I. INTRODUCTION AND SUMMARY

Models with spontaneously broken local symmetries have been suggested as a solution to two of the major problems of elementary-particle theory:

(a) the unification of the weak and electromagnetic interaction;

(b) the elimination of ultraviolet divergences appearing in higher-order effects of the weak interactions.

Recently there have been indications that such models may also elucidate one other outstanding problem:

(c) the explanation of the weak breaking of intrinsic symmetries such as isospin.

The purpose of the present paper is to provide a formal foundation for general theories with spontaneously broken local symmetries.

The formalism described here is based on choice of a particular gauge, the "unitarity gauge," in which the absence of Goldstone bosons and the order-by-order unitarity of these theories is manifest. This is the gauge which was originally used to show that, instead of Goldstone bosons appearing when local symmetries are spontaneously broken.
en, the Yang-Mills fields in the theory acquire a mass.\textsuperscript{4} Also, it is this gauge that was originally used to suggest the occurrence of this “Higgs-Kibble phenomenon” in a combined theory of weak and electromagnetic interactions.\textsuperscript{1} However, the renormalizability of the unitarity gauge is extremely obscure, arising as it does from miraculous cancellations among different diagrams.\textsuperscript{5} (The term “cryptorenormalizable” has been coined to describe such theories.) Indeed, the proof that these theories are renormalizable had to wait for four years after the original suggestion of renormalizability, when a new gauge, the “renormalizable” gauge, was invented by ’t Hooft\textsuperscript{8} and Lee.\textsuperscript{7}

Despite these considerations, there are a number of reasons for wishing to see a systematic development of the theory of spontaneously broken local symmetries in the unitarity gauge. The unitarity gauge has an obvious heuristic value, in immediately revealing the particle content of the theory. It is more convenient than the renormalizable gauge for calculation of physical processes in the tree approximation, just because it is not necessary to cancel out unphysical poles. It can be used in some simple one-loop calculations.\textsuperscript{5} Finally, the canonical quantization of theories with spontaneously broken local symmetries has so far only been possible in the unitarity gauge. This last is important, because once we know that a set of Feynman rules can be derived by canonical quantization from an Hermitian Hamiltonian, we know that the $S$ matrix is unitary, at least in the perturbative sense.

This paper begins in Sec. II with the description of a general notation for renormalizable gauge-invariant theories. The unitarity gauge is then introduced in Sec. III by imposing the condition

$$\left(\theta_{\alpha}\lambda, \phi(x)\right)=0 \text{ for all } \alpha, x,$$

(1.1)

where $\phi(x)$ is a vector whose components comprise all the real scalar fields of the theory, $\theta_{\alpha}$ is the matrix representing the action of the $\alpha$th generator of the gauge group on this scalar-field vector, and $\lambda$ is the value of $\phi$ for which the scalar-meson part of the Lagrangian is stationary in $\phi$. This is a “unitarity gauge,” because the scalar-field components corresponding to unphysical Goldstone bosons of zero mass are just those in the $\theta_{\alpha}\lambda$ directions\textsuperscript{8} which are here made to vanish. We usually think of a gauge as being chosen through conditions on the vector fields $A_{\alpha\mu}$, but of course the scalar as well as the vector fields respond to gauge transformations, so a gauge can be specified by conditions on $\phi$ as well as by conditions on $A_{\alpha\mu}$. A simple proof is given, showing that (1.1) can always be satisfied by a suitable change of gauge.\textsuperscript{9}

The canonical quantization procedure is carried out in Secs. IV–VI, leading to the main result of this paper, the general Feynman rules for theories in which all local symmetries are spontaneously broken. It turns out that the correct Feynman rules are given by the naive prescription of using covariant propagators for $A_{\alpha\mu}$ and $\theta_{\alpha}\phi$ and using the negative of the interaction Lagrangian as the interaction Hamiltonian, provided we add to the effective interaction Hamiltonian an term

$$i\delta^4(0) \ln \text{Det}[\mu^2 \Phi(x)]=1.2,$$

where $\mu^2$ and $\Phi$ are the matrices

$$\mu^2_{\alpha\alpha} = -\left(\theta_{\alpha}\lambda, \theta_{\alpha}\lambda\right),$$

$$\Phi = -\left(\theta_{\alpha}\lambda, \theta_{\alpha}\phi\right).$$

This result is the same as would be found by a path-integral quantization in the unitarity gauge\textsuperscript{10} and since the path-integral formalism manifestly leads to a gauge-invariant $S$ matrix, the unitarity of the $S$-matrix derived by path-integral methods in any gauge, renormalizable as well as unitary, is now assured.

The last two sections deal with special topics. In Sec. VII the class of “simple” theories, in which (1.1) forces $\phi$ to point in the direction of $\lambda$, is introduced and briefly discussed. Section VIII deals with the properties of theories with unbroken local subgroups, and shows how some previous results for photon-neutral-vector-meson mixing can be rederived in a more general way. The modifications in the Feynman rules required in the presence of massless vector mesons are discussed, but not fully worked out.

The formalism developed here has already been used in Ref. 3, and will be employed in future papers on symmetry breaking and other topics.

II. GENERAL GAUGE-ININVARIANT RENORMALIZABLE LAGRANGIANS

We shall consider the general class of renormalizable field theories possessing gauge invariance with respect to some compact semisimple Lie group $G$. For the sake of renormalizability, the contents of the Lagrangian will be limited to a set of spin-zero fields $\phi_{\alpha}(x)$ transforming according to a representation $D_{\alpha}$ of $G$, a set of spin-one-half fields $\psi_{\alpha}(x)$ transforming according to a representation $D_{\alpha}$ of $G$, and a set of spin-one gauge fields $A_{\alpha\mu}(x)$ transforming according to the adjoint representation of $G$. (Either $D_{\alpha}$ or $D_{\alpha}$ or both may be reducible.) The most general gauge-invariant Lagrangian which satisfies the usual power-counting conditions for renormalizability is of the Yang-Mills form\textsuperscript{11}
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\alpha}^{\mu\nu} - \frac{1}{2} (D_\mu \phi, D^\mu \phi) \\
- \bar{\psi} \gamma^\mu D_\mu \psi - \bar{\psi} m \psi - \bar{\psi} (\phi) \phi = \frac{1}{2} (\Gamma, \phi) \phi 
\]

Our notation and conventions are as follows:

(A) The gauge-covariant curl is
\[
F_{\alpha}^{\mu\nu} = \partial_\alpha A_{\mu\nu} - \partial_\nu A_{\mu\alpha} - C_{\alpha\beta\gamma} A_{\beta\mu} A_{\gamma\nu},
\]

where \( C_{\alpha\beta\gamma} \) is the structure constant, defined by the commutation relations for the generators \( T_\alpha \) of the abstract group \( S \):
\[
[T_\alpha, T_\beta] = i C_{\alpha\beta\gamma} T_\gamma.
\]

(Repeated indices are summed over unless otherwise noted.) Without loss of generality, we choose the generators \( T_\alpha \) and the corresponding gauge fields \( A_{\alpha\mu} \) to be Hermitian,
\[
T_\alpha = T_\alpha^\dagger, \\
A_{\alpha\mu} = A_{\alpha\mu}^\dagger
\]

so that the structure constants are real,
\[
C_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}^\dagger.
\]

Also, we have normalized the generators and gauge fields so that \( T_\alpha T_\alpha \) and \( F_{\alpha}^{\mu\nu} F_{\gamma}^{\mu\nu} \) are \( S \)-invariant, and therefore the structure constants are totally antisymmetric,
\[
C_{\alpha\beta\gamma} = -C_{\alpha\gamma\beta} = -C_{\beta\alpha\gamma} = -C_{\beta\gamma\alpha}.
\]

This does not entirely specify the normalization of the generators and structure constants – there is still a free normalization constant for each simple subgroup into which \( S \) may be decomposed. In our Lagrangian, this freedom has been employed to absorb into the generators \( T_\alpha \) all gauge coupling constants, so that \( C_{\alpha\beta\gamma} \) is of first order in the gauge coupling constants.

(B) The gauge-covariant derivative of the scalar field \( \phi_\alpha(x) \) is
\[
(D_\alpha \phi)_\mu = \partial_\mu \phi_\alpha + i (\bar{\theta}_\alpha \gamma_\mu \phi_\alpha A_{\alpha\mu},
\]

where \( \theta_\alpha \) is the matrix representing the abstract generator \( T_\alpha \) in the representation \( D_\alpha \) of \( S \) furnished by the boson fields:
\[
[\theta_\alpha, \theta_\beta] = i C_{\alpha\beta\gamma} \theta_\gamma.
\]

Note that \( \theta_\alpha \), like \( T_\alpha \) and \( C_{\alpha\beta\gamma} \), is taken to be proportional to the various gauge coupling constants. If the \( \phi_\alpha \) were not Hermitian, we could form a real representation of \( S \) by separating the Hermitian and anti-Hermitian parts of \( \phi_\alpha \); hence we shall simply assume from the beginning here that \( \phi_\alpha \) is Hermitian,
\[
\phi_\alpha^* = \phi_\alpha,
\]

so that \( i \theta_\alpha \) is real.

\[
(\theta_\alpha)_{\beta\gamma} = -(\theta_\alpha)_{\gamma\beta}.
\]

In (2.1), we are using a scalar-product notation
\[
(a, b) = a_\mu b^\mu.
\]

The \( \phi_\alpha \) fields are normalized so that scalar products like \( (D_\mu \phi_\alpha, D^\mu \phi_\alpha) \) are \( S \)-invariant, so therefore \( \theta_\alpha \) must be antisymmetric
\[
(\theta_\alpha)_{\beta\gamma} = -(\theta_\alpha)_{\gamma\beta}
\]

and hence Hermitian.

