief motivation for adopting a non-Abelian gauge model of strong interactions. It is suggested here that \( G_s \) is not broken, so that the gluons have zero mass and the quarks exhibit exact \( G_s \) degeneracy, but that the infrared divergences present in asymptotically free gauge theories prevent us in principle from being able to produce quarks or gluons in collisions of ordinary particles. Further work will be needed for us to tell whether this idea is mathematically viable; only the broad outlines of the argument are sketched here.

There is one other problem, of a more technical nature, that arises in dealing with non-Abelian gauge theories of the strong interactions. In our analysis of weak and electromagnetic corrections, we need to make use of the symmetries of the Lagrangian, including symmetries which may be spontaneously broken. It is therefore important that when we formulate a renormalization procedure to deal with the divergences of the strong interactions, we do so in such a way as to respect these symmetries, or at worst, to violate them in a carefully controlled way. In Abelian gauge theories, the introduction of a gluon mass does not violate the conservation of the current to which the gluon field is coupled, so we were able to define renormalization counterterms in Ref. 1 in terms of Green's functions with vanishing fermion mass but nonvanishing gluon mass. Here, in dealing with non-Abelian gluons, we have to adopt a different procedure, and define counterterms (including mass-renormalization counterterms) in terms of Green's functions with all masses set equal to zero, and, to avoid infrared divergences, with external momenta given arbitrary off-shell values. It has recently been shown that this "zero-mass" renormalization procedure offers other advantages, both in clarifying the nature of the assumptions which underlie the renormalization-group method, and in allowing the derivation of asymptotic expansions for Green's functions or Wilson functions at large momenta. We shall make heavy use of such asymptotic expansions in the present work.

The reward for preserving asymptotic freedom in these theories is that one is able to carry out order-\( \alpha \) calculations of symmetry breaking without worrying about strong interactions. Matters are not usually so simple, even in asymptotically free theories. In general, the presence of anomalous dimensions would prevent us from being able to calculate the constant factors appearing in asymptotic expressions for Green's functions at large momentum, so that we might expect to be able to calculate the form of the symmetry-breaking corrections, but not their absolute magnitude. However, it turns out that the strong interactions enter into the order-\( \alpha \) corrections to natural zeroth-order symmetries only through their effect on a particular Wilson coefficient function, which happens to have a vanishing anomalous dimension. For this reason, the correct prescription for calculating order-\( \alpha \) (as opposed to order \( \alpha/\mu_0^2 \)) corrections to the quark mass matrix is just to use the results of perturbation theory, ignoring all effects of the strong interactions.

Section II describes the general class of non-Abelian gauge theories studied in this paper. The effective Lagrangian of the strong interactions is separated out in Sec. III, and it is shown how strong-interaction symmetries, always including parity and strangeness, naturally arise. Section IV deals with the problem of how the strong gauge group \( G_s \) might be broken, and discusses the pos-
sibility that $G_\phi$ is not broken. Section V reviews the zero-mass renormalization procedure used here to eliminate the divergences in strong interactions. In Sec. VI, we turn to the weak and electromagnetic corrections and review the analysis of these effects into gauge-invariant terms representing $G_\psi$ vector-boson exchange, scalar-boson exchange, and various kinds of tadpole graphs. The asymptotic behavior of the hadronic matrix elements appearing in the boson-exchange terms is discussed in Sec. VII, using a slightly improved version of the perturbative analysis of Ref. 1, and it is shown that the leading terms have the symmetry properties required to ensure the cancellation of divergences in corrections to natural symmetries. Section VIII treats the symmetry-breaking corrections which are of order $\alpha$ rather than of order $\alpha/\mu^2$, and shows, as in Ref. 1, that these take the form of ordinary photon-exchange terms (with a natural ultraviolet cutoff) plus corrections to the quark mass matrix, both of which automatically conserve parity and strangeness. Finally, in Sec. IX, we use the new renormalization-group equations based on the "zero-mass" renormalization procedure to show that the perturbative arguments of Sec. VII are justified in asymptotically free gauge theories, and that the order-$\alpha$ corrections to natural symmetries may be calculated in such theories by using second-order perturbation theory, ignoring all effects of strong interactions.

Of course, these very general results will become really interesting only when we have some specific gauge model of the weak and electromagnetic interactions which can be taken seriously as a possible description of the real world. This we do not yet have.

Even with a believable gauge model of weak and electromagnetic interactions, and with an asymptotically free gauge theory of the strong interactions, the perturbative calculations possible in such theories would not immediately lead to predictions for observables like the neutron-proton mass difference. Rather, the output of such calculations would be a set of corrections to the effective quark mass matrix appearing in the strong-interaction Lagrangian. To draw quantitative conclusions, it would be necessary to use these quark mass corrections as an input to further calculations, making use of either explicit bound-state models, or of the methods of current algebra. Such calculations will be the subject of the next paper in this series.

II. STATEMENT OF THE THEORY

We assume that the complete gauge group of our theory is the direct product of a "weak" gauge group $G_\psi$ and a "strong" gauge group $G_\phi$. Associated with $G_\psi$ is a set of gauge fields $W_{\alpha\nu}(x)$, with couplings typically of order $\epsilon$. Similarly, associated with $G_\phi$ is a set of "gluon" gauge fields $A_{\alpha\nu}(x)$, with couplings typically of order unity. Of course, $W_{\alpha\nu}$ is neutral under $G_\phi$, so it has no strong couplings, and $A_{\alpha\nu}$ is neutral under $G_\psi$, so it has no weak or electromagnetic couplings.

The Lagrangian is also supposed to contain a set of spin-$\frac{1}{2}$ fields $\psi_{\alpha\nu}(x)$, with $\alpha$ labeling the transformation properties under $G_\psi$ and $\nu$ labeling the transformation properties under $G_\phi$. As an example, we might consider the "colored quark" model, with

$$\psi_{\alpha\nu} = \left(\begin{array}{c} \phi_R \\ \phi_w \\ \phi_y \\ \lambda_R \\ \lambda_w \\ \lambda_y \end{array}\right).$$

The weak and electromagnetic interactions here act vertically, and the strong interactions act horizontally. More generally, in dealing with theories in which $\psi$ may not furnish an irreducible representation of $G_\psi \otimes G_\phi$, the distinction between row and column indices may be somewhat arbitrary. It will prove extremely convenient in all cases to choose these indices so that for each $\nu$ the fields $\psi_{\alpha\nu}$ furnish an irreducible representation of $G_\psi$, even though for each $\nu$ these fields may not furnish irreducible representations of $G_\phi$. It is also very convenient to subject the $\psi$ fields to a similarity transformation, such that if any two irreducible representations of $G_\psi$ corresponding to different $\nu$-values are equivalent, they are equal.

The gauge-covariant derivative of the fermion fields takes the form

$$D_\alpha \psi = \partial_\alpha \psi + i t_\alpha \partial_\alpha W_{\alpha\nu} + iT_\alpha \psi A_{\beta\lambda},$$

where the $T_\alpha$ and $t_\alpha$ are matrices, which in general may involve the Dirac matrix $\gamma_5$ as well as $I$, and which satisfy the commutation relations of $G_\phi \otimes G_\psi$:

$$[T_\alpha, T_\beta] = iC_{\alpha\beta} T_T,$$

$$[t_\alpha, t_\beta] = iC_{\alpha\beta} T_T,$$

(2.1)

$$[T_\alpha, t_\beta] = iC_{\alpha\beta},$$

and the reality conditions

$$T_\alpha^* = \gamma_5 T_\alpha \gamma_5,$$

$$t_\alpha^* = \gamma_5 t_\alpha \gamma_5.$$

(2.2)

(2.3)

(2.4)

(2.5)

(2.6)

(Repeated indices are summed except where otherwise noted.) Here $C_{\alpha\beta}$ and $C_{\alpha\beta}$ are the real totally antisymmetric structure constants of $G_\psi$ and $G_\phi$, respectively. The elements of $T_\alpha$ and $C_{\alpha\beta}$ are preserved to be of order unity, and are
proportional to strong gauge coupling constants; the elements of \( t_\alpha \) and \( c_{\alpha\beta} \) are presumed to be of order \( e \), and are proportional to weak and electromagnetic gauge coupling constants.

In accordance with our specification above of the indices \( n, N \), the strong generator matrices \( T_A \) must take the form

\[
(T_A)_{nN, mM} = \delta_{nm}(T_A^m)_{nN}, \tag{2.7}
\]

where \( T_A^m \) is the matrix generator for the irreducible representation \( n \) of \( G_S \). If any matrix \( X \) commutes with all \( T_A^m \), then

\[
\sum_{n'} \langle X \rangle_{nN, n'M} (T_A^m)_{n'M, M} = \sum_{n'} \langle T_A^m \rangle_{nN', n'M} (X)_{n'M, M} \tag{not summed}. \tag{2.8}
\]

In this case, Schur's lemma gives

\[
\langle X \rangle_{nN, mM} = \delta_{nm} \langle X \rangle_{nN}, \tag{2.9}
\]

with \( \langle X \rangle_{nN} \) equal to zero unless \( \psi_{n} \) and \( \psi_{n'} \) furnish equivalent representations of \( G_S \). (Recall that \( \psi \) is defined above so that if \( n \) and \( m \) are equivalent, then \( T^n \) and \( T^m \) are equal.) In particular, the weak generator matrices \( t_\alpha \) take the form

\[
(t_\alpha)_{nN, mM} = \delta_{nm} (t_\alpha)_{nN}, \tag{2.10}
\]

with \( t_\alpha \) independent of \( N \) and \( M \).

We make an additional assumption, which seems to be necessary in order to understand in a natural way why parity and strangeness should be conserved to order \( \alpha \). We assume that \( G_S \) is nonchiral, in the sense that the \( 1 + \gamma_5 \) and \( 1 - \gamma_5 \) parts of the fermion fields transform according to equivalent representations of \( G_S \). In accordance with our definitions above, the \( 1 + \gamma_5 \) and \( 1 - \gamma_5 \) parts of the generator matrices \( T_\alpha \) are then equal, or in other words, the matrices \( T_A \) are free of \( \gamma_5 \) terms. For instance, in the colored quark model \( G_S \) might be SU(3), but not SU(3) \( \otimes \) SU(3). Of course, \( G_W \) may be chiral.