(C) The gauge-covariant derivative of the spin-one-half field \( \psi_\alpha(x) \) is
\[
(D_\mu \psi)_\alpha = \partial_\mu \psi_\alpha + i (t_\alpha)_\alpha \gamma_\mu \psi_\alpha A_{\alpha\mu},
\]

where \( t_\alpha \) is the matrix representing the generator \( T_\alpha \) in the representation \( D_\alpha \) of \( S \) furnished by the fermion fields:
\[
[t_\alpha, t_\beta] = i C_{\alpha\beta\gamma} t_\gamma.
\]

Once again, \( t_\alpha \), like \( T_\alpha \), \( C_{\alpha\beta\gamma} \), and \( \theta_\alpha \) is taken to be proportional to the various gauge coupling constants. The \( t_\alpha \) matrices will in general contain terms proportional to the Dirac matrix \( \gamma_\alpha \) as well as the unit Dirac matrix. They may be decomposed into left- and right-handed parts
\[
t_\alpha = t_\alpha^L + t_\alpha^R,
\]

\[
t_\alpha^L = \frac{1}{2} (1 + \gamma_\alpha) t_\alpha,
\]

\[
t_\alpha^R = \frac{1}{2} (1 - \gamma_\alpha) t_\alpha,
\]

each of which separately satisfy the commutation relations of our group:
\[
[t_\alpha^L, t_\beta^L] = i C_{\alpha\beta\gamma} t_\gamma^L,
\]

\[
[t_\alpha^R, t_\beta^R] = i C_{\alpha\beta\gamma}^\dagger t_\gamma^R.
\]

Correspondingly, the field \( \psi_\alpha \) may be decomposed into left- and right-handed parts
\[
\psi = \psi^L + \psi^R,
\]

\[
\psi^L = \frac{1}{2} (1 + \gamma_\alpha) \psi,
\]

\[
\psi^R = \frac{1}{2} (1 - \gamma_\alpha) \psi,
\]

which furnish representations \( D_\alpha \) and \( D_\beta \) of \( S \), in which the generators \( T_\alpha \) are represented respectively by \( t_\alpha^L \) and \( t_\alpha^R \). The fields \( \psi_\alpha(x) \) are normalized in such a way as to make \( \bar{\psi} \gamma^\mu D_\mu \psi \) invariant under \( S \), where \( \bar{\psi} \) is the usual Lorentz invariant adjoint
\[
\bar{\psi} = \bar{\psi}^\dagger \gamma_4.
\]

Since \( t_\alpha \) commutes with \( \gamma_4 \gamma^\mu \), this choice of normalization requires that \( t_\alpha \) be Hermitian

\[
t_\alpha^* = t_\alpha
\]

and therefore also
\[ t_a^L = t_a^R, \quad t_a^{R*} = t_a^R. \]  
\[ \text{(2.24)} \]

In the notation used here, the Dirac matrices satisfy the anticommutation relations

\[ \{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \]

where \(g_{\mu\nu}\) is a diagonal matrix, with elements +1, +1, -1, -1 for \(\mu = \nu = 1, 2, 3, 0\). Also, \(\gamma_4\) and \(\gamma_5\) are defined by

\[ \gamma_4 = -i\gamma_0, \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4. \]

The matrices \(y_1, \gamma_2, \gamma_3, \gamma_4, \) and \(\gamma_5\) are Hermitian, and have unit squares.

(D) The bare-mass matrix \(m_0\) must be \(\gamma_5\)-invariant, in the sense that

\[ [t_a, \gamma_5 m_0] = 0. \]  
\[ \text{(2.25)} \]

Decomposing (2.25) into terms proportional to \(\gamma_5(1 \pm \gamma_3)\) gives

\[ t_a^L m_0 - m_0 t_a^{R*} = t_a^L m_0 - m_0 t_a^R = 0. \]  
\[ \text{(2.26)} \]

Thus \(m_0\) is absent unless the representations \(D_+^g \otimes D_g\) and \(D_+^g \otimes D_g\) of \(g\) contain among their irreducible components the identity representation. When it is present, the matrix \(\gamma_5 m_0\) must be Hermitian,

\[ \gamma_5 m_0 \gamma_5 = m_0^\dagger. \]  
\[ \text{(2.27)} \]

In general, \(m_0\) might contain terms proportional to the Dirac matrix \(\gamma_5\) as well as 1, though such terms could always be eliminated by a suitable choice of \(\psi\) fields.

(E) The function \(\theta(\phi)\) is a real 4th-order polynomial in \(\phi\), and is \(\gamma_5\)-invariant in the sense that

\[ \frac{\partial \theta(\phi)}{\partial \phi_\alpha} (\epsilon_{\alpha})_{\beta\mu} \phi_\beta = 0. \]  
\[ \text{(2.28)} \]

In contrast to the case of the fermions, here there is no way that \(\gamma_5\) invariance can prevent the appearance of a boson mass term quadratic in \(\phi\) and a boson-boson interaction quartic in \(\phi\), while terms in \(\theta(\phi)\) linear or cubic in \(\phi\) may or may not be allowed.

(F) The \(\gamma_5\) invariance of the Yukawa interaction in (1) requires that \(\gamma_5 \Gamma_\alpha\) transform under \(g\) according to the representation \(D_+\), in the sense that

\[ [t_a, \gamma_5 \Gamma_\alpha] = -(\epsilon_{\alpha})_{\beta\mu} \gamma_5 \Gamma_\beta. \]  
\[ \text{(2.29)} \]

The matrices \(\Gamma_\alpha\) include any coupling constants that may appear in the Yukawa interaction, and may contain terms proportional to the Dirac matrix \(\gamma_5\) as well as 1. Decomposing (2.29) into terms proportional to \(\gamma_5(1 \pm \gamma_3)\) gives

\[ t_a^{L\Gamma(\phi_\alpha)} = t_a^{R\Gamma(\phi_\alpha)} = - (\epsilon_{\alpha})_{\beta\mu} \Gamma_\beta, \]
\[ \text{(2.29a)} \]

where

\[ \Gamma_\alpha^L = \frac{1}{2}(1 - \gamma_3) \Gamma_\alpha, \]
\[ \Gamma_\alpha^R = \frac{1}{2}(1 + \gamma_3) \Gamma_\alpha. \]

Thus \(\Gamma_\alpha^L\) or \(\Gamma_\alpha^R\) vanish unless the representations \(D_+^g \otimes D_g\) or \(D_+^g \otimes D_g\) contain one or more of the irreducible components of \(D_g\). The matrix \(\gamma_5 \Gamma_\alpha\) must be Hermitian, so

\[ \Gamma_\alpha^L = \gamma_5 \Gamma_\alpha^R = 0. \]  
\[ \text{(2.30)} \]

or in other words

\[ \Gamma_\alpha^{L\Gamma(\phi_\alpha)} = \Gamma_\alpha^{R\Gamma(\phi_\alpha)} = 0. \]

III. THE UNITARITY GAUGE

In consequence of the gauge invariance of our Lagrangian, the fields \(\psi_\alpha\), \(\phi_\alpha\), and \(A_{\alpha\mu}\) do not all represent independent physical degrees of freedom. This redundancy stands in the way of a straightforward canonical quantization of the theory, so before quantizing we shall choose a gauge.

Many choices are possible, including gauges defined by covariant conditions on the \(A_{\alpha\mu}\) (such as \(\partial_\mu A_{\alpha\mu} = 0\)), which would require quantization according to an indefinite metric. However, in this paper we shall instead choose a gauge designed to make manifest the nature of the Hilbert space of physical state vectors. That is, we shall eliminate the fields which do not correspond to physical particles, and we shall avoid an indefinite metric, so that the unitarity of the \(S\) matrix will be apparent in each order of perturbation theory.

Our gauge is defined as follows: First, define a vector \(\lambda_\alpha\) in the representation space of the scalar fields by the condition

\[ \frac{\partial \theta(\lambda)}{\partial \lambda_\alpha} = 0, \]  
\[ \text{(3.1)} \]

where \(\theta(\phi)\) is the quartic polynomial in the Lagrangian (2.1). The unitarity gauge is then defined by the condition that the scalar field components in the directions \(\theta_\alpha \lambda, \phi(x)\) should vanish

\[ \theta_\alpha \lambda, \phi(x) = 0. \]  
\[ \text{(3.2)} \]

Before exploring the properties of this gauge, we must first convince ourselves that it exists, and then consider to what extent it is unique.