In addition to the gauge and fermion fields, the Lagrangian is also supposed to contain a set of weakly interacting spin-zero fields \( \phi_i(x) \) which are neutral under \( G_S \). (The possibility of introducing strongly interacting spin-zero fields is considered briefly in Sec. III, but our whole line of argument is enormously simplified if we exclude such fields.) The gauge-covariant derivative of the \( \phi_i \) takes the form

\[
(D\phi)_{i\mu} = \partial_\mu \phi_i + \imath \langle \theta_\alpha \rangle_{i\mu} \phi_i W_\mu, \tag{2.11}
\]

where \( \theta_\alpha \) are Hermitian matrices satisfying the commutation relations of \( G_W \):

\[
\langle \theta_\alpha \theta_\beta \rangle = i c_{\alpha\beta\gamma} \theta_\gamma, \tag{2.12}
\]

with \( c_{\alpha\beta\gamma} \) the same as in (2.3). It can be assumed without loss of generality that the \( \phi_i \) are real fields, so that the \( \theta_\alpha \) are imaginary and antisymmetric:

\[
\langle \theta_\alpha \rangle t = \langle \theta_\alpha \rangle i t = -\langle \theta_\alpha \rangle i t. \tag{2.13}
\]

The elements of \( \theta_\alpha \) are supposed to be of order \( e \).

With these assumptions, the most general Lorentz-invariant, Hermitian, gauge-invariant, and renormalizable Lagrangian is of the form (assuming fermion-number conservation)

\[
\mathcal{L} = -\frac{1}{4} (\xi_5)_{AB} G_{AB} G^\mu_\nu - \frac{1}{2} (\xi_\mu)_{\alpha\beta} F_{\alpha\beta}^\mu \tilde{F}_{\mu}^\nu
\]

\[
- \frac{1}{2} (\xi_5)_{\mu i} (D_\mu \phi)_i (D^\mu \phi)_i - \bar{\psi}_i \gamma_\mu D_\mu \psi_i
\]

\[
- \bar{\psi}_m \gamma_\mu \Gamma^i \psi_1 - P(\phi), \tag{2.14}
\]

where \( G_{AB} \) and \( F_{\alpha\beta} \) are the gauge-covariant curlicues

\[
G_{AB} = \partial_\mu S_{AB} - \partial_\nu S_{BA} - C_{AB \Gamma} S_{B \mu} S^{\mu \Gamma}, \tag{2.15}
\]

\[
F_{\alpha\beta} = \partial_\mu W_{\alpha\beta} - \partial_\nu W_{\alpha\beta} - c_{\alpha\beta} W_{\gamma\delta} \gamma_\gamma W_{\gamma\delta}, \tag{2.16}
\]

while \( P(\phi) \) is a real quartic \( G_W \)-invariant polynomial

\[
P^*(\phi) = P(\phi), \tag{2.17}
\]

\[
\frac{\partial P(\phi)}{\partial \phi_i} (\theta_\alpha)_{i\mu} \phi_i = 0, \tag{2.18}
\]

and the \( \xi_\mu \)'s, \( m_\alpha \), and \( \Gamma_i \) are matrices subject to the reality conditions

\[
\xi_\mu = \xi_\mu^*, \tag{2.19}
\]

\[
\xi_\mu = \xi_\mu^* = \xi_\mu, \tag{2.20}
\]

\[
\xi_\mu = \xi_\mu = \xi_\mu, \tag{2.21}
\]

\[
\xi_\mu = \xi_\mu, \tag{2.22}
\]

\[
\gamma_\mu \gamma_\mu = m_\alpha, \tag{2.23}
\]

\[
\gamma_\mu \Gamma_\Gamma \gamma_\mu = \Gamma_\Gamma, \tag{2.24}
\]

and the gauge invariance conditions

\[
C_{AB \Gamma} (\xi_\mu)_{\delta \Gamma} - (\xi_\mu)_{\delta \Gamma} C_{AB \Gamma} = 0, \tag{2.25}
\]

\[
c_{\alpha\beta\gamma} (\xi_\mu)_{\gamma \epsilon} - (\xi_\mu)_{\gamma \epsilon} c_{\alpha\beta\gamma} = 0, \tag{2.26}
\]

\[
[\theta_\alpha, \xi_\mu] = 0, \tag{2.27}
\]

\[
[T_\alpha, \xi_\mu] = [t_\alpha, \xi_\mu] = 0, \tag{2.28}
\]

\[
[T_\alpha, m_\alpha] = [t_\alpha, \gamma_\mu m_\alpha] = 0, \tag{2.29}
\]

\[
[T_\alpha, \Gamma_\Gamma] = 0, \tag{2.30}
\]

\[
[t_\alpha, \gamma_\mu \Gamma_\Gamma \gamma_\mu] = - [\theta_\alpha, \Gamma_\Gamma]. \tag{2.31}
\]

The matrices \( \xi_\mu \), \( m_\alpha \), and \( \Gamma_i \) may all in general contain terms proportional to the Dirac matrix \( \gamma_\mu \) as well as 1. The various \( \zeta \) matrices are required by the gauge invariance conditions (2.25)–(2.28) to be proportional to unit matrices if the
corresponding fields form irreducible representations of the gauge group, and the matrix $m_{\alpha}$ vanishes if the weak gauge group $G_W$ is chiral, but we leave the $\xi$'s and $m_{\alpha}$ as they are the Lagrangian for the sake of generality. All $\xi$'s and $m_{\alpha}$ are taken to be of zeroth order in $e$, while $\Gamma$ is nominally of order $e$, though as we shall see, it may actually be much smaller.

At this point a further simplification is useful, though not strictly necessary. We note that all the matrices $\xi_\alpha$, $m_{\alpha \nu}$, $\Gamma$, and $t_{\alpha}$ commute with all the $T_A$, so they take the form (2.9), and therefore do not connect fermion fields belonging to inequivalent irreducible representations of $G_\phi$. Also, the $T_A$ matrices themselves, while they do not commute with each other, also do not connect different irreducible representations of $G_\phi$. Hence there are one or more absolute conservation laws here, unbroken to all orders in the strong, weak, and electromagnetic interactions, which state the separate conservation of the total number of fermions (minus antifermions) belonging to each inequivalent type of irreducible representation of $G_\phi$. A trivial example is provided by the conservation of leptons: Since the leptons have no strong interactions, they must belong to singlet representations of $G_\phi$, and must therefore be conserved separately from any strongly interacting fermions, which belong to nonsinglet representations of $G_\phi$. If the nonsinglet $\psi_{\nu \alpha}$ fields also furnished inequivalent representations of $G_\phi$ for different $\nu$'s, then in addition to the conservation of lepton and total fermion number, we would have a separate absolute conservation of different kinds of hadrons.

There would be no difficulty at all in dealing with such theories, but our notation later on would have to become a bit complicated, and since we do not know experimentally of any separate absolute global conservations laws for different kinds of hadrons, we shall simply assume from now on that the $\psi_{\nu \alpha}$ for all $\nu$ furnish equivalent irreducible representations of $G_\phi$. (We thus also ignore the leptons.) This assumption is of course satisfied by the colored quark model, 17 where each quark row forms a triplet representation of SU(3).

The weak gauge symmetry is broken by the vacuum expectation values of the $\phi_i$. The lowest-order terms $\lambda_i$ in these vacuum expectation values are determined by the symmetry-breaking conditions\footnote{This gives a zeroth-order mass to the weakly interacting gauge bosons, described by the mass matrix
\begin{equation}
(\mu^2)_{\alpha \beta} = \lambda_i (\delta_{\alpha \beta} \delta_i).
\end{equation}
However, the vacuum expectation values of the $\phi_i$ do not give masses to the strongly interacting vector gluons.

The order of magnitude of $\lambda$ is fixed by the observation that the Fermi weak-interaction constant is of order $e^2/\mu_W^2$. Since $\theta$ is of order $e$, this gives
\begin{equation}
\lambda = \mu_W/e = 1/\sqrt{G_F} \approx 300 \text{ GeV}.
\end{equation}
Also, the zeroth-order fermion mass matrix is
\begin{equation}
m = m_0 + \Gamma_i \lambda_i.
\end{equation}
This is called "zeroth order" because $\Gamma_i$ is nominally of order $e$, while (2.34) requires $\lambda_i$ to be nominally of order $1/e$. However, if the fermion mass matrix $m$ is much less than $\mu_W$, then $\Gamma$ is actually much less than $e$:
\begin{equation}
\Gamma_i = m = m_0 + \Gamma_i \lambda_i \approx m_0 \approx m_0. \text{ (2.36)}
\end{equation}

III. STRONG INTERACTIONS

We can now extract from the theory described in Sec. II a description of what would be phenomenologically recognized as the strong interactions. Since the scalar fields $\phi_i$ interact very weakly but have very large vacuum expectation values, we replace $\phi_i$ everywhere with $\lambda_i$. Dropping all couplings of order $e$ except where they appear multiplied with a factor $\lambda = 1/e$, we obtain the effective Lagrangian of the strong interactions:
\begin{equation}
L_S = -\frac{1}{2} (\xi_\alpha)_{AB} G^{A\mu}_{AB} G^\mu_B
- \overline{\psi} \gamma^\mu (\partial_\mu + i T_A S^A) \psi - \overline{\psi} m \psi,
\end{equation}
with $G^{A\mu}_B$ and $m$ given, respectively, by (2.15) and (2.35).

In order to explore the general features of the strong-interaction theories described by (3.1), let us consider the symmetries of this Lagrangian. Clearly, (3.1) is invariant under the gauge group $G_S$; we return to this symmetry group and its possible breakdown in Sec. IV. To uncover the other symmetries of (3.1), it is extremely convenient first to redefine the fermion fields so that $\xi_\alpha$ is unity and $m$ is diagonal and $\gamma_5$-free, while $T_A$ remains $\gamma_5$-free and commuting with $m$. In the neutral vector-gluon theory, this is easily accomplished because there the $T_A$ matrix is proportional to the unit matrix. The same arguments go through here in much the same way. Note that $\xi_\alpha$ and $m$ commute with all $T_A$, and therefore by Schur's lemma must take the form (2.9)
\begin{equation}
(\xi_\alpha)_{\nu \mu} = \delta_{\nu \mu} (\xi_\alpha)_{\nu \mu},
\end{equation}
\begin{equation}
m_{\nu \mu} = \delta_{\nu \mu} m_{\nu \mu}.
\end{equation}
Note also that $T_A$ takes the form
because we have assumed that the \( \psi_{\alpha \mu} \) for each \( \nu \) furnish the same irreducible representation of \( G_3 \).
Therefore, if we apply to \( \psi \) a linear transformation of the form
\[
\psi \rightarrow S \psi,
\]
\[
(S)_{\alpha \nu, \mu \rho} = \delta_{\alpha \nu} \delta_{\mu \rho}.
\]
we find that the whole effect is to change the submatrices \( \bar{S}_\phi \) and \( \bar{m} \) into
\[
\bar{S}_\phi = \bar{S}' \bar{S},
\]
\[
\bar{m} = \gamma_5 \bar{S}' \gamma_5 \bar{m} \bar{S},
\]
with no change in the generators \( T_A \). (If the \( \psi_{\alpha \mu} \) for different \( \nu \) furnished inequivalent representations of \( G_3 \), we would choose \( \bar{S}_{\alpha \nu} \) to vanish unless \( \psi_{\alpha \mu} \) and \( \psi_{\alpha \nu} \) belong to equivalent representations.) We can now proceed, in precisely the same way as in the neutral vector-gluon theory, to choose \( S \) so that \( \bar{S}_\phi \) is unity while \( \bar{m} \) is real, diagonal, and \( \gamma_5 \)-free. We will assume from now on that this has been accomplished, and drop the bars, so that
\[
(\xi_{\phi})_{\alpha \nu, \mu \rho} = \delta_{\alpha \nu} \delta_{\mu \rho},
\]
and
\[
(m)_{\alpha \nu, \mu \rho} = \delta_{\alpha \nu} \delta_{\mu \rho} m_{\mu \rho} \quad \text{(not summed)},
\]
with \( m_{\mu \rho} \) a set of real numbers.

It is now obvious that the strong-interaction theory described by Eqs. (3.1)–(3.3) conserves parity, provided of course that we make a judicious choice of intrinsic parities. That is, we must attribute positive parity to all fermion fields and identify all \( S_A \) fields as polar vectors.

In addition, the Lagrangian (3.1) with \( \xi_\phi = 1 \) is clearly invariant under any global transformation
\[
\psi \rightarrow U \psi, \quad S_A \rightarrow S_A U^*, \quad (3.4)
\]
with
\[
(U)_{\alpha \nu, \mu \rho} = \delta_{\alpha \nu} U_{\mu \rho},
\]
\[
U^* U = 1, \quad (3.5)
\]
\[
U^* \gamma_\mu U = \gamma_\mu U. \quad (3.6)
\]
These transformations will always include phase transformations of the form
\[
(U)_{\alpha \nu, \mu \rho} = \delta_{\alpha \nu} e^{i \phi_{\mu \rho}} \quad \text{(not summed)}, \quad (3.8)
\]
with \( \phi \), arbitrary real numbers. Such phase transformations simply correspond to the conservation of hadrons of each weak-interaction type. For instance, in the "colored quark" model, the numbers \( N_\phi \), \( N_\lambda \), and \( N_\rho \) of all \( \phi \)-type quarks, \( \lambda \)-type quarks, and \( \rho \)-type quarks, \textit{summed over color}, are conserved. Such conservation laws can of course be reexpressed in terms of the conservation of charge \( Q \), baryon number \( B \), and strangeness \( S \):
\[
Q = \frac{2}{3} N_\rho - \frac{1}{3} N_\lambda - \frac{1}{3} N_\phi, \]
\[
B = \frac{1}{3} N_\rho + \frac{1}{3} N_\lambda + \frac{1}{3} N_\phi, \]
\[
S = - N_\lambda.
\]
If we add a fourth quark row, then a fourth quantum number, the charm, is also conserved.

Apart from phase transformations of the form (3.8), there may be additional global symmetries of the class described by (3.4)–(3.7), if the mass values satisfy suitable symmetry relations. For instance, in the colored quark model, if \( m_\phi = m_\pi \), then the symmetry group includes isospin conservation as well as conservation of \( B \) and \( S \); if \( m_\phi = m_\eta = 0 \) then the symmetry group includes chiral SU(2) \( \otimes \) SU(2) as well as conservation of \( B \) and \( S \) and a \( \gamma_5 \)-phase transformation. [Recall that the masses in (3.1) are color-degenerate.] We are not interested in mass relations of this type which are simply arranged to occur by a choice of parameters in the original Lagrangian [i.e., in \( P(\phi) \) and \( \Gamma_1 \)], for then the higher-order weak and electromagnetic corrections will in general produce infinite symmetry-breaking corrections. However, it often happens that the zeroth-order fermion masses satisfy simple relations like \( m_\phi = m_\eta \) for all choices of the parameters in the Lagrangian, or at least all choices in a finite range. In this case the corresponding symmetries such as isospin spin are taken seriously as "natural" symmetries of the strong interactions, and the higher-order corrections are finite, as we shall see below. It is also possible that approximate symmetries could arise in this way—for instance, \( m_\phi \) and \( m_\eta \) might, by accident, be somewhat smaller than \( m_\lambda \), leading to an approximate SU(2) \( \otimes \) SU(2) strong-interaction symmetry. In this case the higher-order corrections to this symmetry due to weak and electromagnetic interactions would be infinite, except in the approximation of perfect zeroth-order symmetry.

Any of these global symmetries of the strong interactions may be spontaneously broken by purely dynamical effects, along the lines suggested long ago by Nambu and Jona-Lasinio. There will then appear massless "pseudo-Goldstone" bosons in the theory, which are not eliminated by the Higgs mechanism, but pick up finite masses from higher-order effects of the weak and electromagnetic interactions. We rely on experiment to tell us that parity, strangeness, and isospin are not spontaneously broken in this way, and that zeroth-order chiral symmetries, if any, must be spontaneously broken, producing pseudoscalar bosons with van-
lishing zeroth-order masses. Also, any accidental approximate zeroth-order symmetries of the strong interactions may also be subject to a strong spontaneous breakdown, producing pseudo-Goldstone bosons of small but nonzero zeroth-order mass. It is not clear which picture describes the physical pion.

The problem of spontaneous breakdown of natural zeroth-order symmetries will be dealt with in detail in the next paper of this series. It is shown there that such spontaneous symmetry breaking does not affect the Wilson coefficient functions that concern us here. In the present work, we therefore can and will leave open the possibility that the natural zeroth-order symmetries are spontaneously broken by dynamical effects of the strong interactions.

IV. BREAKING STRONG GAUGE INVARIANCE

Now let us return to the gauge symmetry $G_8$. As matters stand at this point, the theory described by (3.1) contains a multiplet of strongly interacting vector bosons of zero mass. This certainly does not sound like the real world, so we have to add some explanations. There seem to be just three possibilities:

(1) We might add scalar fields to the Lagrangian, which are non-neutral under $G_8$, and whose vacuum expectation values break $G_8$ and give the vector gluons a mass. The Higgs phenomenon would as usual prevent the appearance of massless Goldstone bosons. We can avoid any interference between the presence of these scalar fields and the conservation of parity and other "natural" symmetries of the class described by (3.4)-(3.7) by the simple expedient of choosing these scalar fields to belong to representations of $G_8$ such that Yukawa couplings to the fermions are forbidden. Then parity, strangeness, etc. will be automatically conserved by the strong interactions in any renormalizable theory, provided we give the scalar fields positive parity and zero strangeness, etc. We can also avoid mass-mixing of the $G_8$ and $G_6$ gauge bosons by simply choosing these scalar fields to be neutral under $G_6$, though it is not clear that this is necessary. However, a much more serious difficulty arises if we want our theory to be "asymptotically free" in the sense of Ref. 5. It appears that if we add enough scalar fields to the Lagrangian to break $G_8$ completely, so that there are no massless gluons, then the quartic interaction among these fields would prevent a free-field asymptotic behavior. (Note that any scalar fields non-neutral under $G_8$ will have strong interactions with the vector-gluon fields so that strict renormalizability would require them to interact strongly with each other.) Even apart from the question of asymptotic freedom, the spontaneous breaking of $G_8$ raises serious problems of physical interpretation. In order to give the vector gluons reasonably large masses, it is necessary that the vacuum expectation values of the scalar fields be fairly large, so the gauge symmetry $G_8$ is strongly spontaneously broken. It is then a mystery why observed physical particles should be classifiable according to simple representations of $G_8$, like the color singlets of Ref. 17.

(2) The symmetry $G_8$ might be spontaneously broken by purely dynamical effects, without the need to introduce elementary scalar fields in the Lagrangian. Again, the Higgs phenomenon would have to eliminate the Goldstone bosons. However, even if this is possible it still leaves us with the problem of classifying physical hadron states, just as in (1).