A. Existence

We must verify that there always exists an appropriate gauge in which the scalar fields \(\phi_\alpha(x)\) obey the condition (3.2). That is, starting with a field \(\phi(x)\) in any gauge, not necessarily satisfying Eq. (3.2), we wish to find the gauge transformation \(\phi(x) - \tilde{\phi}(x)\) to a new field \(\tilde{\phi}(x)\) which does satisfy Eq. (3.2):

\[ (\epsilon_{\alpha})_{\beta\mu} \tilde{\phi}(x) = 0 \text{ for all } \alpha, x. \]
Under an arbitrary gauge transformation, the field \( \phi(x) \) transforms according to the rule

\[
\phi(x) \rightarrow O(x)\phi(x),
\]

where \( O(x) \) is an arbitrary \( x \)-dependent orthogonal matrix belonging to the representation \( D_\mathcal{B} \) of \( \mathcal{G} \). Hence we want to show that for any field \( \phi(x) \) we can find an \( O(x) \) such that

\[
(\theta_\alpha \lambda, O(x)\phi(x)) = 0 \quad \text{for all} \ x,
\]

We are working on a classical level, prior to quantization, so \( \phi(x) \) may be treated here as a \( c \)-number field. Also, since \( O(x) \) can have an arbitrary \( x \) dependence; the \( x \) dependence of \( O(x) \) and \( \phi(x) \) is irrelevant.\(^{12}\) Thus, our task is to prove that for an arbitrary vector \( \phi \), we can choose an orthogonal matrix \( O_\mathcal{G} \) belonging to the representation \( D_\mathcal{G} \), such that

\[
(\theta_\alpha \lambda, O_\mathcal{G} \phi) = 0 \quad \text{for all} \ \alpha.
\]

To prove that this is possible, consider the quantity

\[
(\lambda, O\phi),
\]

where \( O \) sweeps over all orthogonal matrices belonging to the representation \( D_\mathcal{G} \). Since \( D_\mathcal{G} \) is a real representation of a compact Lie group, this quantity is a real bounded differentiable function defined in the closed manifold of the group \( \mathcal{G} \). Therefore it has at least one maximum and at least one minimum. Let us tentatively choose \( O_\mathcal{G} \) to be any one of the \( O \) matrices for which \( (\lambda, O\phi) \) is an extremum. At any point on the group manifold, the variation of \( O \) is of the form

\[
\delta O = i\epsilon_\alpha \theta_\alpha O,
\]

where the \( \epsilon_\alpha \) are arbitrary infinitesimal parameters. Since \( (\lambda, O\phi) \) is stationary at \( O_\mathcal{G} \), we have

\[
0 = (\lambda, \delta O\phi)|_{O=O_\mathcal{G}} = i\epsilon_\alpha (\lambda, \theta_\alpha O\phi) = -i\epsilon_\alpha (\theta_\alpha \lambda, O_\mathcal{G} \phi)
\]

and, since \( \epsilon_\alpha \) is arbitrary,

\[
(\theta_\alpha \lambda, O_\mathcal{G} \phi) = 0
\]

as was to be proven. Thus our desired gauge may be constructed by starting with \( \phi(x) \) in any arbitrary gauge and then performing a gauge transformation \( \phi(x) \rightarrow O(x)\phi(x) \), with \( O(x) \) chosen at each \( x \) to give the quantity \( (\lambda, O(x)\phi(x)) \) an extremal value.\(^{13}\)

B. Uniqueness

There are two different ways in which the unitarity gauge can fail to be unique. First, there may be several vectors \( \lambda \), with different directions, satisfying the defining Eq. (3.1). Also, given any vector \( \lambda \), there may be several choices of gauge satisfying the gauge condition (3.2).

If \( \lambda \) is a solution of Eq. (3.1) then so is \( \mathcal{O} \lambda \), where \( O \) is any orthogonal matrix belonging to the representation \( D_\mathcal{G} \) of \( \mathcal{G} \). However, since we start with a completely \( \mathcal{G} \)-invariant theory, the solutions \( \lambda \) and \( \mathcal{O} \lambda \) are physically indistinguishable. On the other hand, we might find two different solutions \( \lambda_1 \) and \( \lambda_2 \) which are not related by a \( \mathcal{G} \) transformation. With the method of quantization to be developed here, two such solutions will in general lead to two physically inequivalent Hilbert spaces.

Given a vector \( \lambda \) satisfying Eq. (3.1), and a gauge in which the scalar field \( \phi(x) \) satisfies Eq. (3.2), we may be able to find gauge transformations \( \phi(x) \rightarrow O(x)\phi(x) \) such that the new field \( O(x)\phi(x) \) also satisfies Eq. (3.2). In particular, this will be the case if \( O(x) \) leaves \( \lambda \) invariant:

\[
O(x)\lambda = \lambda
\]

for then

\[
(\theta_\alpha \lambda, O(x)\phi(x)) = (O^{-1}(x)\theta_\alpha \lambda, \phi(x)) = \mathcal{G}_{\alpha \beta}(x)(\theta_\alpha \lambda, \phi(x)) = 0,
\]

where \( \mathcal{G}_{\alpha \beta}(x) \) is the adjoint representation of \( D_\mathcal{G} \):

\[
O^{-1}(x)\theta_\alpha O(x) = \mathcal{G}_{\alpha \beta}(x)\theta_\beta.
\]

The elements of \( \mathcal{G} \) satisfying Eq. (3.3) form a subgroup \( \mathcal{G} \), which as we shall see, consists precisely of all the unbroken symmetries of the theory.\(^{14}\) Thus, if our original group \( \mathcal{G} \) is \( N \)-dimensional, and there is an unbroken \( M \)-dimensional subgroup \( \mathcal{G} \), then the condition (3.2) really imposes only \( N-M \) independent constraints on \( \phi \), leaving us with complete freedom to perform gauge transformations in \( \mathcal{G} \). Conversely, the conditions (3.2) will in general provide \( N \)-dimensional Lie algebra \( \theta_\alpha \), unless there is one or more linear relations among the vectors \( \lambda, \theta_\alpha \lambda \):

\[
C_\alpha \theta_\alpha \lambda = 0.
\]

If Eq. (3.2) really provides only \( N-M \) constraints on \( \phi \), then there must be \( M \) independent linear relations of this form, and the \( M \)-linear combinations \( C_\alpha \theta_\alpha \) will then generate an \( M \)-dimensional subgroup of \( \mathcal{G} \) which leaves \( \lambda \) invariant.

We see that the continuous degrees of freedom in the gauge, which are not removed by the gauge
condition (3.2), are precisely those associated with the unbroken subgroup $S$ which leaves $\lambda$ invariant. It is not clear to me whether there could also be a discrete choice of gauges not associated with $S$ transformations, or what would be the physical significance of such a choice.

C. Physical Significance

Now we must show that the gauge condition (3.2) really accomplishes the purpose for which it was designed, of eliminating all Goldstone bosons from the theory. Let $n_\alpha$ be an orthonormal set of vectors spanning the subspace (in the representation space of the scalar fields) orthogonal to all $e^\alpha_\lambda$:

\begin{align}
(n_\alpha, e^\alpha_\lambda) &= 0, \\
(n_\alpha, n_\beta) &= \delta_{\alpha\beta}, \\
(n_\alpha, n_\alpha) &= 1, \quad \text{if} \quad (u, e^\alpha_\lambda) = 0.
\end{align}

Any scalar field vector satisfying Eq. (3.2) may be written

$$\phi_{\alpha} = \phi_{\alpha}(n^\alpha_\beta),$$

(3.7)

The $\phi_{\alpha}$ are then the independent scalar fields of the theory.

At this point we must recognize that the defining condition for $\lambda$, Eq. (3.1), is to be imposed before we choose our gauge, so that $\phi(\phi)$ must be stationary at $\phi = \lambda$ with respect to variations of $\phi$ not only in the directions $n_\alpha$, but also in the directions $e^\alpha_\lambda$ perpendicular to the $n_\alpha$. Thus Eq. (3.1) really gives us two different conditions: For all $\alpha$ and $\alpha$,

\begin{align}
\frac{\partial \phi(\phi)}{\partial \phi_{\alpha}} n_\alpha &= 0 \quad \text{at} \quad \phi = \lambda, \\
\frac{\partial \phi(\phi)}{\partial \phi_{\alpha}} (e^\alpha_\lambda) &= 0 \quad \text{at} \quad \phi = \lambda.
\end{align}

(3.8) (3.9)

However, Eq. (3.9) is automatically satisfied by virtue of the $S$-invariance condition (2.28). Thus the full content of Eq. (3.1) is contained in Eq. (3.8), or in other words, in the statement that $\phi(\phi)$ has no terms of first order in the “shifted field” $\phi^\prime_{\alpha}$, where

$$\phi_{\alpha} = (n_\alpha, \lambda) + \phi^\prime_{\alpha}.$$

(3.10)

We may recognize this as the condition that $\phi^\prime_{\alpha}$ should have zero vacuum expectation value in the tree approximation, and therefore conclude that $\lambda$ is the vacuum expectation value of $\phi$ in the tree approximation:

$$\langle \phi_{\alpha} \rangle_{\text{vac,tree}} = (n_\alpha, \lambda)$$

(3.11)

Now let us see what happens when we try to prove Goldstone’s theorem. By differentiating Eq. (2.28) with respect to $\phi_{\alpha}$, we find for an arbitrary $\phi$ (not necessarily in the unitarity gauge) that

$$\frac{\delta^2 \phi(\phi)}{\delta \phi_{\alpha} \delta \phi_{\beta}} (e^\alpha_\mu) \phi_{\epsilon} + \frac{\delta^2 \phi(\phi)}{\delta \phi_{\alpha} \delta \phi_{\beta}} (e^\alpha_\mu) \phi_{\epsilon} = 0.$$

Now setting $\phi = \lambda$ and using Eq. (3.1), we have

$$\frac{\delta^2 \phi(\lambda)}{\delta \lambda_{\alpha} \delta \lambda_{\beta}} (e^\alpha_\mu) \lambda_{\epsilon} = 0.$$  \hspace{1cm} (3.12)

In theories with a global rather than a local invariance group, this would require the tree approximation to the mass matrix of the scalar fields to have an eigenvector $e^\alpha_\lambda$ with eigenvalue zero. Indeed there are proofs that in such theories the propagator of the scalar field has poles which remain at zero mass to all orders in perturbation theory. However, in our present theory the second derivative appearing in Eq. (3.12) has nothing to do with the mass matrix of the scalar fields, which in the tree approximation is just the coefficient of the term in $\phi(\phi)$ quadratic in the shifted fields $\phi_{\alpha}^\prime$:

$$M_{\alpha\beta} = \frac{\delta^2 \phi(\lambda)}{\delta \lambda_{\alpha} \delta \lambda_{\beta}} (e^\alpha_\mu) \lambda_{\epsilon} \phi_{\alpha}^\prime.$$  \hspace{1cm} (3.13)

There are no linear relations between the $n_\alpha$ and $e^\alpha_\lambda$ vectors, so Eq. (3.12) tells us nothing whatever about the eigenvalues of the mass matrix (3.13). One of these eigenvalues may vanish, but there is no reason to expect it.

Our conclusions remain unchanged when we consider higher-order effects. If all eigenvalues of the scalar-boson mass matrix are nonzero in zeroth-order perturbation theory, then they must all be nonzero for sufficiently small values of the coupling constants. Of course, one of the eigenvalues of the mass matrix might vanish for some sufficiently large values of the coupling constants, but again, there is no particular reason to expect it, and certainly there is no Goldstone theorem here which requires a vanishing mass.

These remarks illuminate a difficulty that might be anticipated in some theories. When the representation $D_\phi$ is sufficiently complicated we must expect that the true direction of the vacuum expectation value of the scalar field vector $\phi$ will differ from the direction of the vacuum expectation value $\lambda$ in the tree approximation. Should we then correct the gauge condition (3.2) in each order of perturbation theory by using in place of $\lambda$ the true value of $\phi_{\alpha}$ calculated up to that order? Note that this is possible, because the above proof of the existence of a gauge satisfying (3.2) did not depend on $\lambda$ having any particular direction. Nevertheless, it appears that the gauge defined by (3.2) is perfectly adequate for use in all orders.

IV. QUANTIZATION

Having chosen our gauge, we shall now quantize the theory. For simplicity, it will be assumed
from now on that all gauge symmetries are broken, so that there is no nontrivial subgroup \( S \) of \( G \) which leaves the direction \( \lambda \) [defined by Eq. (3.1)] invariant. This way, we avoid the well-known complications of quantizing theories, like quantum electrodynamics, with massless spin-one particles. After quantization, we can always turn off the symmetry breaking of some subgroup \( S \) of \( G \), as discussed in Sec. VIII.

The canonical variables here will be chosen as the components \( \phi_i(x) \) of the spin-one-half fields, the spatial components \( A_{\alpha i}(x) \) of the gauge fields, and the independent components \( \phi_i(x) \) of the scalar-field vector constrained by the unitarity gauge condition (3.2). The canonical "momenta" conjugate to these variables are

\[
\chi_{\alpha i} = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_i)} = \bar{\phi}_i \,, \quad \Pi_{\alpha i} = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_{\alpha i})} = F_{\alpha i}^0 \,, \quad \pi_a = \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi_a)} = (n_a, D_0 \phi) \,.
\]

The fields \( A_{\alpha i}^0 \) are not independent dynamical variables, but rather may be expressed in terms of the above fields and conjugates by using the field equations. The field equation for \( A_{\alpha i}^0 \) reads

\[
0 = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_{\alpha i})} - \frac{\partial \mathcal{L}}{\partial A_{\beta \nu}} \right) = -\partial_\mu F_{\beta i}^\mu + F_{\alpha i}^{\mu, \nu} C_{\alpha \beta \gamma} A_{\beta \gamma} + i(\theta_3 \phi, D^0 \phi) \,.
\]

Setting \( \nu = 0 \) gives

\[
0 = -\nabla \cdot \vec{F}_i + \vec{F}_i \cdot \vec{A}_\gamma C_{\alpha \beta \gamma} + i(\theta_3 \phi, D^0 \phi) \,.
\]

To evaluate the last term, we note that our assumption that all gauge symmetries are broken means that no linear combination \( C_{\alpha \mu \lambda} \) leaves the vector \( \lambda \) invariant, so that the vectors \( \theta_\alpha \lambda \) are independent. The vectors \( n_a \) and \( \theta_a \lambda \) form a complete set, so that

\[
1 = n_a n_a^T = (\theta_a \lambda) (\mu^{-2})_{a \beta} (\theta_a \lambda)^T ,
\]

where \( \mu^{-2} \) is the inverse of the matrix

\[
\mu^{a \beta} = (\theta_a \lambda, \theta_\beta \lambda) .
\]

(It will be seen later that \( \mu \) is the zeroth-order vector-meson mass matrix.) Inserting (4.6) in (4.5) gives

\[
0 = -\nabla \cdot \vec{F}_i + \vec{F}_i \cdot \vec{A}_\gamma C_{\alpha \beta \gamma} + i(\theta_3 \phi, n_a) (n_a, D^0 \phi) + i(\theta_3 \phi, D^0 \phi) - i(\theta_3 \phi, (\mu^{-2})_{a \beta} (\theta_a \lambda, D^0 \phi) .
\]

At this point, we use the unitarity gauge condition, which tells us that for all \( x \),

\[
(\theta_\gamma \lambda, \phi(x)) = 0
\]

and therefore also

\[
(\theta_\gamma \lambda, \phi_\mu) = 0
\]

It follows then that

\[
(\theta_\gamma \lambda, D^0 \phi) = i(\theta_\gamma \lambda, \theta_\delta \phi) A_{\delta 0}^0 .
\]

The unitarity gauge condition also lets us replace \( \phi \) with \( n_a \phi_a \). Using (4.3) and (4.9) in (4.8) gives our solution for \( A_{\alpha 0} \):

\[
A_{\alpha 0} = \Omega^{-1} (\phi) \left[ \right. -\nabla \cdot \vec{F}_i + \vec{F}_i \cdot \vec{A}_\gamma C_{\alpha \beta \gamma} \\
- i(\theta_3)_{a \beta} \pi_a \phi_a \left. \right] ,
\]

where

\[
(\theta_\alpha \beta = (n_a, \theta_\alpha n_a)
\]

and \( \Omega^{-1} \) is the inverse of the symmetric matrix

\[
\Omega_{\beta \delta}(\phi) = (\theta_\beta \phi, (\theta_\delta \lambda) (\mu^{-2})_{a \beta} (\theta_a \lambda) ; \theta_\delta \phi) \,.
\]

Our next task is to evaluate the Hamiltonian density:

\[
\mathcal{H} = \chi_{\alpha i} n_i + \bar{\phi}_\alpha \cdot \vec{A}_\alpha + \pi_a \phi_a - L .
\]

The time derivatives \( \dot{\vec{A}}_\alpha \) and \( \dot{\phi}_a \) may be expressed in terms of canonical variables and their conjugates by using (4.2) and (4.3)

\[
\dot{\vec{A}}_\alpha = \bar{\vec{A}} \, a_{\alpha 0} + C_{\alpha \beta} \bar{A}_\beta \, \vec{A}_\gamma C_{\alpha \beta \gamma} + \vec{F}_i \phi ,
\]

\[
\dot{\phi}_a = -i(\theta_\alpha \beta) \psi_a \phi_a + \pi_a .
\]

Using (4.13) and (4.14) in (4.12), and noting the cancellation of terms involving \( \phi \), we have then

\[
\mathcal{H} = \bar{\vec{A}} \, a_{\alpha 0} + \bar{\vec{A}} \, a_{\alpha 0} A_{\alpha 0} \vec{A}_\gamma + \vec{F}_i \phi ,
\]

\[
+ \pi_a [-i(\theta_\alpha \beta) \phi_a \phi_a + \pi_a - \frac{1}{2} \bar{\vec{F}}_i \phi ,
\]

\[
+ \frac{1}{2} F_{\alpha i j} F_{\alpha i j} - \frac{1}{2} (D_0 \phi, D_0 \phi) + \frac{1}{2} (D_i \phi, D_i \phi) + \vec{\phi}_\gamma \cdot \vec{D} \psi + i \bar{\psi}_\gamma \gamma^\dagger \psi_a \phi_a,
\]

\[
+ \frac{1}{2} m_{\phi} \psi + \theta(\phi) + \bar{\gamma}(\phi, \phi) .
\]

Using (4.6), (4.9), (4.11), and (4.3), the sixth term can be evaluated as

\[
(D_0 \phi, D_0 \phi) \pi_a \phi_a + \Omega_{\alpha \beta}(\phi) A_{\alpha 0} A_{\beta 0}
\]

while the seventh term is

\[
(D_i \phi, D_i \phi) = \bar{\psi}_\gamma \psi_a \phi_a + 2i \bar{\psi}_\gamma \psi_a (\theta_a \beta) \phi_b + \bar{A}_\alpha \cdot \bar{\vec{A}} \, a_{\alpha 0} \phi_a (\theta_a \beta) \phi_b ,
\]

\[
+ \bar{A}_\alpha \cdot \bar{\vec{A}} \, a_{\alpha 0} (\phi_a - (\theta_a \beta) \phi_b) .
\]