(3) There is an entirely different and extremely attractive possibility that the strong gauge invariance is not broken at all. The vector gluons would then remain massless, and the spectrum of physical hadron states would exhibit $G_8$ as an exact symmetry. For instance, in the colored quark model, there would be an octet of massless neutral vector gluons, and the physical quark masses would be color-degenerate. At first glance this proposal appears absurd. All our experience teaches us that there are no massless hadrons, and no exact non-Abelian symmetries of the hadron states are known. However, we may be able to find a path past these obstacles by considering them in conjunction with another famous puzzle, the apparent absence of free quarks. Suppose that all ordinary hadrons observed so far are $G_8$-singlet bound states of quarks and/or gluons, and that there is some systematic reason why $G_8$ non-singlets cannot in principle be produced in collisions of $G_8$-singlet particles. Then $G_8$ invariance would have no observable consequences for the spectrum of the known particles, and the quarks and massless vector gluons, being $G_8$ non-singlets, could not be produced in any reaction. But why should it be impossible to produce $G_8$ non-singlets in ordinary reactions? The massless vector gluons themselves provide a possible answer. It is well known that the infrared divergences for massless Yang-Mills theories cannot be summed by the usual Bloch-Nordsieck techniques. It is also well known that these infrared divergences do not appear in reactions among gauge-neutral particles, which for $G_8$ would include all ordinary hadron reactions. It seems at least possible then that in reactions involving nonsinglet particles, the infrared divergences add up to give a factor $\exp(-\alpha)$ multiplying the S matrix, thus preventing...
the reaction. However, this way of stating the problem is too simple. Indeed, in quantum electrodynamics it is well known that the cross section for reactions like $\gamma + \gamma \to e^+ e^-$ are zero if summed to all orders in $g$, because the infrared divergences produce just such a factor $\exp(-\alpha)$. However, we do not conclude from this that pair production is impossible in photon-photon collisions—rather, we conclude that pair production must be accompanied by an indefinite number of very soft photons. The real question then is whether the rate vanishes for production of a definite number of hard massless vector gluons and quarks plus an indefinite number of very soft gluons in collisions of $G_z$-singlet particles. At first sight, it might be thought that this question is answered in the negative by the theorems of Kinoshita and Lee and Nauenberg, which state that infrared divergences cancel in all such total rates in each order of perturbation theory. However, there are reasons for doubt that these theorems apply here. For one thing, these authors introduce an infrared cutoff, such as a finite particle mass, and then study the behavior of the rates as this cutoff is removed, while in non-Abelian gauge theories it seems impossible to introduce an infrared cutoff without radically changing the physical content of the theory. (Introducing a gluon mass by hand or simply cutting off the integrals would spoil the gauge invariance of the theory, so that either renormalizability or unitarity would be lost, depending on the gauge. Introducing a gluon mass through the Higgs phenomenon would preserve gauge invariance, unitarity, and renormalizability, but then when the parameters in the scalar field Lagrangian were adjusted so as to turn off the vector-gluon mass, there would appear scalar as well as vector particles of zero mass.) Even more important, there are definite reasons for believing that perturbation theory is inapplicable to this problem in asymptotically free theories. In such theories there is an effective gauge coupling constant $g(\kappa)$, which describes the characteristic strength of the gauge coupling at momenta of order $\kappa$, and which vanishes as $\kappa \to 0$. For $\kappa \to 0$, the effective coupling can either approach a finite value, expected to be of order unity, or it can grow indefinitely, approaching infinity at $\kappa = 0$. In either case, perturbation theory would seem entirely incapable of dealing with the infrared divergences in asymptotically free theories. In particular, the second case, where $g(0) = \infty$, suggests a simple physical picture of why the naive quark model (including premature scaling) should work so well, even though it is impossible to produce free quarks or gluons. According to this picture, the effective coupling $g(\kappa)$ is not very strong when $\kappa$ is of the order of typical hadron masses, so that ordinary hadrons are rather loosely bound structures of quarks, and gluons, and should be describable by simple bound-state models. However, when an attempt is made to remove a quark or gluon, the effective coupling $g(\kappa)$ must be taken with $1/\kappa$ of the order of the distance that the quark or gluon has traveled away from its fellow, so that the binding force would become increasingly large the further away any quark or gluon were pulled. In this picture, premature scaling works well because $g(\kappa)$ not only approaches zero as $\kappa \to \infty$, but it is not very large even at a few GeV. Of course, all these qualitative remarks need to be supported by a detailed analysis presumably based on renormalization-group methods, showing how to sum graphs in non-Abelian gauge theories in the limit of large distances. Work on this problem is in progress.

In what follows, the question of how and whether $G_2$ is broken will be simply left open. As will be seen in the next paper of this series, the properties of the Wilson functions are unaffected by a possible spontaneous breakdown of $G_2$.

V. STRONG RENORMALIZATION

Before going on to consider the effects of weak and electromagnetic corrections, it will be necessary for us to specify the renormalization prescriptions to be used in dealing with the strong interactions. For our present purposes, it turns out to be extremely convenient to make use of a recently suggested "zero-mass renormalization procedure," which allows an easy derivation of asymptotic expansions at large momenta.

According to this renormalization procedure, we define all counterterms in terms of Feynman graphs evaluated with all masses set equal to zero and, to avoid infrared divergences, with all external momenta given nonzero values parameterized by an arbitrary scale parameter $\sigma$, called $\mu$ in Ref. 13. (In contrast, in Ref. 1 the renormalization counterterms were evaluated for zero fermion mass but nonzero neutral vector-gluon mass.) In particular, the gauge coupling constant is defined in terms of $\psi\psi$ or $SS$ vertices evaluated at zero mass and with external momenta proportional to $\mu$, and the field renormalization constants are defined in terms of proper self-energy parts evaluated at zero mass and with external momentum $p^2 = \sigma^2$, just as in a strictly massless gauge theory.

However, we really are interested here in a theory with massive fermions, so it is necessary to make special arrangements for mass renormalization. For this purpose we introduce a divergent
matrix $\delta \bar{\psi}$ by specifying that the renormalized bilinear product of two fermion fields should take the form
\[ \langle \bar{\psi}_{a,N} \psi_{m,N} \rangle_R = \langle \delta \bar{\psi} \rangle_{n,N,n,N'} \langle \bar{\psi}_{n,N'} \psi_{m,N'} \rangle. \] (5.1)

That is, $\delta \bar{\psi}$ is to be chosen so that all matrix elements of this renormalized operator should be finite. Using simple power-counting arguments, we may observe that $\delta \bar{\psi}$ insertions produce divergences in general Green's functions only when the insertion is made into a fermion self-energy part, in which case the divergence is only logarithmic, and therefore mass- and momentum-independent. Hence it is sufficient to require that the operator (5.1) should have finite matrix elements when inserted into a fermion self-energy part with any given fermion momentum and mass. This of course still leaves a great deal of freedom in our choice of the $\delta \bar{\psi}$ matrix, and so in accordance with our "zero-mass renormalization procedure," we define $\delta \bar{\psi}$ so that
\[ [\Gamma'_{n',m'}(\sigma^2; \bar{\psi}_{n,N} \psi_{m,N})]_{n=0} = \langle \delta \bar{\psi}^{-1} \rangle_{n,N,n',m'}. \] (5.2)

where $\Gamma_{n',m'}(\sigma^2; \bar{\psi}_{n,N} \psi_{m,N})$ is the total Green's function (including square roots of external-line propagators) for an incoming $\bar{\psi}_{n,N}$ line of momentum $p_0$, an outgoing $\psi_{m,N}$ line of momentum $p_1$, and a single zero-momentum $V$-vertex insertion. It follows immediately that the corresponding Green's function for the renormalized operator (5.1) is
\[ [\Gamma_{n',m'}(\sigma^2; \bar{\psi}_{n,N} \psi_{m,N})]_{n=0} = \langle \delta \bar{\psi}^{-1} \rangle_{n,N,n',m'}. \]

and since this is finite, all matrix elements of (5.1) are finite. Having defined the $\delta \bar{\psi}$ matrix, we may now define the renormalized fermion mass matrix as
\[ (m_R)_{n,N,n',m'} = \langle \delta \bar{\psi}^{-1} \rangle_{n,N,n',m'}(m)_{n,N,n',m'}. \] (5.3)
so that, suppressing indices,
\[ \bar{\psi} m \bar{\psi} = \langle \delta \bar{\psi} m_R \bar{\psi} \rangle_R. \] (5.4)

The elements of $m_R$ are not directly related to the poles in the fermion propagator (except in lowest order), but they do constitute a set of finite numbers which, together with the renormalized gauge coupling constants, serve to parameterize our theory.

It is fortunate that the symmetries of the strong-interaction Lagrangian (3.1) lead to great simplifications in the structure of the important $\delta \bar{\psi}$ matrix. The strong interactions do not produce transitions between different fermion rows, nor do they distinguish different fermion rows except through their mass values. (Recall that the $\phi_N$ are assumed for each $n$ to furnish the same irreducible representation of $G_3$.) Since we evaluate $\langle \delta \bar{\psi} \rangle$ with $m = 0$, we see immediately that
\[ \langle \delta \bar{\psi} \rangle_{n,N,m'} = 0 \delta_{N,N'} \delta_{m,m'}. \] (5.5)

In practice, we will always be interested here in contracted bilinear products of the form $(\bar{\psi} X \psi)$, where $X$ is a $G_3$-invariant matrix, and hence takes the form
\[ (X)_{n,n'} = \delta_{n,n'} X_{nm}. \]

Thus the product $\bar{\psi} X \psi$ is a $G_3$-invariant matrix $X$. In particular, the renormalized mass matrix $m_R$ is $G_3$-invariant, so
\[ \langle \bar{\psi} m_R \bar{\psi} \rangle = Z_R \delta \bar{\psi} \delta \bar{\psi} \] (5.6)

with $Z_R$ a single real divergent number, not a matrix. The renormalized bilinear products therefore take the extremely simple form
\[ \langle \bar{\psi} X \psi \rangle_R = Z_R \langle \bar{\psi} \rangle \bar{\psi} \] (5.7)

with the same divergent factor $Z_R$, for all $G_3$-invariant matrices $X$. In particular, the renormalized mass matrix $m_R$ is $G_3$-invariant, so
\[ \langle \bar{\psi} m_R \bar{\psi} \rangle_R = Z_R \langle \bar{\psi} m_R \bar{\psi} \rangle_R, \]

and comparing with (5.4), we have then our rule for mass renormalization
\[ m_R = Z_R \delta \bar{\psi}^{-1}. \] (5.8)

The fact that this relation is a simple proportionality implies that any linear relation among the unrenormalized masses will also apply to the renormalized masses.33 In particular, the $m_R$ exhibit the "natural" zeroth-order symmetries of the strong interactions as well as $m$. This would of course not be the case in general if we defined the renormalized masses in the conventional way, as the location of poles in fermion propagators. We shall see other advantages of this "zero-mass renormalization procedure" as we go along.