Also, we can add a divergence \(-\nabla \cdot (P A_{\alpha 0})\) to \( \mathcal{H} \), so that the first term becomes

\[
-\bar{A}_\alpha \nabla \cdot \vec{F}_i .
\]

Using (4.10) to express \( A_{\alpha 0} \) in terms of independent fields and their canonical conjugates, we have
at last
\[ 3\mathcal{K} = \frac{1}{2} \Omega^{-1} a_0^\dagger (\phi)[ \vec{\nabla} \cdot \vec{P}_a - C_{\alpha\gamma} \gamma_a \vec{P}_b \cdot \vec{K}_\gamma + i(\theta_a)_{ab} \pi_a \phi_b] [\vec{\nabla} \cdot \vec{P}_b - C_{\beta\epsilon} \epsilon_b \vec{P}_c \cdot \vec{K}_\epsilon + i(\theta_b)_{ac} \pi_c \phi_c] \\
+ \frac{1}{2} F_a \cdot \vec{P}_a + \frac{1}{4} \pi_a \phi_a + \frac{1}{4} \partial_i A_{\alpha j} - \partial_j A_{\alpha i} - C_{\alpha \beta} A_{\beta i} A_{\gamma j} [\partial_i A_{\alpha j} - \partial_j A_{\alpha i} - C_{\beta \epsilon} \epsilon_b A_{\beta i} A_{\epsilon j}] \\
+ \frac{1}{2} \pi a \phi a + i \vec{K}_a \cdot \vec{P}_a \phi_a + i \vec{K}_a \cdot \vec{K}_\beta [\Omega_{\alpha \beta}(\phi) - (\theta_a)_{ac} (\theta_b)_{bc} \phi_c] + \vec{\gamma} \cdot \vec{\gamma} \phi + i \vec{\gamma} \cdot \vec{P}_a \phi a \cdot \vec{K}_a \\
- i \vec{\gamma} \cdot \vec{\gamma} \phi + \Omega_0^{-1} a_0 (\phi)[ \vec{\nabla} \cdot \vec{P}_a - \vec{P}_b \cdot \vec{K}_\gamma + i(\theta_a)_{ab} \pi_a \phi_b] + \vec{m}_0 \phi + \phi(\phi) + \vec{\Omega}(\Gamma, \phi) \phi . \] (4.17)

The quantization procedure is completed by writing down the canonical commutation relations:
\[ [A_a(\vec{x}, t), P_b(\vec{y}, t)] = i \delta_{ab}[\delta_{ij}(\vec{x} - \vec{y})] , \] (4.18)
\[ [\phi_a(\vec{x}, t), \pi_b(\vec{y}, t)] = i \delta_{ab}[\delta_{ij}(\vec{x} - \vec{y})] , \] (4.19)
\[ \{\pi(\vec{x}, t), \vec{\Omega}(\vec{y}, \bar{\tau})\} = i \delta_{ij}(\vec{x} - \vec{y}) \] . (4.20)

V. THE INTERACTION REPRESENTATION

In order to develop a perturbation series, we must extract from \( \phi \) its zeroth-order vacuum expectation value \( \lambda \), and write
\[ \phi = \lambda + \phi' , \] (5.1)
so that the shifted field \( \phi' \) has zero vacuum expectation value in the tree approximation. Note that in this approximation, the \( \Omega \) matrix (4.11) is just
\[ \Omega_{ab}(\lambda) = \mu^2 c_{ab} . \] (5.2)

\[ 3\mathcal{K}' = 3\mathcal{K} - 3\mathcal{K}_0 = \frac{1}{2}[\Omega^{-1} a_0^\dagger (\lambda + \phi') - \mu^2 c_{ab} \vec{\nabla} \cdot \vec{P}_a \vec{\nabla} \cdot \vec{P}_b + \Omega^{-1} a_0^\dagger (\lambda + \phi') \vec{\nabla} \cdot \vec{P}_b - [- C_{\alpha\gamma} \gamma_a \vec{P}_b \cdot \vec{K}_\gamma + i(\theta_a)_{ab} \pi_a \phi_b] \\
+ \frac{1}{2} \Omega^{-1} a_0^\dagger (\lambda + \phi') [\partial_i A_{\alpha j} - \partial_i A_{\alpha j} - C_{\beta \epsilon} \epsilon_b A_{\beta i} A_{\epsilon j}] \] (4.17)

Also note that
\[ (\theta_a)_{ab} \lambda_b = (m_a, \theta_a) 0 . \] (5.3)

The free-field Hamiltonian \( 3\mathcal{K}_0 \) is just that part of (4.17) quadratic in the canonical variables and their conjugates:
\[ 3\mathcal{K}_0 = \frac{1}{2}\mu^2 a_0 \vec{\nabla} \cdot \vec{P}_a + \frac{1}{2} F_a \cdot \vec{P}_a \\
+ \frac{1}{2} \pi a \phi a + \frac{1}{2} \partial_i A_{\alpha j} - \partial_i A_{\alpha j} (\theta_a)_{ab} \vec{P}_b \cdot \vec{K}_\gamma + i(\theta_a)_{ab} \pi_a \phi_b \] (4.17)

Also note that
\[ (\theta_a)_{ab} \lambda_b = (m_a, \theta_a) 0 . \] (5.4)

The free-field Hamiltonian \( 3\mathcal{K}_0 \) is just that part of (4.17) quadratic in the canonical variables and their conjugates:
\[ 3\mathcal{K}_0 = \frac{1}{2}\mu^2 a_0 \vec{\nabla} \cdot \vec{P}_a + \frac{1}{2} F_a \cdot \vec{P}_a \\
+ \frac{1}{2} \pi a \phi a + \frac{1}{2} \partial_i A_{\alpha j} - \partial_i A_{\alpha j} (\theta_a)_{ab} \vec{P}_b \cdot \vec{K}_\gamma + i(\theta_a)_{ab} \pi_a \phi_b \] (4.17)

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+ \frac{1}{2} \pi a \phi a + \frac{1}{2} \partial_i A_{\alpha j} - \partial_i A_{\alpha j} (\theta_a)_{ab} \vec{P}_b \cdot \vec{K}_\gamma + i(\theta_a)_{ab} \pi_a \phi_b \] (4.17)

Also note that
\[ (\theta_a)_{ab} \lambda_b = (m_a, \theta_a) 0 . \] (5.4)

We recognize this as the Hamiltonian for free particles with spin zero, one-half, and one, and with mass matrices \( M, m, \) and \( \mu, \) respectively.

The interaction Hamiltonian here is
\[ \hat{P}_a = \hat{m}_0 \phi + \frac{1}{2} M^2 a_0 \phi_0 \phi_0' , \] (5.5)

We now perform a "unitary" transformation to the interaction representation, in which the time derivatives of all canonical variables are determined by \( 3\mathcal{K}_0 \) alone. Distinguishing interaction-representation variables by superscript \( I \), we have for the vector fields
\[ \hat{A}_a^I = \frac{\partial \mathcal{K}'_0}{\partial A_a^{i+}} - \partial_j \left( \frac{\partial \mathcal{K}'_0}{\partial (\theta_j A_a^{i+})} \right) \] (5.6)

It proves extremely convenient to define the time component of the vector potential in the interaction representation by
\[ A_a^{i+} = -\frac{\partial \mathcal{K}'_0}{\partial (\theta_j A_a^{i+})} \] (5.7)

This is not what we would get by applying the general "unitary" transformation, that takes us from the Heisenberg to the interaction representation,
to the Heisenberg representation quantity $A_{ab}$ given by (4.10). For notational convenience, we also introduce the curvilinear coordinate $f_{ab} = \partial_x A_{\alpha \beta} - \partial_x A_{\beta \alpha}$.

Equation (5.6) now reads simply

$$P_{ab} = f_{ab}$$

(5.10)

and Eqs. (5.7) and (5.8) become just the space and time components of the free-field equation:

$$\partial \phi_{ab} = -\gamma_{ab}$$

(5.11)

Similarly (but more simply) the equations for the spin-zero and spin-one-half fields and their canonical conjugates yield the free-field equations

$$\square \phi_{ab} - M^2_{ab} \phi_{ab} = 0,$$

(5.12)

$$\gamma_{ab} \partial_t \phi_{ab} + m_{ab} \phi_{ab} = 0$$

(5.13)

and the formulas

$$\pi = \partial_t \phi,$$

(5.14)

$$\chi = \langle \phi \rangle.$$  

(5.15)

In writing down the propagators of the interaction-representation fields, it is convenient first to define covariant propagators:

$$\langle T \{ f_{ab}(x), f_{ab'}(y) \} \rangle = \langle T \{ g_{ab}(x), g_{ab'}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.16)

$$\langle T \{ \phi_{ab}(x), \phi_{ab'}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \phi_{ab'}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.17)

$$\langle T \{ \phi_{ab}(x), \partial \phi_{ab}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \partial \phi_{ab}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.18)

$$\langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.19)

$$\langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.20)

$$\langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.21)

$$\langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \partial_{ab} \phi_{ab}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.22)

with $\Delta_f$ defined for a general mass matrix $m$ by

$$\Delta_f(x-y; 2\pi) = (2\pi)^{-2} \int d^2 p \delta(p^2)\delta(p^2 + m^2) \hat{\phi}(x-y; p) e^{i\phi}.$$ 

(5.23)

A straightforward calculation then shows that the true vacuum expectation values of time-ordered products are given by

$$\langle T \{ f_{ab}(x), f_{ab'}(y) \} \rangle = \langle T \{ g_{ab}(x), g_{ab'}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.24)

$$\langle T \{ f_{ab}(x), f_{ab'}(y) \} \rangle = \langle T \{ g_{ab}(x), g_{ab'}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.25)

$$\langle T \{ f_{ab}(x), f_{ab'}(y) \} \rangle = \langle T \{ g_{ab}(x), g_{ab'}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.26)

$$\langle T \{ \phi_{ab}(x), \phi_{ab'}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \phi_{ab'}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.27)

$$\langle T \{ \phi_{ab}(x), \phi_{ab'}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \phi_{ab'}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.28)

$$\langle T \{ \phi_{ab}(x), \partial \phi_{ab}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \partial \phi_{ab}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.29)

$$\langle T \{ \phi_{ab}(x), \partial \phi_{ab}(y) \} \rangle = \langle T \{ \phi_{ab}(x), \partial \phi_{ab}(y) \} \rangle \Delta_{ab}(x-y; M),$$

(5.30)

The fermion propagators here have the same form as usual.

VI. DERIVATION OF THE FEYNMAN RULES

As a first step toward a Lorentz-invariant formulation of this theory, let us use Eqs. (5.8), (5.10), and (5.14) to eliminate canonical “momenta” from the interaction Hamiltonian $3C$ in the interaction representation. Dropping the superscript $I$, Eq. (5.5) becomes
\[ \mathcal{L}' = \frac{1}{2} [\mu^2 \Omega^{-1}(\lambda + \phi')] \mu - \mu^2] A_{\alpha \bar{\beta}} A_{\alpha \bar{\beta}} - [\Omega^{-1}(\lambda + \phi')] \mu A_{\alpha \bar{\beta}} [-C_{\alpha \gamma} f_{\delta \epsilon \gamma} \phi_\delta \phi_\epsilon] - C_{\delta \epsilon} f_{\delta \epsilon A_{\gamma}} + i(\theta A) \phi_\delta \phi_\epsilon \]

Now, we note that the objects appearing in (6.1) may be divided into two classes:

(A) The quantities \( A_{\alpha \bar{\beta}}, f_{\alpha \bar{\gamma}}, \phi_\alpha, \) and \( \phi_\delta, \) all of whose propagators are given according to Eqs. (5.24)–(5.30) by the covariant \( T^* \) products defined in Eqs. (5.16)–(5.23).

(B) The quantities \( A_{\alpha \bar{\gamma}}, f_{\alpha \bar{\gamma}}, \) and \( \phi_\gamma, \) which enter into propagators containing noncovariant local terms (given by Eqs. (5.24), (5.26), and (5.29)) in addition to the covariant \( T^* \) products.

We also note that the quantities of type (B) enter in the interaction (6.1) only linearly and quadratically. It is therefore possible to sum up the effects of the noncovariant terms in the propagators precisely, by using techniques developed long ago by Lee and Yang.\(^{16}\)

Let us denote the general field variables of type (B) as \( \beta_\mu(x), \) and write their propagators as

\[ \langle T^* \{ \beta_\mu(x), \beta_\nu(y) \} \rangle_{\text{cov}} = \mathcal{L}^* \langle T^* \{ \beta_\mu(x), \beta_\nu(y) \} \rangle + i \delta_{\mu \nu} \delta^2(x - y), \]

with \( \mathcal{L}^* \) understood as the covariant propagators defined in Eqs. (5.16)–(5.23). The interaction here may be written as

\[ \mathcal{L}' = \mathcal{S}(\alpha) + \mathcal{F}(\alpha) + \mathcal{G}(\alpha), \]

where \( \alpha \) stands for all the fields of type (A), which have purely covariant propagators. Following Lee and Yang,\(^{16}\) we can represent the \( T^* \) and \( \mathcal{S} \) terms in (6.2) by straight and wiggly lines, respectively, and evaluate the effects of the \( \mathcal{S} \) terms by summing chains of wiggly lines. The effect of inserting a chain of wiggly lines between two straight lines is to replace \( \mathcal{S}(\alpha) \) with

\[ \mathcal{S}(\alpha) = \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \cdots = \mathcal{S}(\alpha) [1 - \mathcal{S}(\alpha)]^{-1}. \]

The effect of inserting a chain of wiggly lines between a straight line and an \( \mathcal{F} \) vertex is to replace \( \mathcal{F}(\alpha) \) with

\[ \mathcal{F}(\alpha) = \mathcal{F}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \cdots = \mathcal{F}(\alpha) [1 - \mathcal{S}(\alpha)]^{-1} \]

or

\[ \mathcal{F}(\alpha) = \mathcal{S}(\alpha) \mathcal{F}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \cdots = [1 - \mathcal{S}(\alpha)]^{-1} \mathcal{S}(\alpha) \mathcal{F}(\alpha). \]

Note that these are equal, because \( \mathcal{S} \) and \( \mathcal{S}(\alpha) \) are symmetric matrices. Finally, it is possible for wiggly lines to form a ring, producing a contribution

\[ \frac{1}{2} \delta^4(0) [\text{Tr} (\mathcal{S}(\alpha) \mathcal{S}(\alpha)) + \frac{1}{2} \text{Tr} (\mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha) \mathcal{S}(\alpha)) \cdots] = -\frac{1}{2} \delta^4(0) \text{Tr} \ln[1 - \mathcal{S}(\alpha) \mathcal{S}(\alpha)] \]

Thus, we can drop the noncovariant \( \mathcal{S} \) term in the propagators (6.2) if we replace the interaction (6.3) with an effective interaction:

\[ \mathcal{L}'_{\text{eff}} = -\mathcal{L}' - (i/2) \delta^4(0) \text{ln} \det[1 - \mathcal{S}(\alpha)], \]

with

\[ \mathcal{L}' = \mathcal{S}(\alpha) + (\mathcal{S}(\alpha) [1 - \mathcal{S}(\alpha)]^{-1})_{\mu} \beta_\mu + \frac{1}{2} (\mathcal{S}(\alpha) [1 - \mathcal{S}(\alpha)]^{-1})_{\mu \nu} \beta_\mu \beta_\nu. \]

(\text{It will turn out that } \mathcal{L}' \text{ is the interaction Lagrangian.})

In our case, the index \( N \) labeling the variables of class \( \mathcal{S} \) may be considered to run over the values \( \alpha, \alpha_i, \) and \( \alpha_\ell, \) with

\[ \beta_\alpha = \mu \partial_\alpha A_{\beta \bar{\sigma}}, \quad \beta_\alpha = f_{\alpha \bar{\gamma}}, \]

(6.6)
With this normalization of $\beta$ variables, Eqs. (5.24)–(5.30) give the nonvanishing elements of the $\mathfrak{A}$ matrix here as

$$\mathfrak{A}_{ab} = -\delta_{ab}, \quad \mathfrak{A}_{a1,1} = -\delta_{a0} \delta_{1f}, \quad \mathfrak{A}_{00} = -\delta_{2b}$$

or more compactly

$$\mathfrak{A} = -1.$$  

Also, inspection of (6.1) shows that the $\mathfrak{S}$ matrix, the $\mathfrak{F}$ vector, and the function $\mathfrak{s}$ here are

$$\mathfrak{S}_{a} = \left[ \mu \Omega^{-1}(\lambda + \phi') \mu - 1 \right] a \mathfrak{s},$$  

$$\mathfrak{G}_{a} = \frac{\mathfrak{G}_{b}}{\sqrt{2}} \frac{\mathfrak{G}_{i}}{\sqrt{2}} \frac{\mathfrak{G}_{a}}{\sqrt{2}} \frac{\mathfrak{G}_{b}}{\sqrt{2}} \frac{\mathfrak{G}_{i}}{\sqrt{2}},$$

$$\mathfrak{S} = -\frac{1}{2} C_{\alpha a} A_{n} A_{Y_{b}} A_{Y_{b}} + \frac{1}{2} C_{\alpha a} A_{n} A_{Y_{b}} A_{Y_{b}} A_{b} f_{b} + i \sigma_{\sigma} \psi_{\alpha} \phi_{\sigma} \phi_{\alpha} \Omega_{ab}(\lambda + \phi' + \mu^{2} a_{b})$$

In calculating $\mathfrak{S}$, it will be very convenient to introduce a supermatrix notation, with all $\alpha$ indices lumped together in the left column and/or upper row, and all $\alpha$, $i$, and $a$ indices in the right column and/or lower row. In this notation, the above formulas for $\mathfrak{S}$ and $\mathfrak{F}$ become simply

$$\mathfrak{S} = \left[ \mu \Omega^{-1} \mu - 1 \right] \mu \Omega^{-1} \mathfrak{e}^{T}, \quad \mathfrak{F} = \left[ i \mu \Omega^{-1} i \psi^{T} \right],$$

where

$$\mathfrak{e}_{a, b} = C_{ab} A_{Y_{b}}, \quad \mathfrak{e}_{a, b} = -i \sigma_{\sigma} \psi_{\alpha} \phi_{\sigma} \phi_{\alpha}$$

It is easy to check that

$$[1 - \mathfrak{S}]^{-1} = [1 + \mathfrak{S}]^{-1} = \left[ \mu^{-1}(\Omega + \mathfrak{e}^{T} \mathfrak{e}) \mu^{-1} - \mathfrak{e}^{-1} \mathfrak{e}^{T} \right].$$