VI. SYMMETRY BREAKING IN ORDER $\epsilon^2$

Now let us consider the calculation of the weak and electromagnetic correction $\delta S_{\text{P}}$ to the $S$-matrix element for a transition from a general hadron "in" state $I$ to a general hadron "out" state $F$. The strong interactions will be taken into account in $\delta S_{\text{P}}$, to all orders, but the weak and electromagnetic interactions will be included only to second order in $\epsilon$.

The general outlines of this calculation were laid out in Ref. 1, and hardly any changes will be needed here. Just as in Ref. 1, the only weak and electromagnetic interactions in the original Lagran-
gian (2.14) that concern us here are the interaction of the gauge field \( W_{\alpha\gamma} \) (previously called \( A_{\alpha\gamma} \)) and the scalar fields \( \phi_i \) with hadrons

\[
J_{\alpha\gamma} W_{\alpha\gamma} + S_i \phi_i, \tag{6.1}
\]

where

\[
J^\mu_{\alpha\gamma} = -i \gamma^\mu t_{\alpha\gamma} \phi_i, \tag{6.2}
\]

\[
S_i = -\overline{\phi}_i \Gamma_i \phi_i, \tag{6.3}
\]

plus terms of first-order in \( e \) which can produce \( \phi' \) tadpole graphs. We work in a strong-interaction Heisenberg representation based on the effective Lagrangian (3.1)-(3.3), so that the fermion fields obey the strong-interaction field equation

\[
(\gamma^\mu \partial_\mu + i T_A \gamma^5 S_{\mu\nu} + m) \psi = 0. \tag{6.4}
\]

Recalling that \( T_A \) commutes with \( \gamma_\mu, \Gamma_i, \) and \( t_\alpha \), it follows that \( J^\mu_{\alpha\gamma} \) and \( S_i \) obey the commutation relations

\[
[\gamma^\mu \gamma^\nu, t_{\alpha\beta}] = -i \gamma^\mu \gamma^\nu [t_{\alpha\beta}, \gamma^5], \tag{6.5}
\]

\[
[\gamma^\mu \gamma^\nu, s_i, \gamma^5] = -i \gamma^\mu \gamma^\nu [s_i, \gamma^5], \tag{6.6}
\]

\[
[\gamma^\mu \gamma^\nu, \gamma^5] = -i \gamma^\mu \gamma^\nu, \tag{6.7}
\]

(We assume that the theory has been constructed so that all Adler-Bell-Jackiw anomalies cancel.) In addition, \( J^\mu_{\alpha\gamma} \) and \( S_i \) obey the commutation relations

\[
[J^\mu_{\alpha\gamma}(x, t), J^\nu_{\beta\delta}(\vec{x}, \vec{t})] = -i \delta^\mu(\vec{x} - \vec{y}) e_{\alpha\beta\gamma\delta}, \tag{6.8}
\]

\[
[S_i(x, t), S_j(\vec{y}, \vec{t})] = \delta^5(\vec{x} - \vec{y}) \delta_{ij} S_J(\vec{y}, \vec{t}), \tag{6.9}
\]

where the \( \delta \) denotes \( c \)-number Schwinger terms. Equations (6.5)-(6.7) are just the same as in Ref. 1, so the calculation presented in Sec. III of that paper goes through in precisely the same way here. After cancellation of all gauge-dependent terms, the correction \( \delta S_{\mu\nu} \) of second order in \( e \) is found to be a sum of five gauge-invariant terms:

\[
\delta S_{\mu\nu} = \delta_{AB} S_{\mu
u} + \delta_{c} S_{\mu
u} + \delta_{A} S_{\mu
u} + \delta_{B} S_{\mu
u} + \delta_{T} S_{\mu
u}, \tag{6.10}
\]

\[
\delta_{AB} S_{\mu
u} = (2\pi)^6 \delta^{(5)}(P_F - P_T) \int d^4 k \langle \phi, \mu \rangle \langle \phi, -\mu \rangle \int d^4 k \langle \phi, \mu \rangle \langle \phi, -\mu \rangle, \tag{6.11}
\]

\[
\delta_{c} S_{\mu
u} = (2\pi)^6 \delta^{(5)}(P_F - P_T) \langle \phi, \mu \rangle \langle \phi, -\mu \rangle \int d^4 k \langle \phi, \mu \rangle \langle \phi, -\mu \rangle, \tag{6.12}
\]

\[
\delta_{A} S_{\mu
u} = (2\pi)^6 \delta^{(5)}(P_F - P_T) \langle \phi, \mu \rangle \langle \phi, -\mu \rangle \int d^4 k \langle \phi, \mu \rangle \langle \phi, -\mu \rangle, \tag{6.13}
\]

\[
\delta_{B} S_{\mu
u} = (2\pi)^6 \delta^{(5)}(P_F - P_T) \langle \phi, \mu \rangle \langle \phi, -\mu \rangle \int d^4 k \langle \phi, \mu \rangle \langle \phi, -\mu \rangle, \tag{6.14}
\]

\[
\delta_{T} S_{\mu
u} = (2\pi)^6 \delta^{(5)}(P_F - P_T) \langle \phi, \mu \rangle \langle \phi, -\mu \rangle \int d^4 k \langle \phi, \mu \rangle \langle \phi, -\mu \rangle, \tag{6.15}
\]

and \( M \) and \( f \) are the scalar-boson mass matrix and trilinear interaction coefficient:

\[
M^2_{i\lambda} = \frac{\partial^3 P(\phi)}{\partial \phi_{i\lambda} \partial \phi_{i\lambda}} \bigg|_{\phi = \lambda}, \tag{6.16}
\]

\[
f_{i\lambda} = \frac{\partial^3 P(\phi)}{\partial \phi_{i\lambda} \partial \phi_{i\lambda} \partial \phi_{i\lambda}} \bigg|_{\phi = \lambda}. \tag{6.17}
\]

VII. ASYMPTOTIC EXPANSION OF THE MATRIX ELEMENTS

We must now examine the asymptotic behavior as \( \kappa \to \infty \) of the matrix elements (6.14) and (6.15). The tool we shall use is the Wilson operator-product expansion, which seems ideally suited to this purpose. In this section and Sec. VIII, we will rely on perturbation theory to estimate the asymptotic behavior of the Wilson coefficient functions, without worrying about whether powers of logarithms might add up when the perturbation series is summed in such a way as to change our results. Our reasoning will again follow pretty closely that of Ref. 1, with some differences due to both changes in the field content of the theory and improvements in the argument. Later, in Sec. IX, we will reconsider the problem more carefully, using the powerful apparatus of the renormalization group.
Suppose we take \( k \) in Eqs. (6.14) and (6.15) to be a Euclidean momentum four-vector
\[ k^0 = i k_t, \quad k_t \text{ real,} \]
and let the Euclidean modulus \( k \) go to infinity with \( k^0/k \) fixed, where
\[ \kappa = \left( (k_k)^2 + (k_\mu)^2 + (k_q)^2 + (k_{\mu q})^2 \right)^{1/2}. \]
In this limit, the direction-averaged and Lorentz-index- contracted matrix elements have the asymptotic behavior
\[
\eta^\mu \int d\Omega_\kappa \, \mathcal{F}_\mu (k) = - \sum_x \langle F \mid O_{\alpha B}^\mu \mid I \rangle \mathcal{V}_{\alpha B}^\mu (k),
\]
\[
\int d\Omega_\kappa \, \mathcal{C}_\mu (k) = - \sum_x \langle F \mid O_{\alpha B}^\mu \mid I \rangle \mathcal{V}_{\alpha B}^\mu (k),
\]
where \( d\Omega_\kappa \) is the element of solid angle in four-dimensions (with \( d\Omega_\kappa \) equal to \( 2\pi^2 \)); the \( O_{\alpha B}^\mu \) are renormalized local operators (that is, products of elementary fields and derivatives of fields, with infinite \( Z \) factors supplied where necessary to make the matrix elements finite); and the \( U \) and \( V \) are finite \( c \)-number functions of \( \kappa \). (It should be noted that the currents \( J_\mu^\kappa \) and \( S_i \) are already renormalized operators, because the matrices \( i_\alpha \) and \( T_i \) are proportional to unrenormalized coupling constants, which contain the counterterms needed to make matrix elements finite.) In perturbation theory, the asymptotic behavior of the coefficient functions is given by
\[
U_{\alpha \mu}^\kappa (k) = O(\kappa^{\alpha \mu} \times \text{powers of } \ln \kappa),
\]
\[
\mathcal{V}_{\alpha \mu}^\kappa (k) = O(\kappa^{\alpha \mu} \times \text{powers of } \ln \kappa),
\]
\[
\kappa = 2 - \frac{1}{2} F_N - G_N - D_N,
\]
where \( F_N, G_N \) and \( D_N \) are, respectively, the number of fermion fields, gluon fields, and derivatives of fields appearing in the operator \( O_\mu \). We shall consider in Sec. IX how the infinite series in \( \ln \kappa \) appearing in (7.5) and (7.6) can be summed, but for the moment the perturbative estimates (7.5)–(7.7) are good enough.

The operators \( O_{\alpha B}^\mu \) appearing in (7.3) and (7.4) are \( G_\kappa \)-invariant (because \( J_{\alpha B} \) and \( S_i \) are \( G_\kappa \)-invariant operators) and Lorentz-invariant (because we average over space-time directions and contract four-vector indices). This course leaves an infinite number of possible terms in (7.3) and (7.4), but inspection of Eqs. (6.9)–(6.11) suggests that the interesting terms are those with \( \alpha_\mu = -2 \), because these are the only terms which could potentially make divergent contributions to \( \delta S_{F_T} \), or, after cancellation of divergences, which could make contributions to \( \delta S_{F_T} \) proportional to \( \ln \mu^2 \) rather than \( 1/\mu^2 \). If we therefore wish to keep only those terms in the asymptotic expansions with \( \alpha_\mu \geq -2 \), we must take into account only those Lorentz- and \( G_\kappa \)-invariant operators satisfying the condition
\[
4 - \frac{3}{2} F_N - G_N - D_N \geq 0.
\]
A complete list of such operators is easily compiled. There are just three entries
\[
\langle \bar{\psi} X^{(1)} \psi \rangle,
\]
\[
\langle \bar{\psi} \gamma^\mu X^{(2)} (i \gamma_5) S_A \psi \rangle,
\]
\[
X_{AB}^{(3)} G_{AB}^{\mu \nu} G_{\mu \nu}^{\nu},
\]
where the \( X \)'s are arbitrary matrices satisfying the \( G_\kappa \) invariance conditions
\[
[T_{AB} X^{(1)}] = 0,
\]
\[
[T_{AB} X^{(2)}] = 0,
\]
\[
C_{ABD} X^{(3)} - C_{A\kappa \mu \kappa} X^{(4)} = 0.
\]
(Lorentz invariance restricts the Dirac matrices that can appear in \( X^{(1)} \) and \( X^{(2)} \) to just 1 and \( \gamma_5 \).)

In compiling this list we exclude operators like \( \epsilon_{\mu \nu \lambda \kappa} G_{\mu \nu}^{\kappa} G_{\mu \nu}^{\kappa} \) which take the form of a four-dimensional divergence and therefore vanish between states \( I, F \) of equal four-momentum.

Now, the \( G_\kappa \) term (7.11) is totally uninteresting, because it is not only \( G_\kappa \)-invariant, but also obviously conserves parity and conserves any global symmetry of the type described by Eqs. (3.4)–(3.7). We are only interested in calculating terms in \( \delta S_{F_T} \) which break these symmetries, so we can drop operators of the type (7.11) from further consideration. In addition, we can also drop operators of the type (7.10), because by using the field equation (6.4) any such term can be written in terms of operators of the type (7.9). Thus we conclude that it is only the operators of the type (7.9) that need to be kept in the Wilson expansion here. (The same conclusion was reached in Ref. 1 through a more laborious line of reasoning.) In effect, then, (7.3) and (7.4) become
\[
\eta^\mu \int d\Omega_\kappa \, \mathcal{F}_\mu (k) = - \langle F \mid \bar{\psi} U_{AB}(k) \psi \mid I \rangle,
\]
\[
\int d\Omega_\kappa \, \mathcal{C}_\mu (k) = - \langle F \mid \bar{\psi} V_{AB}(k) \psi \mid I \rangle,
\]
where \( U_{AB}(k) \) and \( V_{AB}(k) \) are finite matrix functions with suppressed indices running over the various fermion field types, and may involve terms proportional to the Dirac matrices 1 and \( \gamma_5 \). They satisfy the \( G_\kappa \) invariance conditions
\[
[T_{AB} U_{AB}] = [T_{AB} V_{AB}] = 0
\]
as well as the reality and crossing-symmetry conditions.
\[ U_{ab} = \gamma_4 U_{ab}^* \gamma_4 = U_{ba}, \]
(7.15)
\[ V_{ij} = \gamma_4 V_{ij} \gamma_4 = V_{ji}. \]
(7.16)

The label \( R \) on the right-hand side of (7.12) and (7.13) indicates that infinite \( Z \) factors are supplied where needed to make all matrix elements of the operators finite. As shown in Sec. V, the \( G_z \) invariance of the matrices \( U_{ab} \) and \( V_{ij} \) implies that the bilinear products in (7.3) and (7.4) are renormalized by a single multiplicative factor \( Z_{\psi \phi} \):
\[ (\bar{\psi} U_{ab} \psi)_R = Z_{\psi \phi} (\bar{\psi} U_{ab} \psi), \]
(7.17)
\[ (\bar{\psi} V_{ij} \phi)_R = Z_{\psi \phi} (\bar{\psi} V_{ij} \phi). \]
(7.18)

(In Ref. 1 this \( Z \) factor was absorbed into the \( U \) and \( V \) matrices.)

As matters stand at this point, the \( U \) and \( V \) matrices depend on the zeroth-order fermion mass matrix \( m \), so it is not immediately possible to make any simple statement about their \( G_\phi \)-transformation properties. To analyze the \( m \)-dependence of \( U \) and \( V \), we note that differentiation with respect to \( m \) lowers the asymptotic behavior of the Wilson functions in each order of perturbation theory by one power of \( k \), and that these functions are finite and have finite first and second derivatives at \( m = 0 \). (These properties apply here only because we use the "zero-mass renormalization procedure" described in Sec. V; otherwise they would be spoiled by the mass dependence of the renormalization counterterms.) We can write each Wilson function as the sum of its value at \( m = 0 \) plus an integral of its first derivative, and would therefore normally expect them to approach their \( m = 0 \) values as \( k \to \infty \). However, by doing a little simple \( \gamma_4 \) bookkeeping, it is easy to see that \( U_{ab} \) and \( V_{ij} \) must be odd in \( m \), and therefore vanish as \( m = 0 \). (Let us call a matrix \( M_{\gamma_4} \)-even or \( \gamma_4 \)-odd according as \( \gamma_4 \) commutes or anticommutes with \( \gamma_4 M \). The Feynman graphs for \( U_{ab} \) only contain the \( \gamma_4 \)-even vertex factors \( \gamma_4 t_a \) and \( \gamma_4 T_A \), while the graphs for \( V_{ij} \) also involve an even number of the \( \gamma_4 \)-odd vertices \( T_i \). However, \( U_{ab} \) and \( V_{ij} \) are \( \gamma_4 \)-odd, because Lorentz invariance allows them to include terms proportional to only the Dirac matrices \( 1 \) and \( \gamma_4 \). Hence they must both be odd in the \( \gamma_4 \)-odd part of the fermion propagator, i.e., in the fermion mass.) Since the \( m = 0 \) terms are missing in \( U_{ab} \) and \( V_{ij} \), these functions can be written as the sum of terms of first order in \( m \), plus integrals of second derivatives with respect to \( m \). We conclude that \( U_{ab} \) and \( V_{ij} \), approach terms of first order in \( m \) as \( k \to \infty \), with asymptotic behavior reduced from (7.5)–(7.7) by one power of \( k \):
\[ U_{ab}(k) = O(k^{-2} \times \text{powers of } \ln k), \]
(7.19)
\[ V_{ij}(k) = O(k^{-2} \times \text{powers of } \ln k), \]
(7.20)

Returning now to the question of the \( G_\phi \)-transformation properties of the Wilson functions, we note that \( G_\phi \) is broken in the Lagrangian (3.1) only by the term \( \Gamma_4 \lambda_4 \) in \( m \). Since \( U_{ab} \) and \( V_{ij} \) become of first order in \( m \) for \( k \to \infty \), they are asymptotically linear in \( \lambda_4 \):
\[ U_{ab}(k) \sim U_{ab}^0(k) + U_{ab}^0(k) \lambda_4, \]
(7.21)
\[ V_{ij}(k) \sim V_{ij0}(k) + V_{ij0}(k) \lambda_4, \]
(7.22)

with \( G_\phi \)-covariant coefficients.

These results can be used to demonstrate the cancellation of all divergences in weak and electromagnetic corrections to natural strong-interaction symmetries, using precisely the same line of reasoning as in Ref. 1. There is no need to repeat these arguments here; our discussion of the order-\( \alpha \) terms in Sec. VIII will remind the reader of how they go.

**VIII. WEAK CORRECTIONS OF ORDER \( \alpha \)**

One of the most intriguing outgrowths of the unification of weak and electromagnetic interactions is the observation\(^1,2\) that weak interactions can produce corrections to some (but not all) natural strong-interaction symmetries, of the same order of magnitude as the electromagnetic corrections. Such "order-\( \alpha \) corrections" have already been discussed in detail in Ref. 1; we shall here repeat the same discussion in an abbreviated and slightly improved form.

Let us go through Eqs. (6.8)–(6.13) to pick out those corrections of second order in \( \epsilon \) which are of order \( \alpha \) rather than of order \( \alpha/\mu_\omega^2 \). The A1 term will be saved for last, as it is the most interesting. The \( \phi 1 \) term makes no contribution of order \( \alpha \), because it involves the scalar current \( S_1 \), and (2.36) shows that it is therefore suppressed by a factor \((m/\mu_\omega)^2\). The \( A \phi \) term is suppressed by the same factor, because \( S_{il}(k) \) in (6.11) vanishes like \( 1/k^4 \) as \( k \to \infty \), so that the denominator \((k^2 + \mu_\omega^2)^{-1}\) effectively yields a factor \( \mu_\omega^{-2} \). The \( AT \) term is immediately seen on inspection of (6.12) to produce corrections to the strong-interaction \( S \) matrix, which are the same as would be produced by a change in the zeroth-order fermion mass matrix
\[ \delta_{AT} m = \frac{1}{2\epsilon^2} \Gamma_4 (\theta_\phi^\dagger \theta_\phi) \int_0^\infty (k^2 + \mu_\omega^2)^{-1} a_\alpha k^2 \, dk. \]

The divergent part of this expression is shown in Ref. 1 to give a contribution equivalent to a change in the Yukawa coupling constants in \( \Gamma_\omega \), so it cannot produce corrections to natural zeroth-order symmetries. The finite remainder gives
\[
\delta_{AT} = \frac{1}{32\pi^2} \Gamma_{\alpha \beta}(\partial_\alpha \partial_\beta)(\ln \mu^2)_{\alpha \beta}.
\]

This is of order \(\alpha m\), because \(\partial\) is of order \(e\) and \(\Gamma\) is of order \(m\). The \(T1\) term (6.13) also gives order-\(\alpha\) contributions, provided the scalar mass matrix \(M\) is not much larger than \(\mu_w\). We can drop the \((S\alpha)\) term in (6.13), because it is smaller than the other terms by a factor \((m/\mu_w)^2\), so that the order-\(\alpha\) corrections produced by the \(T1\) term is equivalent to a change in the fermion mass matrix

\[
\delta_{T} = -\frac{1}{32\pi^2} \Gamma_{\alpha \beta} M^{-1}_{\alpha \beta}
\]

\[
\times \left[ f_{BM} \int d^4k (k^2 + M^2)^{-1}_{\alpha \beta} + 6(\partial_\alpha \partial_\beta)(\mu^2 \ln \mu^2)_{\alpha \beta} \right].
\]

As shown in Ref. 1, the divergent part of this expression gives a contribution equivalent to a change in the parameters of the polynomial \(P(\phi)\), so it cannot produce corrections to natural zeroth-order symmetries. The finite remainder is

\[
\delta_{T} = -\frac{1}{32\pi^2} \Gamma_{\alpha \beta} M^{-1}_{\alpha \beta}
\]

\[
\times \left[ f_{BM} (M^2 \ln M^2)_{\alpha \beta} + 6(\partial_\alpha \partial_\beta)(\mu^2 \ln \mu^2)_{\alpha \beta} \right].
\]

(8.2)

In so far as corrections to natural zeroth-order symmetries are concerned, both (8.1) and (8.2) are independent of the mass units used to calculate the logarithms.

Now let us return to the \(\Lambda\) term (6.9). We isolate the photon, writing the propagator

\[
\left( \frac{1}{k^2 + \mu^2} \right)_{\alpha \beta} = \delta_{\alpha \beta} \delta_{\alpha \beta} \left( \frac{1}{k^2 + \Lambda^2} \right) + \left( \frac{1}{k^2 + \Lambda^2} \right)_{\alpha \beta},
\]

(8.3)

where \(\Lambda\) is a large but otherwise arbitrary mass; \(n_\alpha\) is the eigenvector of \(\mu^2\) corresponding to the photon

\[
(\mu^2)_{\alpha \beta} n_\beta = 0, \quad n_\alpha n_\alpha = 1
\]

and \(\mu^2\) is the positive definite matrix

\[
(\mu^2)_{\alpha \beta} = (\mu^2)_{\alpha \beta} + n_\alpha n_\beta \Lambda^2.
\]

The first term in (8.3) produces a correction

\[
\delta_{m} = \delta_{m} + \delta_{AT} m + \delta_{T} m.
\]

(8.4)

\[
\int d^4k \, T_{\alpha \beta} (k^2 + \mu^2)_{\alpha \beta},
\]

which is just the usual photon-exchange term, with a cutoff \(\Lambda\). The second term in (8.3) produces a

correction

\[
\delta_{m} = (2\pi)^4 \delta^{\mu}(P_P - P_P) n_\alpha n_\beta
\]

\[
\times \int d^4k \, T_{\alpha \beta} (k^2 + \mu^2)_{\alpha \beta}.
\]

(8.5)

All the elements of \(\mu^2\) are large because the photon is given a fictitious mass \(\Lambda\), so the only terms in (8.5) which make contributions of order \(\alpha\) are those arising from the terms in the \(\beta\)-matrix element which vanish as \(k \to 0\) no faster than \(1/k^2\).

These terms are given (taking account only of terms which can affect natural strong-interaction symmetries) by Eq. (7.12), and so produce corrections equivalent to a change in the zeroth-order fermion mass matrix

\[
\delta_{m} = Z_{\overline{\psi}} \int_0^\infty U_{\alpha \beta} (k^2 + \mu^2)_{\alpha \beta} \Lambda^2 dk.
\]

(8.6)

It was in Ref. 2 that the divergent part of this expression (aside from the factor \(Z_{\overline{\psi}}\)) gives a contribution equivalent to a change in the bare mass matrix \(m_\alpha\) and the Yukawa coupling constants in \(\Gamma\), and therefore has no effect on natural zeroth-order symmetries. The divergent factor \(Z_{\overline{\psi}}\) is needed to give \(\phi m_\psi\) finite matrix elements.

[Recall that (5.8) gives the ratio of unrenormalized to renormalized mass here as just \(Z_{\overline{\psi}}\).] Similar divergent factors are contributed in (8.1) and (8.2) by the unrenormalized Yukawa coupling constants \(\Gamma\).

Apart from the photon-exchange term (8.4), all order-\(\alpha\) corrections to natural zeroth-order symmetries are now seen to consist purely of corrections to the effective zeroth-order fermion mass matrix:

\[
\delta_{m} = \delta_{m} + \delta_{AT} m + \delta_{T} m.
\]

(8.7)

By applying the same sort of matrix transformation to \(\psi\) as in Sec. III, we can again reduce \(m + \delta_{m}\) to a real, diagonal \(\gamma_5\)-free matrix, so strangeness and parity conservation will be conserved in order \(\alpha\). On the other hand, relations like \(m_P = m_w\) need not be respected by the order-\(\alpha\) corrections. It has already been suggested in Ref. 1 that these order-\(\alpha\) effects of the weak interactions are responsible for the nonem permeable corrections to isotopic spin that seem to be present in \(\Delta I = 1\) mass differences and in \(\eta\) decay.

**IX. RENORMALIZATION-GROUP CALCULATION OF \(\delta_{m}\)**

Now let us ask how, if we are given some specific gauge model of strong, weak, and electromagnetic interactions, could we go about calculating the effective order-\(\alpha\) fermion mass shift (8.7)? Inspection of Eqs. (8.1) and (8.2) shows that the
AT and T 1 terms are immediately calculable in terms of “known” mass and interaction matrices, with no complications entering due to the strong interactions. This leaves the A' term (8.6), in which the strong interactions enter through the unknown Wilson coefficient function \( U_{\alpha\beta}(\kappa) \). We are only interested here in contributions to \( \delta m \) of order \( \alpha m \), not \( \alpha m^2/\mu^2 \), so we shall only need to consider the asymptotic behavior of this function as \( \kappa \rightarrow -\infty \).

For this purpose, we make use of a new set of renormalization-group equations derived recently by use of the “zero-mass” renormalization technique of Sec. V. As usual in renormalization-group calculations the results will involve an effective strong gauge coupling constant \( g(\kappa) \), defined by the conditions

\[
\frac{d}{d\kappa} \kappa_0 \gamma_{\kappa_0}(\kappa) = \beta_0 g(\kappa),
\]

\[
\beta_0 \gamma_{\kappa_0}(\kappa_0) = \gamma_{\kappa_0},
\]

where \( \kappa_0 \) is any arbitrary fixed \( \kappa \) value; \( \gamma_{\kappa_0} \) is the \( \alpha \)th renormalized gauge coupling constant; and \( \beta \) is a function defined by

\[
\beta_0 \gamma_{\kappa_0}(\kappa_0) = \frac{\partial}{\partial \kappa} \ln Z_{\kappa_0},
\]

where \( \sigma \) is the renormalization-point scale parameter of Sec. V (called \( \mu \) in Ref. 13), and the derivative is taken with constant unrenormalized coupling constants. In addition, our new formulas also involve a \( \kappa \)-dependent effective fermion mass matrix:

\[
m(\kappa) = m_R \exp \left\{ - \int_{\kappa_0}^{\kappa} \left[ 1 + \gamma_0 g(\kappa') \right] \frac{d\kappa'}{\kappa'} \right\},
\]

where \( \gamma_0 \) is a function defined by

\[
\gamma_0 g(\kappa') = \frac{\partial}{\partial \kappa} \ln Z_{\kappa_0},
\]

the derivative again being taken with constant unrenormalized coupling constants. If \( \Gamma_\kappa(\kappa; \gamma_{\kappa_0}, m_R, \sigma) \) is any renormalized Green’s function, in which all momenta are allowed to vary together with the same scale parameter \( \kappa \), then from the new renormalization-group equations we learn that (suppressing the \( \sigma \) argument)

\[
\Gamma_\kappa(\kappa; \gamma_{\kappa_0}, m_R) = (\kappa/\kappa_0)^{D_\kappa} \exp \left\{ - \int_{\kappa_0}^{\kappa} \gamma_0 g(\kappa') \frac{d\kappa'}{\kappa'} \right\} \times \Gamma_\kappa(\kappa_0; \gamma_{\kappa_0}, m_R),
\]

where \( D_\kappa \) is the dimensionality of \( \Gamma_\kappa \) (in the usual sense of dimensional analysis), and \( \gamma_0 \) is an “anomalous dimension” function, defined by

\[
\gamma_0 = \frac{\partial}{\partial \kappa} \ln Z_{\kappa},
\]

Here \( Z_\kappa \) is the cutoff-dependent factor relating the renormalized amplitude \( \Gamma_\kappa \) to the corresponding unrenormalized amplitude \( \Gamma R \).

\[
\Gamma_\kappa = Z_\kappa \Gamma R.
\]

(In general, \( Z_\kappa \) and \( \gamma_0 \) might be matrices connecting different Green’s functions, but this complication is spared us here.) The same applies to Wilson coefficient functions, except that \( \gamma_0 \) is replaced by an anomalous dimension associated with the currents, \( m \) minus an anomalous dimension associated with the operator appearing in the operator-product expansion.

Returning now to the coefficient function \( U_{\alpha\beta}(\kappa) \) of present interest, we note that there are no anomalous dimensions associated with the conserved or partially conserved currents \( J_{\alpha\beta} \) while the operator appearing in the Wilson expansion here is just \( \bar{\psi} \psi \), so the anomalous dimension appearing here in place of \( \gamma_0 \) is given by

\[
\gamma_U = -\gamma_0.
\]

Also, \( U \) has the dimensions of \( (\text{mass})^{-1} \) so

\[
D_U = -1.
\]

The formula for \( U_{\alpha\beta} \) corresponding to (9.6) thus reads

\[
U_{\alpha\beta}(\kappa; \gamma_{\kappa_0}, m_R) = (\kappa/\kappa_0)^{-1} \exp \left\{ - \int_{\kappa_0}^{\kappa} \gamma_0 g(\kappa') \frac{d\kappa'}{\kappa'} \right\} \times U_{\alpha\beta}(\kappa_0; \gamma_{\kappa_0}, m_R).
\]

Now let us consider the limit \( \kappa \rightarrow -\infty \). Equation (9.4) shows that as long as the anomalous dimension \( \gamma_0 \) stays above the value \( -1 \), the effective mass \( m(\kappa) \) will vanish as \( \kappa \rightarrow -\infty \). As indicated in Sec. VII, the Wilson functions are odd in \( m_R \) and differentiable with respect to \( m_R \) at \( m_R = 0 \). So for \( m_R = 0 \), we have

\[
U_{\alpha\beta} = m_R \left[ \frac{\partial U_{\alpha\beta}}{\partial m_R} \right]_{m_R = 0}.
\]

Thus Eq. (9.9) gives, for \( \kappa \rightarrow -\infty \),

\[
U_{\alpha\beta}(\kappa; \gamma_{\kappa_0}, m_R) = (\kappa/\kappa_0)^{-1} \exp \left\{ - \int_{\kappa_0}^{\kappa} \gamma_0 g(\kappa') \frac{d\kappa'}{\kappa'} \right\} \times m_R \left[ \frac{\partial}{\partial m_R} U_{\alpha\beta}(\kappa_0; \gamma_{\kappa_0}, m_R) \right]_{m_R = 0}.
\]

Referring back to Eq. (9.4), we see that the anomalous dimensions cancel, leaving us with the simple result
The cancellation occurs only because the operator $\bar{\psi}\psi$ appearing in the Wilson expansion is the same as the mass operator in the Lagrangian. For instance, the coefficient function for the operator $\bar{\psi}q^2\psi$ is also asymptotically of first order in the fermion mass, but the exponents here would not cancel.

Everything now depends on how $g(\kappa)$ behaves as $\kappa \to \infty$. As is well known, the solution of Eq. (9.1) either approaches infinity as $\kappa \to \infty$ or else approaches some zero $\kappa_\omega$ of the function $\beta$, perhaps $\kappa_\omega = 0$. If $g(\kappa)$ approaches infinity as $\kappa \to \infty$ there is little further we can say. On the other hand, if $g(\kappa)$ approaches $\kappa_\omega$ as $\kappa \to \infty$, and if $g(\kappa)$ is already close to $\kappa_\omega$ for $\kappa$ of order $\mu$, then we can calculate $\delta A(m)$ by using (9.10) in (8.6), with $g(\kappa)$ set equal to $\kappa_\omega$. This result is rather interesting in its own right; as shown in Ref. 1, if $U_{\alpha\beta}(\kappa)$ behaves like $\kappa^{-n}$ for large $\kappa$ then the effective mass shift is of the form

$$
U_{\alpha\beta}(\kappa; g_\mu, m_\mu) - (\kappa/\kappa_0)^{-n} m_{\mu R} \times \left[ \frac{\partial}{\partial \mu} U_{\alpha\beta}(\kappa; g(\kappa), m_{\mu R}) \right]_{\mu \to \infty}.
$$

This result was anticipated in Ref. 1, relying on arguments based on the observation of Bjorken scaling in inelastic electron scattering. Now we see that "asymptotic freedom," which is supposed to explain Bjorken scaling, also yields the result for order-$\alpha$ corrections to natural symmetries that was less directly suggested by Bjorken scaling.

In closing, let us return to the question of whether we were justified in our use of perturbation-theoretic estimate of asymptotic behavior in Secs. VII and VIII. From the firm ground of the renormalization-group approach, we can look back and see just three ways that we might have founndered:

(a) It might be that the effective coupling $g(\kappa)$ tends to infinity as $\kappa \to \infty$. In this case, there is no reason to believe any of the asymptotic estimates made here.

(b) It might be that the operator $\bar{\psi}\psi$ has such a large negative anomalous dimension at $g(\kappa)$ that the effective mass $m(\kappa)$ does not vanish as $\kappa \to \infty$ (see Eq. (9.4)). This would invalidate the Gell-Mann-Low and Callan-Symanzik asymptotic solutions, and although Eq. (9.9) would still be valid, it could not be used to estimate the asymptotic behavior of the Wilson function.

(c) It might be that other operators besides (7.9) have such large asymptotic dimensions at $\kappa = \infty$ that their corresponding Wilson functions fall off no more rapidly that $1/\kappa^2$ as $\kappa \to \infty$. In this case these operators would have to be included in the evaluation of the $\Lambda^1$ term, and it would be a mystery why parity and strangeness are conserved to order $\alpha$.

We see that our previous analysis is correct if $g(\kappa)$ approaches a constant $g_\omega$ as $\kappa \to \infty$, and if all anomalous dimensions at $g_\omega$ are sufficiently small in absolute value. Both conditions are satisfied in asymptotically free theories, where $g_\omega = 0$, and all anomalous dimensions at $g_\omega$ therefore vanish.

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1688 (1973); H. Georgi and S. L. Glashow, ibid. 7, 2457 (1973); S. Weinberg, ibid. 7, 2887 (1973); B. W. Lee (unpublished); A. Dacãñ (unpublished); P. Schattner (unpublished).


4D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); Phys. Rev. D 8, 3633 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); also see G. 't Hooft (unpublished). It has also been shown by S. Coleman and D. J. Gross [Phys. Rev. Lett. 31, 851 (1973)] that non-Abelian gauge theories are the only stable field theories that can be asymptotically free.


7A. De Rújula and S. L. Glashow (unpublished), and Ref. 1.

8I. Bars (unpublished).


10This suggestion was made independently in Ref. 6 and by D. J. Gross and F. Wilczek (unpublished), in the context of theories that are asymptotically free and hence infrared unstable. The same suggestion was made by H. Fritzsch, M. Gell-Mann, and H. Leutwyler (unpublished), but not connected with asymptotic freedom.


24In Ref. 17 it is also suggested that the known hadrons are all color singlets, and that quarks cannot be produced in collisions of such singlets. The extension of this idea to color octets of neutral vector gluons was made by H. Fritzsch and M. Gell–Mann in *Proceedings of the XVI International Conference on High Energy Physics, Chicago–Batavia, Ill.*, 1972, edited by D. Jackson and A. Roberts (NAL, Batavia, 1972) following an unpublished suggestion of J. Weiss. The role of infrared divergences in providing a mechanism for suppression of quark and gluon production was proposed by the authors cited in Ref. 11.

25Dynamical mechanisms that might prevent the production of free quarks have been suggested by K. Johnson (unpublished) and A. Casher, J. Kogut, and L. Susskind Phys. Rev. Lett. 31, 792 (1973). The difference here is that we seek the origin of this mechanism in the existence of massless gluons, and we rely on this mechanism to prevent the production of gluons as well as quarks.

26Strictly speaking, the gluons are all \( G_q \) nonsinglets if and only if \( G_q \) is semisimple in the strict sense of having no invariant Abelian subgroups, including U(1). It is interesting that these are the only theories which can be asymptotically free; see Ref. 5.


32For a discussion of why this condition is important, see Sec. II of Ref. 1.


35It has been emphasized by J. C. Pati and A. Salam (unpublished) that the problem of anomalies is less severe when the strong interactions are described by a non-Abelian gauge model, with generator matrices \( T_A \) having zero trace.
Reggeization of Elementary Particles in Renormalizable Gauge Theories: Vectors and Spinors

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The question of the Reggeization of the elementary particles in renormalizable gauge theories with the Higgs-Kibble mechanism is examined. It is concluded that the massive non-Abelian gauge vector mesons and $J = \frac{1}{2}$ fermions of these theories lie on Regge trajectories in every case in which a counting criterion due to Mandelstam is satisfied. Several explicit examples are presented which verify that the necessary factorization condition of the Born helicity matrices is satisfied at $J = 1$ and $J = \frac{1}{2}$, supporting this conclusion.

I. INTRODUCTION

The idea that the elementary particles of a Lagrangian field theory, which at first sight give rise to nonanalytic Kronecker $\delta$ terms in the complex angular momentum plane, in fact lie on Regge trajectories dates back more than ten years to a series of papers of Gell-Mann et al.1 Although their ideas initiated a sizable industry which specialized in summing Feynman diagrams,2 with the objectives of exhibiting possible Regge asymptotic behavior of the $S$-matrix elements and demonstrating that the elementary particles of the theory lie on these Regge trajectories, their program was to a large extent unsuccessful. The only positive result1 was the Reggeization of the fermion in a particular model with a spin-$\frac{1}{2}$ particle coupled to vector mesons by means of a conserved vector current. Subsequently, Mandelstam3 presented criteria to establish when an elementary particle must Reggeize, which indeed agreed with the conclusion that the fermion of the vector-spinor theory should lie on a Regge trajectory. His arguments also indicated that the elementary particles of other Lagrangian theories need not Reggeize, and explained the results of the theoretical experiments carried out by computing Feynman diagrams. The Mandelstam criteria are to a large extent kinematical and suggest where one should look for Reggeization. These criteria involve comparing, for the particular model being studied, the number of arbitrary parameters appearing in the scattering amplitudes with the number of constraints satisfied by these amplitudes. The dynamical criteria of the program are very general ones requiring that the theory lead to amplitudes which satisfy analyticity and unitarity; they appear to be related to the renormalizability of the Lagrangian field theory.

Indeed Teplicitz and collaborators4 10 have exhibited a number of models which satisfy Mandelstam’s counting criteria but fail to Reggeize because of the absence of renormalizability, which implies a violation of unitarity bounds in each order of perturbation theory. A particularly interesting example is the failure to Reggeize the gauge vector mesons of a (nonrenormalizable) massive Yang-Mills theory.

Since renormalizable theories of massive Yang-Mills fields are now known,7 we have reexamined the question of the Reggeization of gauge vector mesons. Our study was further motivated by the fact that the zero-slope limit of certain dual models8 leads to amplitudes which are identical to those constructed from particular renormalizable Yang-Mills Lagrangians. Although our work has not given us any special insight into this aspect of