The last two terms in (6.5) are then

$$\frac{1}{2}[9(1 - \mathfrak{S})^{-1}]_{i \mu} \sigma_{\alpha} \beta_{i} = \frac{1}{2}\left[(1 + \mathfrak{S})^{-1} - 1\right]_{i \mu} \sigma_{\alpha} \beta_{i}$$

and

$$[(1 - \mathfrak{S})^{-1} \mathfrak{F}]_{i \mu} \beta_{i} = \left[(1 + \mathfrak{S})^{-1} \mathfrak{F}\right]_{i \mu} \beta_{i}$$

Putting this together with (6.16), we find for the first term in (6.4)

$$-\mathcal{L} = -\frac{1}{2} A_{a} A_{b} A^{b}_{a} \left[ (\theta_{a})_{ab} (\theta_{a})_{ac} \phi_{a} \phi_{c} - \Omega_{ab}(\lambda + \phi') + \mu^{2} a_{b} \right]$$

$$+ \frac{1}{2} C_{a b} A_{a} A_{b} A_{Y_{b}} A_{Y_{b}} - \frac{1}{2} C_{a b} A_{a} A_{b} A_{Y_{b}} A_{Y_{b}} + i \sigma_{\sigma} \psi_{\alpha} \phi_{\sigma} \phi_{\alpha} A_{a}$$

(6.17)
with $\Omega$ given by Eq. (4.11):

$$\Omega_{\alpha\beta}(\phi) = (\theta_{\alpha} \phi, \theta_{\beta} \lambda) \mu^{-q}(\theta_{\beta} \lambda, \theta_{\alpha} \phi).$$

This is covariant, and in fact is just what would be obtained by the naive procedure of using the negative of the interaction part of the Lagrangian as our interaction Hamiltonian. However, we also must deal with the second term in (6.4). To calculate the determinant, we note that

$$[1 - \mathbf{g}][1 + \mathbf{g}]^{-1} = \begin{bmatrix} \mu^{-1/2} & 0 \\ 0 & 1 \end{bmatrix} [1 + \Omega^{-1/2} e^T e \Omega^{-1/2} - \Omega^{-1/2} e^T] \begin{bmatrix} \Omega^{1/2} \mu^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrix in the middle is of the form

$$\begin{bmatrix} 1 + A^T A & A^T \\ A & 1 \end{bmatrix}, \quad A = -e \Omega^{-1/2},$$

and therefore, according to Appendix A, has unit determinant. Thus, by the product rule,

$$\text{Det}(1 - \mathbf{g}) = \text{Det}(\mu^{-1/2} \Omega) \text{Det}(\Omega^{1/2}) = \text{Det}(\mu^{-2} \Phi) \text{Det}(\mu^{-2} \Phi) = [\text{Det}(\mu^{-2} \Phi)]^2,$$

where $\Phi$ is the matrix

$$\Phi_{\beta \alpha} = -(\theta_{\beta} \lambda, \theta_{\alpha} \phi). \quad (6.18)$$

The effective interaction Hamiltonian is then

$$\mathcal{H}_{\text{eff}} = -\mathcal{L} + i \delta'([0] \ln [1 + \chi(x)]$$

where $\mathcal{L}$ is the just the result that would be obtained by a path-integral quantization of this theory in a gauge in which the propagators are given by the $T^*$ products (5.16)–(6.22).

**VII. SIMPLE THEORIES**

As an example, suppose we are dealing with a scalar field representation $D_\phi$ so simple that every vector in the representation can be rotated into a fixed direction by applying a gauge transformation in the group $S$. For the gauge group SU(2) or U(2), this is the case if $D_\phi$ consists of just a single complex doublet, as in the old model of leptons, but it is not the case if $D_\phi$ contains, say, a doublet and a triplet, or two doublets with different charge assignments. If $\phi(x)$ at each point can be rotated into a fixed direction, then this must also be the direction of the vacuum expectation value of $\phi(x)$ in the tree approximation. Thus there is a "gauge" in which

$$\phi_\beta(x) = [1 + \chi(x)] \lambda_\beta$$

where $\chi(x)$ is a scalar field, the only one present in this gauge. This obviously is a "unitarity gauge," because the antisymmetry of $\theta_\alpha$ gives

$$(\theta_{\alpha} \lambda, \phi(x)) = [1 + \chi(x)] (\theta_{\alpha} \lambda, \lambda) = 0.$$

In these simple theories, the existence proof for the unitarity gauge presented in Sec. III is unnecessary.

Equation (7.1) allows an instant evaluation of the matrix (6.18)

$$\Phi_{\beta \alpha} = -(1 + \chi)(\theta_{\beta} \lambda, \theta_{\alpha} \lambda)$$

$$= (1 + \chi) \mu^2 \delta_{\beta \alpha}. \quad (7.2)$$

The effective Hamiltonian (6.19) is then simply

$$\mathcal{H}_{\text{eff}} = -\mathcal{L} + i \delta'(0) \ln [1 + \chi(x)]$$

where $\mathcal{L}$ is the number of $\theta_\alpha$ generators.

**VIII. THEORIES WITH UNBROKEN SUBGROUPS**

The quantization program carried out in Secs. IV–VII is strictly applicable only to theories in which all gauge symmetries are broken. Let us now briefly return to the general case, and consider a theory with an unbroken subgroup $S$.

The generators of $S$ are simply those independent linear combinations $C_\alpha T_\alpha$ of the generators of $S$ for which $1 + i \epsilon C_\alpha T_\alpha$ leaves $\lambda$ invariant:

$$C_\alpha T_\alpha \lambda = 0. \quad (8.1)$$

Each such set of coefficients $C_\alpha$ forms an eigenvector of the vector-meson mass matrix with eigenvalue zero:

$$\mu^2_{\alpha \beta} C_\beta = -(\theta_{\alpha} \lambda, \theta_{\beta} \lambda) C_\beta = 0.$$
Since $\mu^2$ is a real symmetric matrix, we can find a complete set of orthonormal eigenvectors
\begin{equation}
\mu^2_{\alpha\beta} = C_{\alpha\beta} = \mu_{\alpha\beta} C_{\alpha\beta} , \tag{8.2}
\end{equation}
\begin{equation}
C_{\alpha\beta} C_{\gamma\delta} = \delta_{\rho\gamma} C_{\alpha\beta} \delta_{\rho\delta} , \tag{8.3}
\end{equation}
\begin{equation}
\sum_{\alpha} C_{\alpha\beta} C_{\alpha\gamma} = \delta_{\beta\gamma} , \tag{8.4}
\end{equation}
among which must be all the vectors $C_{\alpha}$ satisfying (8.1). Thus, if we define new generators
\begin{equation}
T_{\alpha} = C_{\alpha\beta} T_{\beta} , \tag{8.5}
\end{equation}
then the generators $\mathbf{8}$ are just those $T_{\alpha}$ for which $\mu_{\alpha} = 0$ and $C_{\alpha\beta}$ satisfies (8.1):
\begin{equation}
C_{\alpha\beta} \delta_{\lambda\gamma} = 0 \quad \text{for} \quad T_{\alpha} \in \mathbf{8} . \tag{8.5}
\end{equation}
It is very convenient to use the $C_{\alpha\beta}$ also to define new canonically normalized vector fields
\begin{equation}
\mathbf{A}_{\alpha\beta} \equiv C_{\alpha\beta} A_{\alpha\beta} . \tag{8.6}
\end{equation}
Using (8.4), this gives
\begin{equation}
A_{\alpha\beta} = \sum_{\alpha} C_{\alpha\beta} \mathbf{A}_{\alpha\beta} . \tag{8.7}
\end{equation}
Thus the vector-meson mass term in $\mathcal{H}_0$ may be written
\begin{equation}
\frac{1}{2} \mu^2_{\alpha\beta} A_{\alpha\beta} A_{\alpha\beta} = \frac{1}{2} \sum_{\alpha} \mu^2_{\alpha\beta} \mathbf{A}_{\alpha\beta} \mathbf{A}_{\alpha\beta} . \tag{8.8}
\end{equation}
The fields $\mathbf{A}_{\alpha\beta}$ thus describe particles of definite mass. In particular, if $T_{\alpha}$ is one of the generators of the unbroken subgroup $\mathbf{8}$, then $\mu_{\alpha} = 0$ so $\mathbf{A}_{\alpha\beta}$ describes a particle of zero mass. Also, the interaction of gauge fields with spin-zero or spin-one-half fields is described by the matrices
\begin{equation}
t_{\alpha} A_{\alpha\beta} = \sum_{\alpha} t_{\alpha} \mathbf{A}_{\alpha\beta} , \tag{8.9}
\end{equation}
\begin{equation}
\theta_{\alpha} A_{\alpha\beta} = \sum_{\alpha} \theta_{\alpha} \mathbf{A}_{\alpha\beta} , \tag{8.10}
\end{equation}
where
\begin{equation}
t_{\alpha} = C_{\alpha\beta} t_{\beta} , \tag{8.11}
\end{equation}
\begin{equation}
\theta_{\alpha} = C_{\alpha\beta} \theta_{\beta} . \tag{8.12}
\end{equation}
This formalism is particularly useful when we do not have a complete model of the symmetry-breaking mechanism.\(^{14}\) For instance, consider the gauge group $\text{SU}(2) \times \text{U}(1)$ used in the old model of leptons,\(^1\) with gauge coupling constants $g$ and $g'$ associated with the $\text{SU}(2)$ generators $I_i$ and the $\text{U}(1)$ generator $Y$, respectively. The charge is defined here by
\begin{equation}
Q = I_3 - Y . \tag{8.13}
\end{equation}
Remember that we have agreed to absorb the gauge coupling constants into the generators, so this should be written
\begin{equation}
Q = -\frac{1}{g} T_3 - \frac{1}{g'} T_Y , \tag{8.14}
\end{equation}
where
\begin{equation}
\mathbf{T} = g\mathbf{I} , \quad T_Y = g' Y . \tag{8.15}
\end{equation}
But $Q$ is the generator of the unbroken subgroup $\mathbf{8}$, so we can read off from (8.14) the $C_{\alpha\beta}$ coefficients appearing in Eq. (8.5):
\begin{equation}
C_{\alpha\beta} \approx \frac{1}{g} , \quad C_{\gamma\delta} \approx \frac{1}{g'} . \tag{8.15}
\end{equation}
or, recalling that $C$ must be orthogonal,
\begin{equation}
C_{\alpha\beta} \approx \frac{g}{(g^2 + g'^2)^{1/2}} , \quad C_{\gamma\delta} \approx \frac{-g'}{(g^2 + g'^2)^{1/2}} . \tag{8.16}
\end{equation}
The canonically normalized massless vector field coupled to $Q$ is then
\begin{equation}
A_{\mu} = \frac{1}{(g^2 + g'^2)^{1/2}} \left( g A_{\alpha\mu} - g' A_{\gamma\mu} \right) . \tag{8.17}
\end{equation}
Also, in this model the other neutral generator is defined by a vector orthogonal to (8.16):
\begin{equation}
C_{\alpha\beta} \approx \frac{g'}{(g^2 + g'^2)^{1/2}} , \quad C_{\gamma\delta} \approx \frac{-g}{(g^2 + g'^2)^{1/2}} . \tag{8.18}
\end{equation}
so the canonically normalized neutral field of non-zero mass is
\begin{equation}
Z_{\mu} = \frac{1}{(g^2 + g'^2)^{1/2}} \left( g A_{\alpha\mu} + g' A_{\gamma\mu} \right) . \tag{8.19}
\end{equation}
The operators to which $A_{\mu}$ and $Z_{\mu}$ couple are respectively
\begin{equation}
A_{\mu} : \quad C_{\alpha\beta} T_3 + C_{\gamma\delta} T_Y = \frac{gg'}{(g^2 + g'^2)^{1/2}} (I_3 - Y) , \tag{8.20}
\end{equation}
\begin{equation}
Z_{\mu} : \quad C_{\alpha\beta} T_3 + C_{\gamma\delta} T_Y = \frac{gg'}{(g^2 + g'^2)^{1/2}} (I_3 + Y) . \tag{8.21}
\end{equation}
We see in particular that the electric charge of a particle with $I_3 - Y = \pm 1$ is
\begin{equation}
e = \frac{gg'}{(g^2 + g'^2)^{1/2}} . \tag{8.22}
\end{equation}
These results are the same as those derived previously\(^1\) under specific assumptions as to the mechanism of symmetry breaking.

The direct quantization of theories with an unbroken gauge symmetry subgroup $\mathbf{8}$ presents well-known difficulties. We can however break $\mathbf{8}$ weakly by adding an additional multiplet $\Delta \phi$ of scalar fields whose zeroth-order vacuum expectation value $\Delta \lambda$ is small, but is not annihilated by the $\overline{Q}_{\alpha}$ generators belonging to $\mathbf{8}$. Since the $\theta$ matrices act separately on $\phi$ and $\Delta \phi$, the matrix $\Phi$ now is
\[
\Phi_B = (\theta_B \lambda, \theta_B \phi) + (\theta_B \Delta \lambda, \theta_B \Delta \phi) .
\]
(8.23)

This matrix can be put in a supermatrix form
\[
\Phi = \begin{pmatrix} \Phi_{BB} & \Phi_{UB} \\ \Phi_{UB}^t & \Phi_{UU} \end{pmatrix},
\]
where \( U \) refers to the generators \( T^i \) of \( \mathcal{G} \) which span \( \delta \), and \( B \) refers to the other, strongly broken generators. (Note that \( \Phi \) is symmetric). Both \( \Phi_{UB} \) and \( \Phi_{UU} \) receive contributions only from the second term in (8.23), so as long as \( \Delta \lambda \ll \lambda \), we have
\[
\Phi_{UB} \ll \Phi_{BB}, \quad \Phi_{UU} \ll \Phi_{BB} ,
\]
so to lowest nonvanishing order in terms of order \( \Delta \lambda \),
\[
\text{Det}(U \Phi) - (\text{Det} \Phi_{BB})(\text{Det} \Phi_{UU})
\]
and Eq. (6.19) becomes
\[
\mathcal{L}_{\text{eff}} = -\frac{1}{2} i \delta^4(0) \ln \text{Det} \Phi_{BB} + i \delta^4(0) \ln \text{Det} \Phi_{UU} .
\]
(8.24)

The last term involves \( \Delta \phi \), not \( \phi \), and is presumably what is needed to provide a smooth limit as various gauge boson masses tend to zero. Thus, if we put aside such complications, we may conjecture that the determinant of \( \Phi \) is to be evaluated taking account only of the broken generators for which \( \theta_B \lambda \neq 0 \). In particular, in simple theories the number \( N \) in Eq. (7.3) is the number of broken generators of \( \mathcal{G} \).

The last remarks are intended as suggestive rather than conclusive. A more thorough study of gauge theories with unbroken gauge subgroups would be useful.

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APPENDIX A: EVALUATION OF A DETERMINANT

We wish to calculate the determinant of a matrix \( S \), expressed as a supermatrix
\[
S = \begin{bmatrix} 1 + A^T A & A^T \\ A & 1 \end{bmatrix},
\]
where \( A \) is an arbitrary real matrix, not necessarily square. There is probably some easy way to do this calculation, but I have been able to manage it only by considering the eigenvalues \( \lambda \) of \( S \). Writing the eigenvector \( \Psi \) as
\[
\Psi = \begin{bmatrix} u \\ v \end{bmatrix},
\]
the eigenvalue equation reads
\[
S \Psi = \lambda \Psi
\]
or in more detail
\[
(1 + A^T A) u + A^T v = \lambda u ,
\]
\[
A u + v = \lambda v.
\]
For every possible value of \( \lambda \), other than \( \lambda = 1 \), we may solve for \( v \):
\[
v = \frac{1}{\lambda - 1} Au
\]
and write the eigenvalue equation in terms of \( u \) alone
\[
\lambda A^T A u = (\lambda - 1)^2 u .
\]
Thus \( u \) must be an eigenvector of \( A^T A \), with a necessarily positive eigenvalue \( \alpha > 0 \). For each such eigenvalue \( \alpha \), there will be two eigenvalues \( \lambda \) of \( S \), given by the two roots of the quadratic equation
\[
(\lambda - 1)^2 = \lambda \alpha .
\]
These two roots have product unity. Thus, the eigenvalues \( S \) are either unity or come in reciprocal pairs \( \lambda, 1/\lambda \). Since \( S \) is real and symmetric, it can be put in the form
\[
S = O \begin{bmatrix} 1 \\
\lambda_1 \\
\ddots \\
\lambda_1^{-1} \\
\lambda_2 \\
\ddots \\
\lambda_2^{-1} \\
\ddots \end{bmatrix} O^T ,
\]
where \( O \) is an orthogonal matrix. Inspection now yields our result
\[
\text{Det} S = 1 .
\]

Note added in proof. M. Grisaru has pointed out to me that the result derived in Appendix A can be obtained much more easily by factoring the matrix \( S \) into a product of two matrices having ones on the main diagonal and zeros either above or below it.
In subsequent terms, combinations of bosons and fermions are given in [56].

1) The vacuum amplitude for the transformation of the Goldstone bosons in a local group is discussed in Ref. 4.

2) The problem does not arise in the class of "simple" theories discussed here in Sec. VII, which includes most of the models considered in Refs. 1, 6, and 7.

3) The coefficient of the logarithm in the SU(2) × U(1) model of leptons was given as \( \frac{3}{2} \delta(0) \) in Ref. 5. This was a mistake, arising from neglect of the factor \( \delta \) in Eq. (6.4). The number of broken generators in this model is three, and the correct coefficient of the logarithm is therefore \( 3 \delta(0) \).

Reduction Formulas for Charged Particles and Coherent States in Quantum Electrodynamics

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A weak asymptotic limit is proposed for a charged field as an operator on the space of asymptotic states. This leads to a modified Lehmann-Symanzik-Zimmermann reduction formula and a determination of the singularity near the mass shell of the Green's function of a charged particle in the presence of other charged particles. Coherent states of the electromagnetic field are also reduced out. The resultant expression for S-matrix elements in terms of vacuum expectation values of time-ordered fields yields a slight elaboration of the Feynman rules which allows a perturbative calculation that is free of infrared and Coulombic divergences order by order. As an application, the amplitude for scattering of a Dirac particle by an external Coulomb potential is calculated to second order in the external potential, with a finite result.

I. INTRODUCTION

The exact amplitude for scattering of a nonrelativistic particle by a Coulomb potential is given in textbooks on quantum mechanics: