this experiment. Terms similar to those neglected in Eq. (A1) have also been neglected here.

If we let \( \omega \Delta
\) corresponding to an electron energy \( E_1 \) before radiation and \( E_2 \) after radiation, then the probability that an electron of initial energy \( E_0 \) will have an energy \( E' \) after radiating, scattering, and again radiating will be

\[
\tau(E_0, E')dE' = dE' \int_{E_0}^{E_1} \int_{E_2}^{E_1} \omega(E_0, E_1)
\times \sigma(E_1, E_2) \omega'(E_2, E')dE_2dE_1 \quad (A5)
\]

where \( \sigma(E_1, E_2) \) is the theoretical scattering cross section for electrons of initial energy \( E_1 \) and final energy \( E_2 \) and \( \omega \) and \( \omega' \) are the probabilities for radiation before and after scattering, respectively.

For an elastic cross section, \( \sigma(E_1, E_2) \) is a delta function and the integrals of Eq. (A5) can be evaluated approximately to yield Eq. (1) of the text if \( E_4 \) in that equation is replaced by \( E_6' \). Again, terms of the same order as those neglected in Eq. (A1) were neglected in Eq. (1), in addition to terms depending on \( (E_6 - E_2) \) but which were considerably smaller than those that were retained.

If \( \sigma(E_1, E_2) \), as a theoretical inelastic cross section, is considered to be a series of many delta functions (elastic cross sections), Eq. (2) of the text results, where the summation has been replaced by the integral sign of that expression. In deducing Eq. (2), it was assumed that the shape of \( \sigma(E_1, E_2) \) as a function of \( E_2 \) with fixed \( E_1 \) does not change with \( E_1 \). This assumption gives adequate accuracy for this work.

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**High-Energy Behavior in Quantum Field Theory**

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An attack is made on the problem of determining the asymptotic behavior at high energies and momenta of the Green's functions of quantum field theory, using new mathematical methods from the theory of real variables. We define a class \( A_n \) of functions of \( n \) real variables, whose asymptotic behavior may be specified in a certain manner by means of certain "asymptotic coefficients." The Feynman integrands of perturbation theory (with energies taken imaginary) belong to such classes. We then prove that if certain conditions on the asymptotic coefficients are satisfied then an integral over \( k \) of the variables converges, and belongs to the class \( A_{n+1} \) with new asymptotic coefficients simply related to the old ones. When applied to perturbation theory this theorem validates the renormalization procedure of Dyson and Salam, proving that the renormalized integrals actually do always converge, and provides a simple rule for calculating the asymptotic behavior of any Green's function to any order of perturbation theory.

**I. INTRODUCTION**

In many respects, the central formal problem of the modern quantum theory of fields is the determination of the asymptotic behavior at high energies and momenta of the Green's functions of the theory, the vacuum expectation values of time-ordered products. Complete knowledge of the asymptotic properties of these functions would allow us to test the renormalizability of a given Lagrangian, to count the number of subtractions that must be performed in dispersion theory, etc. We shall attack this problem from a rather new direction, which allows a solution in perturbation theory, and which provides an analytic tool that may prove useful in solving the problem in the exact theory.

One might hope to find a solution either kinematically, using only assumptions of covariance, causality, etc., or dynamically, by using the field equations that actually determine the Green's functions. The first method has been successfully applied to the 2-field functions, the particle propagators, and yields the result that the true propagators are asymptotically "larger" than the bare propagators. However, because the theory of several complex variables is so difficult and incomplete, this approach seems unpromising for expectation values of three or more fields. For this reason, and also because we would eventually like to obtain renormalizability conditions on the Lagrangian, we propose to attack the problem on the dynamical level.

Now, what are the equations that, in principle, would determine the Green's functions. In perturbation theory we know that the Green's functions appear as multiple integrals, the integrand being constructed according to the Feynman rules. In a nonperturbative approach the Green's functions are again given by multiple integrals, but with integrands that themselves depend on the

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II. A SIMPLE EXAMPLE

Our task will be to define precisely what we mean by the asymptotic behavior of functions of several real variables, and to show how the convergence and asymptotic properties of multiple integrals may be determined directly from the asymptotic behavior of their integrands. Before proceeding to the general case, we shall discuss a simple example that will illustrate the problems to be faced and our approach in solving them.

Consider the integral \( \Sigma(p') \), defined by

\[
\Sigma(p') = \int_{-\infty}^{\infty} d\rho'' f(p', \rho''),
\]

\[
f(p', \rho'') = \frac{\rho''}{(p'^2 + m^2)(\rho' - p')^2 + \mu^2}
\]

Except for the fact that \( p', \rho'' \) are one-dimensional real variables, instead of four-vectors, \( \Sigma(p') \) is just the lowest order "fermion" self-energy insertion, in a theory with "fermions" of mass \( m \), and interaction \( \psi^\dagger \phi; \) \( p' \) is the fermion momentum, and \( \rho'' \) and \( \rho'' - p' \) the momenta of the virtual fermion and boson lines.

Now, of course, the function \( f(p', \rho'') \) is so simple that one sees immediately that \( \Sigma(p') \) converges, and can compute,

\[
\Sigma(p') = \frac{\pi p' \rho''^2 + (\mu - m)^2}{\mu \rho''^2 + 2 \rho''^2 \mu^2 + (\mu^2 - m^2)^2}
\]

so that

\[
\Sigma(p') = O(p''^{-1}) \quad \text{as} \quad p'' \to \infty.
\]

Usually we are not able to proceed so directly; we may have to deal with complicated functions of many real variables which may not even be entirely known. Therefore, we wish to find some way of characterizing the asymptotic behavior of \( f(p', \rho'') \) so that, with no further information, we may obtain the asymptotic behavior (4) of its integral.

It is very convenient to introduce a vector notation, writing

\[
f(P) = \frac{P \cdot V}{[(P \cdot V)^2 + m^2][(P \cdot V)^2 + \mu^2]}
\]

where

\[
P = (p', \rho''), \quad V = (0,1), \quad V' = (-1, +1).
\]

Suppose we consider the behavior of \( f(P) \) as \( P \) tends to infinity along some fixed line. It is apparent that this behavior depends strongly on the direction of the line. If we let \( P = L \eta + C \), where \( L \) and \( C \) are fixed vectors, and \( \eta \to \infty \), then

\[
f(L \eta + C) = O(\eta^{-\alpha(L)})
\]

where

\[
\alpha(L) = \begin{cases} -3 & \text{if } L \cdot V \neq 0, \ L \cdot V' \neq 0 \\ -2 & \text{if } L \cdot V = 0, \ L \cdot V' \neq 0 \\ -1 & \text{if } L \cdot V' = 0, \ L \cdot V = 0. \end{cases}
\]

This behavior is indicated in Fig. 1. To be a little more
precise, if we confine $C$ to a finite region $W$ in the $(p', p'')$ plane, then for any $L$ there exist positive numbers $b(L, W)$ and $M(L, W)$, such that
\[
|f(p, p')| \leq M(L, W)\eta^a \eta^b(L)
\]  
for $C$ in $W$ and $\eta \geq b(L, W)$.

Now it is unfortunately the case that simply stating (8) and (9) would not allow us to tell anything about the asymptotic behavior of $\Sigma(p')$. It is necessary also to give some information about the behavior of $b(L, W)$. This may be most easily accomplished, if we introduce two positive variables, $v_1$ and $v_2$, which will be allowed to go to infinity independently. Let us set
\[
P = L_1L_2 + L_2L_2 + C,
\]
where $L_1$ and $L_2$ are fixed and independent, and $C$ is again confined to a finite region $W$ in the $(p', p'')$ plane.

It is easy to see that then
\[
f(L_1L_1 + L_2L_2 + C) = O(\eta^a \eta^b(L))
\]  
and, in other words, there exist positive numbers $b_1(L_1L_1, W)$, $b_2(L_1L_2, W)$, $M(L_1L_2, W)$ such that
\[
|f(L_1L_1 + L_2L_2 + C)| \leq M(L_1L_2)\eta^a \eta^b(L)
\]  
whenever $C$ is in $W$, and
\[
v_1 \geq b_1(L_1L_1, W), \quad v_2 \geq b_2(L_1L_2, W).
\]

Proof: A simple calculation shows that if $L_1 \cdot V \neq 0$,
\[
\frac{|P \cdot V|}{(P \cdot V)^2 + \mu^2} \leq \eta^{-1} \eta^{-1} \frac{4}{|L_1 \cdot V|}
\]
for $\eta \geq 2|C \cdot V|/|L_1 \cdot V|$, (14)

while if $L_1 \cdot V = 0$ (and hence $L_2 \cdot V \neq 0$),
\[
\frac{|P \cdot V|}{(P \cdot V)^2 + \mu^2} \leq \eta^{-1} \frac{4}{|L_1 \cdot V|}
\]
for $\eta \geq 2|C \cdot V|/|L_2 \cdot V|$. (15)

Furthermore, if $L_1 \cdot V = 0$ (and hence $L_2 \cdot V \neq 0$),
\[
\frac{1}{(P \cdot V)^2 + \mu^2} \leq \eta^{-2} \frac{4}{|L_1 \cdot V|^2}
\]
for $\eta \geq 2|C \cdot V|/|L_2 \cdot V|$, (16)

while if $L_1 \cdot V = 0$ (and hence $L_2 \cdot V \neq 0$),
\[
\frac{1}{(P \cdot V)^2 + \mu^2} \leq \eta^{-2} \frac{4}{|L_2 \cdot V|^2}
\]
for $\eta \geq 2|C \cdot V|/|L_2 \cdot V|$. (17)

Multiplying (14) or (15) by (16) or (17), and referring back to (8), shows that (11) is correct.

It is the special circumstance summarized in (11) that establishes $f(P)$ as a member of the class (later called $A_n$ in the case of $n$ variables) of functions with which we shall deal, and that allows us to obtain (4). We shall say in this particular case that $\alpha(L, p)$ is the "asymptotic coefficient" associated with the line $L_1$, and that $\alpha(L_1, L_2) = -3$ is the coefficient associated with the whole $(p', p'')$ plane $\{L_1, L_2\}$. (We use $\{L_1, L_2, \cdots \}$ to denote the subspace spanned by the vectors $L_1, L_2, \cdots$.

It is worth emphasizing how much stronger (11) is than the statement (7). According to (7) alone we can easily see that as $\eta \to \infty$,
\[
f(L_1L_1 + L_2L_2 + C) = O\{\eta^a(\log \eta + L_1 + L_2)\},
\]
or more fully, that
\[
|f(L_1L_1 + L_2L_2 + C)| \leq M(L_1L_1 + L_2L_2)\eta^a(\log \eta + L_1 + L_2),
\]
for $C$ restricted to $W$ and
\[
\eta \geq b(L_1L_1 + L_2L_2).
\]

Furthermore, it is obvious that for any $L_1, L_2$, the vector $L_1L_2 + L_2L_2$ will be orthogonal to $V$ or $V'$ for sufficiently large $\eta$, so that for $\eta$ large enough,
\[
\alpha(L_1L_1 + L_2L_2) = -3.
\]
What is not obvious, and indeed is not contained in (7), is that we can find numbers $b_1(L_1L_1, W), b_2(L_1L_2, W)$, and $M(L_1L_2, W)$, such that (12) holds, or alternatively, by comparison with (19) that
\[
M(L_1L_1 + L_2L_2, W) \leq M(L_1L_2, W)\eta^a(L_1),
\]
\[
b(L_1L_1 + L_2L_2, W) \leq b_2(L_1L_2, W),
\]
\[
\alpha(L_1L_1 + L_2L_2) = -3.
\]

We shall now consider the general case of functions $f(p', p'', \cdots)$ of $n$ real variables $p', p'', \cdots$, which for convenience we unite into a vector $P$ in the $n$-dimensional linear vector space $R_n$. We will define a class $A_n$ of such functions, whose asymptotic behavior for high $P$ may be specified in a certain manner, by means of certain "asymptotic coefficients." (The integrands of covariant perturbation theory, constructed according to the Feynman rules, with the $n$ real variables taken as all
components of all internal and external momenta, are shown in Sec. V to belong to \( A_n \), providing that all energy integration contours may be rotated up to the imaginary axis.) The exact definition of the classes \( A_n \) and of the “asymptotic coefficients” is chosen in just such a way that we will be able to prove the asymptotic theorem, which says that if a function belongs to \( A_n \), any sufficiently convergent integral over \( k \) of its arguments belongs to \( A_{n-k} \) and which provides a rule for calculating the convergence properties and asymptotic coefficients of the integral in terms of the asymptotic coefficients of the integrand.

**Definition**

The function \( f(P) \) is said to belong to the class \( A_n \) if to every subspace \( S \subset R_n \) there corresponds a pair of coefficients, a “power” \( \alpha(S) \) and a “logarithmic power” \( \beta(S) \), and for any choice of \( m \leq n \) independent vectors \( L_1 \ldots L_m \) and finite region \( W \) in \( R_n \) we have

\[
 f(L_1 \eta_1 \ldots \eta_m + L_2 \eta_2 + \ldots + L_m \eta_m + C) \\
 = O(\eta_1 \ldots \eta_m |L_1| \ldots |L_m| \eta_1 \ldots \eta_m) \\
 \times \ldots \times O(\eta_1 \ldots \eta_m |L_1| \ldots |L_m| \eta_1 \ldots \eta_m),
\]

(1)

if \( \eta_1 \ldots \eta_m \) tend independently to infinity with \( C \) confined to \( W \). (Here \( \alpha(\{L_1 \ldots L_r\}) \) and \( \beta(\{L_1 \ldots L_r\}) \) are the asymptotic coefficients associated with the subspace \( \{L_1 \ldots L_r\} \) spanned by the vectors \( L_1 \ldots L_r \).) More precisely, there exists a set of numbers \( \delta_1 \ldots \delta_m > 0 \) and \( M > 0 \) (depending on \( L_1 \ldots L_m \) and \( W \) but not of course on the \( \eta_1 \ldots \eta_m \)), such that

\[
 |f(L_1 \eta_1 \ldots \eta_m + L_2 \eta_2 + \ldots + L_m \eta_m + C)| \\
 \leq M \eta_1 \ldots \eta_m |L_1| \ldots |L_m| \eta_1 \ldots \eta_m \\
 \times (\ldots (|L_1| \ldots |L_m| \eta_1 \ldots \eta_m)|L_1| \ldots |L_m| \eta_1 \ldots \eta_m),
\]

(2)

provided that \( C \in W \), and that

\[
 \eta_1 \geq \delta_1, \ldots, \eta_m \geq \delta_m. \tag{3}
\]

It may readily be observed that \( \alpha(S) \), \( \beta(S) \) are the values of \( \alpha(L) \), \( \beta(L) \) for \( L \) a “typical” vector in the subspace \( S \). Suppose we let only \( \eta_1 \) go to infinity in (1), keeping \( \eta_2 \ldots \eta_m \) fixed and satisfying

\[
 \eta_1 \geq \delta_1, \ldots, \eta_m \geq \delta_m. \tag{4}
\]

It then follows from (1) that for \( S = \{L_1 L_2 \ldots L_m\} \)

\[
 f([L_1 \eta_1 \ldots \eta_m-1 + L_2 \eta_2 + \ldots + L_m \eta_m+1 + C] \\
 + \ldots + L_m \eta_m+1 + L_m \eta_m+1 + C]) \\
 = O(\eta_1 \ldots \eta_m |L_1| \ldots |L_m| \eta_1 \ldots \eta_m),
\]

(5)

so that we can take

\[
 \alpha(L_1 \eta_1 \ldots \eta_m-1 + L_2 \eta_2 + \ldots + L_m \eta_m) = \alpha(S), \tag{6}
\]

and likewise for \( \beta \). The conditions (4) just ensure that the vector \( L = L_1 \eta_1 \ldots \eta_m-1 + L_2 \eta_2 + \ldots + L_m \eta_m \) does not take on some special direction for which \( \alpha(L) > \alpha(S) \).

We shall refer to \( \alpha(S) \), \( \beta(S) \) therefore as the asymptotic coefficients of \( f(P) \) for \( P \to \infty \) along typical directions in \( S \).

In the special example given in Sec. II, we saw that \( \alpha(R_2) = -3 \), where \( R_2 \) was the whole \( \rho' \), \( \rho'' \) plane. Furthermore, this was also the value of \( \alpha(L) \) for almost all vectors \( L \) in \( R_2 \); the only exceptions being \( L \sim (1,0) \) (with \( \alpha = -2 \)) and \( L \sim (1,1) \) (with \( \alpha = -1 \)), as shown in Fig. 1. By taking \( L = \eta_1 L_1 + \eta_2 L_2 \), where \( L_1 \) and \( L_2 \) are any fixed vectors and \( \eta_1 \) is sufficiently large, we could always avoid these two special directions.

We now consider an integral of \( f(P) \), given as

\[
 f_{L_1 \ldots L_m}(P) = \int_{-\infty}^{\infty} dy_1 \ldots \\
 \times \int_{-\infty}^{\infty} dy_i f(P + L_i y_i + \ldots + L_m y_m). \tag{7}
\]

In the example of Sec. II, for instance, we had

\[
 \Sigma(P') = f_L(L P'), \quad L' = (0,1), \quad L = (1,0). \tag{8}
\]

We shall say that the integral (7) “exists” if every subsequent integration converges in the iterated integral

\[
 \int_{-\infty}^{\infty} dy_1 \ldots \int_{-\infty}^{\infty} dy_i \mid f(P + L_i y_i + \ldots + L_m y_m) \mid. \tag{9}
\]

In this case, by a simple application of Fubini’s theorem, (with \( f \) integrable on the subspace \( I \subset R_n \)), but only on the subspace \( I \subset R_n \), that span. We therefore write in this case

\[
 f_{I}(P) = f_{L_1 \ldots L_m}(P) \quad \text{for} \quad I = \{L_1 \ldots L_m\}
\]

\[
 = \int_{P \in I} d^k P \cdot f(P + P'). \tag{10}
\]

Furthermore \( f_{I}(P) \) does not change if we add to \( P \) any vector in \( I \); in other words, \( f(P) \) depends only on the projection of \( P \) along the subspace \( I \). It is convenient to choose some particular subspace \( E \) such that \( R_n = I + E \), with \( I \) and \( E \) independent, and restrict \( P \) to \( E \). In applications to perturbation theory, a vector in \( I \) or \( E \) will have as its components the momentum components of the internal or external particle lines. [In Sec. II, \( I = \{L'\}, \quad E = \{L\} \). (III-8.)] We may now state the general asymptotic theorem.

**Theorem**

If a function \( f(P) \) belongs to \( A_n \), with asymptotic coefficients \( \alpha(S), \beta(S) \) for \( S \) any non-null subspace of \( R_n \), and if \( f(P) \) is integrable over any finite region of \( R_n \), then if \( D_I < 0 \), where

\[
 D_I = \max \{\alpha(S') + \dim S'\}, \tag{11}
\]

\[ S' \subset I \]

then the following statements hold:

(a) \( f_I(P) \) exists;
(b) \( f_I(P) \in A_{n-k} \), with asymptotic coefficient \( \alpha_I(S) \) given for any \( S \subseteq E \) by
\[
\alpha_I(S) = \max_{\Lambda(I)S' = S} \left[ \alpha(S') + \dim S' - \dim S \right].
\] (12)

Here \( S' \subseteq I \) means that \( S' \) is a subspace of \( I \), including the possibility \( S'=I \); \( \dim S' \) is the dimensionality of \( S' \); \( \Lambda(I)S' = S \) means that the projection of \( S' \) along \( I \) on \( E \) is \( S \) (this last is discussed in detail in the Appendix). The “max” in (11) and (12) means that we take the maximum over all subspaces \( S' \) satisfying \( S' \subseteq I \) and \( \Lambda(I)S' = S \), respectively. Actually, by using the Heine-Borel theorem we will be able to show in the next section that only a finite number of \( S' \) need be taken into account, so that “max” is finite.

Let us now see how this theorem may be applied to the example of Sec. II. There \( I \) was one-dimensional, so that the only \( S' \subseteq I \) is \( S' = I \). Thus
\[
D_I = \alpha(I) + \dim I = -3 + 1 = -2 < 0,
\] (13)
so that
(a) \( \Sigma(p') \) exists;
(b) \( \Sigma(p') \in A_1 \), which means that we can write
\[
\Sigma(p') = O(p^{\alpha(I)B} \log p^{B(1D)}).
\] (14)

Note that \( \Lambda(I)S' = E \) is satisfied for \( S' = R_0 = I + E \), and for \( S' \) any line in \( R_0 \) except \( I \). Thus by (12) and (II-8),
\[
\alpha_I(E) = \max \left[ -3 + 2 - 1, \quad (S' = R_0) \\
-1 + 1 - 1, \quad (S' = \{(1,1)\}) \\
-2 + 1 - 1, \quad (S' = \{(1,0)\}) \\
-3 + 1 - 1 \right] \quad (S' \text{ other lines})
= -1,
\] (15)
which agrees with (II-4).

It is interesting to compare statement (a) of the asymptotic theorem with what we should expect if we attempted to estimate the degree of divergence of integrals by just “counting powers” of the arguments \( p', p'' \), \( \cdots \). Naively, we should expect the degree of divergence of the integral \( f_I \) of \( f \) over any subspace \( I' \), to be given by the asymptotic power \( \alpha(I') \) of \( f(P) \) for \( P \rightarrow \infty \) along typical directions in \( I' \), plus one unit for each integration performed. We shall call this quantity
\[
D_{I'} = \alpha(I') + \dim I',
\] (16)
the superficial divergence of the integral of \( f \) over \( I' \). According to part (a) of the asymptotic theorem, however, the sufficient condition for existence of the integral of \( f \) over \( I \) is not just that the integral converge superficially, (i.e., \( D_I < 0 \)), but that all subintegrations also converge superficially (i.e., \( D_{S'} < 0 \) for \( S' \subseteq I \)), since according to (11),
\[
D_I = \max_{S' \subseteq I} D_{S'}.
\] (17)

It must be stressed that by all subintegrations we mean all iterated integrals for all possible linear recombinations of the integration variables. This is the mathematical foundation of the perturbative renormalization theory, to be discussed in Sec. V.

IV. THE ASYMPTOTIC THEOREM: PROOF AND POSSIBLE EXTENSIONS

Our proof is by strong mathematical induction, and divided into the following steps:

(A) We prove by purely geometrical reasoning that if the theorem holds whenever \( \dim I \leq k \) (where \( k \geq 1 \)) then it also holds for \( \dim I = k + 1 \), so that it is only necessary to prove the theorem in the case \( \dim I = 1 \).

(B) We consider the case \( I = \{L\} \) (where \( L \) is some vector \( e_{R_a} \)) and show (trivially) that \( f_I(P) \) converges absolutely if \( D_I < 0 \).

(C) We describe a method of covering the infinite interval of integration of \( f_I \) with a finite number of subintervals \( J \).

(D) We show that if \( D_I < 0 \) that the sum of the integrals over the intervals \( J \) belongs to \( A_{n-1} \), with \( \alpha_I \) given by part (b) of the asymptotic theorem.

(A): Assume the theorem holds whenever the subspace of integration has dimensionality \( \leq k \). Let \( I \) be a \( (k+1) \)-dimensional subspace of \( R_a \). We decompose \( I \) into
\[
I = S_1 + S_2,
\] (1)
where \( S_1 \) and \( S_2 \) are some (non-null) independent subspaces of \( R_a \) with dimensions \( k_1, k_2 \) necessarily \( \leq k \). (We could always choose \( S_1 \) or \( S_2 \) to be one-dimensional, but the proof is then less illuminating.) Let \( f \in A_\tau \). The integral \( f_I \) can be written \( f_I = (f_{S_1})_{S_1} \), or in other words,
\[
f_I(P) = \int_{P \in S_1} d^{4}P' f_{S_2}(P + P').
\] (2)

By our induction hypothesis we can apply the asymptotic theorem to both the \( S_2 \) and \( S_1 \) integrations, obtaining the following results:

(a1) \( f_{S_2} \) converges absolutely if \( D_{S_2}(f) < 0 \), where
\[
D_{S_2}(f) = \max_{S' \subseteq S_2} \left[ \alpha(S') + \dim S' \right].
\] (3)

(b1) If \( D_{S_2}(f) < 0 \), then \( f_{S_2} \in A_{n-1} \), with
\[
\alpha_{S_2}(S') = \max_{\Lambda(S_2)S' = S'} \left[ \alpha(S') + \dim S' - \dim S' \right].
\] (4)

(a2) If \( f_{S_2} \in A_{n-1} \), then \( f_I \) as given by (2) converges absolutely if \( D_{S_1}(f_{S_2}) < 0 \), where
\[
D_{S_1}(f_{S_2}) = \max_{S' \subseteq S_1} \left[ \alpha_{S_1}(S') + \dim S' \right].
\] (5)

(b2) If \( f_{S_2} \in A_{n-1} \) and \( D_{S_1}(f_{S_2}) < 0 \), then \( f_I \) as given
by (2) belongs to \( A_{n-k-1} \), with
\[
\alpha_t(S) = \max_{\Lambda(S)_*S' = S} \left[ \alpha_{S*}(S') + \dim S' - \dim S \right].
\] (6)
The whole integral clearly converges if both \( D_{S*}(f) < 0 \) and \( D_{S*}(f_S) < 0 \), i.e., if \( D_I < 0 \) where
\[
D_I = \max_{\Lambda(S)_*S' = S} \left[ D_{S*}(f), D_{S*}(f_S) \right].
\] (7)
Inserting (3), (4), and (5) into (7), we obtain
\[
D_I = \max_{S'' \subseteq S' \subseteq I} \left[ \alpha(S'') + \dim S'' \right],
\] (8)
where “\( \max^{*} \)” runs over all \( S'' \) satisfying \( S'' \subseteq S \) or \( \Lambda(S_2)_*S'' \subseteq S \). According to statement (D) in the Appendix, this means that \( \max^{*} \) runs over all \( S'' \) satisfying the condition \( S'' \subseteq S_1 + S_2 \), so that (8) becomes
\[
D_I = \max_{S'' \subseteq I} \left[ \alpha(S'') + \dim S'' \right],
\] (9)
which completes the proof of part (a) of the theorem. Turning to part (b), we see that if \( D_I < 0 \) then combining (b1), (b2), and (7) we have \( f e A_{n-m-1} \). Combining (4) and (6), the asymptotic power of \( f_I \) is
\[
\alpha_I(S) = \max_{\Lambda(S)_*S' = S} \left[ \dim S' - \dim S \right] + \max_{\Lambda(S_2)_*S'' = S'} \left[ \alpha(S'') + \dim S'' - \dim S \right]
\] (10)
\[
= \max_{\Lambda(S_1)_*S'' = S'} \left[ \alpha(S'') + \dim S'' - \dim S \right] \quad \text{(11)}
\]
By statement (E) in the Appendix the double condition \( \Lambda(S_1)_*S' = S \) and \( \Lambda(S_2)_*S'' = S' \) can be replaced by
\[
S = \Lambda(S_1 + S_2)_*S'' = \Lambda(D)_*S',
\] (12)
so that (10) proves part (b) of the theorem.

(B): We wish to consider \( I = \{ L \} \), and
\[
f_I(P) = f_L(P) = \int_{-\infty}^{\infty} f(P+L)dy,
\] (13)
where \( f e A_n \). According to (III-1), we have
\[
f(P+L) = O(y^{\omega(L)} (\ln y)^{\beta(L)}) \quad \text{as} \quad y \to \infty,
\] (14)
so clearly (12) converges absolutely if
\[
\alpha(L) + 1 < 0.
\] (15)
Since the only non-null subspace of \( I \) is \( I \) itself, we have according to (III-11),
\[
D_I = \alpha(L) + 1,
\] (16)
so that part (a) of the theorem is verified.

(C): Suppose we choose any sequence \( L_1 \cdots L_m \) of vectors \( eR_n \) (independent of each other and of \( L \)) and a finite region \( W \) in \( R_n \); our task is then to prove that if
\[
f e A_n, \ f e A_{n-1}, \text{ or in other words,}
\]
\[
f_I(P) = O(\eta_1^{a(L)}(\ln \eta_1)^{\beta(L)} \cdots \eta_m^{a(L)}(\ln \eta_m)^{\beta(L)})
\] (17)
for
\[
P = L_1 \eta_1 + \cdots + L_m \eta_m + C,
\] (18)
where \( \eta_1 \cdots \eta_m \) tend independently to infinity with \( C \) confined to \( W \). We must also verify (III-12), showing that
\[
\alpha_L(S) = \max_{\Lambda(L)_*S' = S} \left[ \alpha(S') + \dim S' - \dim S \right].
\] (19)
This to end we will first describe a decomposition of the interval \( -\infty < y < \infty \) in (12) into a finite set of intervals \( J \), each of which contribute a “term” to (18). Consider the sequence of \( m+1 \) independent vectors,
\[
L_1 + u_1 L, L_2 + u_2 L, \cdots, L_r + u_r L, L_1, L_2, \cdots, L_m,
\] (20)
where \( 0 \leq r \leq m \) and \( u_1 \cdots u_r \) are a set of \( r \) real variables. Since \( f e A_n \), there must exist a set of numbers
\[
b_1(u_1 \cdots u_r) > 1 \quad (0 \leq l \leq r), \quad M(u_1 \cdots u_r) > 0,
\] (21)
such that
\[
|f((L_1 + u_1 L)\eta_1 + \cdots + (L_r + u_r L)\eta_r + (L_{r+1} + \cdots + L_m)\eta_r + \cdots + L_m \eta_m + C)|
\] (22)
\[
= M(u_1 \cdots u_r) \eta_1^{\alpha(L_1)}(\ln \eta_1)^{\beta(L_1)} \cdots \eta_r^{\alpha(L_r)}(\ln \eta_r)^{\beta(L_r)}(\ln \eta_{r+1}^{\alpha(L_{r+1})}(\ln \eta_{r+1})^{\beta(L_{r+1})})
\] (23)
\[
\times \cdots \times \eta_m^{\alpha(L_m)}(\ln \eta_m)^{\beta(L_m)}(\ln \eta_{m+1})^{\beta(L_m+1)}(\ln \eta_{m+2})^{\beta(L_{m+2})}
\] (24)
\[
\times (\ln \eta_{m+3})^{\beta(L_{m+3})}(\ln \eta_{m+4})^{\beta(L_{m+4})}
\] (25)
where all \( \eta_i \geq b_1(u_1 \cdots u_r) \) and \( C e W \).

Now let us consider the closed interval \( -b_0, b_0 \). [When we refer to an interval as \( (a,b) \) we mean the set of all \( u \) with \( a \leq u \leq b \); by “\( b_0 \)” we mean the \( b \) function with \( l = r = 0 \).] Every point \( u \) on this line is in the interior of a closed interval, \( [u - b_1^{-1}(u), u + b_1^{-1}(u)] \). Therefore, by the Heine-Borel theorem8 we can find a finite set of points \( U \) with \( |U| \leq b_0 \) each \( U \) contained in an interval \( U_i = \lambda_i \eta_i, U_i + \lambda_i \eta_i \), such that the intervals \( U_i = \lambda_i \eta_i, U_i + \lambda_i \eta_i \) cover the entire closed interval \( -b_0, b_0 \), and such that \( 0 < \lambda_i \leq b_1^{-1}(U_i) \). Now consider any particular \( i \), and the closed interval \( [-b_0(U_i), b_0(U_i)] \) Again we may use the Heine-Borel theorem, and obtain a finite set of points \( U_{ij} \) with \( |U_{ij}| \leq b_0(U_i) \), and a set of closed intervals \( U_{ij} = \lambda_{ij} \eta_{ij} U_i + \lambda_{ij} \eta_{ij} \), which cover the finite line \( [-b_0(U_i), b_0(U_i)] \), such that \( \lambda_{ij} \leq b_0(U_{ij}) \). Continuing in this fashion, we find \( m \) finite sets of points \( U_{it}, U_{i't}, \cdots, U_{i'm} \) and numbers \( \lambda_{it}, \lambda_{i't}, \cdots, \lambda_{i'm} \) such that for any \( r \leq m \), the

---

\[(U_{i_1} \cdots i_r, \lambda_{i_1} \cdots i_r, U_{i_1} \cdots i_r + \lambda_{i_1} \cdots i_r) \text{ cover the line}\]
\[\begin{align*}
&(-b_0(i_1 \cdots i_{r-1}, b_0(i_1 \cdots i_{r-1}),) \\
&0 < \lambda_{i_1} \cdots i_r \leq 1/b_0(i_1 \cdots i_r), \\
&\abs{U_{i_1} \cdots i_r} \leq b_0(i_1 \cdots i_{r-1})
\end{align*}\]
\tag{20}
\[\begin{align*}
&\text{where we introduce the notation} \\
&b_l(i_1 \cdots i_r) \equiv b_l(U_{i_1} U_{i_1 l} \cdots U_{i_1 i_r}). \tag{23}
\end{align*}\]

We shall use (20), (21), (22) to split up the integration in (12). Take any set of \(\eta_1 \cdots \eta_m > 1\). We will define a set of intervals \(J_{i_1} \cdots i_r(\eta) (r \leq m)\) to consist of all \(y\) that may be written
\[y = U_{i_1} \eta_{i_1} \cdots \eta_m + U_{i_1} \eta_{i_1} \cdots \eta_m + \sum \eta_{i_{r+1}} \cdots \eta_m, \tag{24}\]
where \(\lambda_{i_1} \cdots i_r \geq \abs{y} = \pm \varepsilon \geq b_0(i_1 \cdots i_r). \tag{25}\)
For the case \(r = 0\) the intervals \(J_{i_1} (\eta)\) are defined to consist of all \(y\) with
\[\pm y = \abs{y} \geq b_0(i_1 \cdots i_r) \eta_1 \cdots \eta_m, \tag{26}\]
and the intervals \(J_{i_1} \cdots i_m\) are defined to consist of all \(y\) that may be written
\[y = U_{i_1} \eta_{i_1} \cdots \eta_m + U_{i_1} \eta_{i_1} \cdots \eta_m + \cdots + \sum i_{r+1} \cdots i_m + z, \tag{27}\]
with \(\abs{z} \leq b_0(i_1 \cdots i_r). \tag{28}\)

It is easy to see that every real \(y\) belongs to at least one of these intervals. For if \(y\) is not in \(J_{i_1} (\eta)\), then \(\abs{y} \leq b_0(i_1 \cdots i_r) \eta_1 \cdots \eta_m\). However, the finite line \((-b_0(i_1 \cdots i_r) \eta_1 \cdots \eta_m + b_0(i_1 \cdots i_r) \eta_1 \cdots \eta_m)\) is covered by the intervals \((U_{i_1} \eta_{i_1} \cdots \eta_m - \lambda_{i_1} \cdots \eta_m, U_{i_1} \eta_{i_1} \cdots \eta_m + \lambda_{i_1} \cdots \eta_m)\). Therefore, we may set
\[y = U_{i_1} \eta_{i_1} \cdots \eta_m + y', \quad \abs{y'} \leq \eta_1 \cdots \eta_m b_0(i_1 \cdots i_r), \tag{29}\]
for some \(y'\). This implies that if \(\pm y' \geq \eta_2 \cdots \eta_m b_0(i_1)\) then \(y \in J_{i_1} (\eta)\) according to (24), (25). On the other hand, if \(\abs{y'} \leq \eta_2 \cdots \eta_m b_0(i_1)\), we can again place it in a covering interval, so that
\[y = U_{i_1} \eta_{i_1} \cdots \eta_m + U_{i_1} \eta_{i_1} \cdots \eta_m + y'' , \quad \abs{y''} \leq \eta_2 \cdots \eta_m \lambda_{i_1} \cdots i_r. \tag{30}\]
Thus if \(\pm y'' \geq \eta_2 \cdots \eta_m b_0(i_1)\), \(y \in J_{i_1} (\eta)\). This process can be continued up to the final alternative, which is \(y \in J_{i_1} \cdots i_m (\eta)\). It therefore follows that
\[\begin{align*}
&\int f(A) \leq \sum_{r=0}^{m} \sum_{i_1 \cdots i_r} \int_{J_{i_1} \cdots i_r(\eta)} \abs{y} \abs{f(P + Ly)} \\
+ \sum_{i_1 \cdots i_m} \int_{J_{i_1} \cdots i_m (\eta)} \abs{y} \abs{f(P + Ly)} \tag{29}
\end{align*}\]
(D): Now we shall examine the asymptotic behavior of each term in (29) with \(P\) given by (17).
We can always take the original \( \beta(S) \) to be non-negative integers, for since the asymptotic coefficients only set an upper bound on the behaviour of \( f(P) \), we are free to increase them as needed. [Of course, our arbitrarily increasing the \( \beta(S) \) to be non-negative integers causes the final formula for \( \beta_L(S) \) to lose its significance.] Therefore we can find numbers \( c(i_1 \cdots i_r) > 1 \), \( N(i_1 \cdots i_r) > 0 \) such that

\[
\begin{align*}
\int_{J_{i_1 \cdots i_r}} dy |f(P+Ly)| & \leq M(i_1 \cdots i_r) N(i_1 \cdots i_r) \\
\times & \eta_1^{\alpha(i_1 \cdots i_r)} (\ln \eta_1)^{\beta(i_1 \cdots i_r)} (\ln \eta_2)^{\beta(i_1 \cdots i_r-1)} \\
\times & \eta_2^{\alpha(i_1 \cdots i_r)} (\ln \eta_2)^{\beta(i_1 \cdots i_r)} (\ln \eta_3)^{\beta(i_1 \cdots i_r-1)} \\
\times & \eta_3^{\alpha(i_1 \cdots i_r)} (\ln \eta_3)^{\beta(i_1 \cdots i_r)} (\ln \eta_4)^{\beta(i_1 \cdots i_r-1)} \\
\times & \cdots \\
\times & \eta_m^{\alpha(i_1 \cdots i_r)} (\ln \eta_m)^{\beta(i_1 \cdots i_r)} (\ln \eta_{m-1})^{\beta(i_1 \cdots i_r-1)},
\end{align*}
\]

if \( \alpha(i_1 \cdots i_r) = \alpha(i_1 \cdots i_r)+1 \)

\[
\begin{align*}
\times & \eta_1^{\alpha(i_1 \cdots i_r)} (\ln \eta_1)^{\beta(i_1 \cdots i_r)} \\
\times & \eta_2^{\alpha(i_1 \cdots i_r)} (\ln \eta_2)^{\beta(i_1 \cdots i_r)} \\
\times & \eta_3^{\alpha(i_1 \cdots i_r)} (\ln \eta_3)^{\beta(i_1 \cdots i_r)} \\
\times & \cdots \\
\times & \eta_m^{\alpha(i_1 \cdots i_r)} (\ln \eta_m)^{\beta(i_1 \cdots i_r)} (\ln \eta_{m-1})^{\beta(i_1 \cdots i_r)},
\end{align*}
\]

if \( \alpha(i_1 \cdots i_r) > \alpha(i_1 \cdots i_r)+1 \),

whenever

\[
\eta_l \geq b(i_1 \cdots i_r) \quad (l \neq r), \quad \eta_l \geq c(i_1 \cdots i_r).
\]

[Note that the interval \( J_{i_1 \cdots i_r}(\eta) \) does not contribute unless \( \eta_l \geq b(i_1 \cdots i_r) b_0(i_1 \cdots i_r) \).

Now let us consider the two infinite intervals

\[
\begin{align*}
\int_{J_{i_1 \cdots i_r}(\eta)} dy |f(P+Ly)| &= \int_{0 \leq \eta_1 \cdots \eta_m} f(P \pm Ly) dy, \\
\int_{J_{i_1 \cdots i_r}(\eta)} dy |f(P+Ly)| &= \int_{0 \leq \eta_1 \cdots \eta_m} f(P \pm Ly) dy.
\end{align*}
\]

After making a change of variable,

\[
\begin{align*}
\int_{J_{i_1 \cdots i_r}(\eta)} dy |f(P+Ly)| &= \eta_1 \cdots \eta_m \int_{0 \leq \eta_1 \cdots \eta_m} f(P \pm Ly) dy. \\
\int_{J_{i_1 \cdots i_r}(\eta)} dy |f(P+Ly)| &= \eta_1 \cdots \eta_m \int_{0 \leq \eta_1 \cdots \eta_m} f(P \pm Ly) dy.
\end{align*}
\]

Therefore, applying (19) for \( r=0 \), \( \eta_0 = - \) we have

\[
\begin{align*}
\int_{J_{i_1 \cdots i_r}(\eta)} dy |f(P+Ly)| & \leq M N \eta_1^{\alpha(i_1 \cdots i_r)} (\ln \eta_1)^{\beta(i_1 \cdots i_r)} \\
\times & \cdots \eta_m^{\alpha(i_1 \cdots i_r)} (\ln \eta_m)^{\beta(i_1 \cdots i_r)} (\ln \eta_{m-1} L),
\end{align*}
\]

whenever

\[
\eta_l \geq b_l \quad (1 \leq l \leq m).
\]

Here \( N \) is the finite positive number

\[
N = \int_0^{\infty} x^{\alpha(L)} (\ln x)^{\beta(L)} dx,
\]

where we recall that \( \alpha(L)+1 < 0 \) by hypothesis.

Finally, we consider the case \( \gamma \epsilon J_{i_1 \cdots i_r}(\eta) \). Combining (27) and (17), we have

\[
P+Ly = (L_1+U_1L) \eta_1 \cdots \eta_m + (L_2+U_1U_2L) \eta_2 \cdots \eta_m \\
+ \cdots + (L_m+U_1 \cdots \eta_m L) \eta_m + Lx + \),
\]

where \( |z| \leq b(i_1 \cdots i_m) \). Suppose we now define a new finite region \( R' \), consisting of all vectors

\[
C' = Lz + C,
\]

where \( C \epsilon R \) and \( |z| \leq b(i_1 \cdots i_m) \). Since \( f \epsilon A_n \), we can find numbers \( M'/(i_1 \cdots i_m) > 0 \), \( b'(i_1 \cdots i_m) > 1 \) such that

\[
\begin{align*}
|f((L_1+U_1L) \eta_1 \cdots \eta_m + (L_2+U_1U_2L) \eta_2 \cdots \eta_m + \cdots + (L_m+U_1 \cdots \eta_m L) \eta_m + Lx + C')| & \\
& \leq M'/(i_1 \cdots i_m) \eta_1^{\alpha(i_1 \cdots i_m)} (\ln \eta_1)^{\beta(i_1 \cdots i_m)} \\
& \times \cdots \eta_m^{\alpha(i_1 \cdots i_m)} (\ln \eta_m)^{\beta(i_1 \cdots i_m)} (\ln \eta_{m-1} L),
\end{align*}
\]

wherever \( C' \epsilon R' \) and \( \eta_l \geq b'(i_1 \cdots i_m) \). Therefore,

\[
\begin{align*}
\int_{J_{i_1 \cdots i_m}(\eta)} dy |f(P+Ly)| & \leq 2b(i_1 \cdots i_m) M'/(i_1 \cdots i_m) \\
\times & \eta_1^{\alpha(i_1 \cdots i_m)} (\ln \eta_1)^{\beta(i_1 \cdots i_m)} \\
\times & \cdots \eta_m^{\alpha(i_1 \cdots i_m)} (\ln \eta_m)^{\beta(i_1 \cdots i_m)},
\end{align*}
\]

provided that

\[
\eta_l \geq b'(i_1 \cdots i_m) \quad (1 \leq l \leq m).
\]

All we need do to finish the proof is to inspect (38), (42), and (47), together with the corresponding conditions on \( \eta_l \), (39), (43), and (48), and use (29). We see that \( f \epsilon A_{n+1} \), with

\[
\begin{align*}
\alpha_L(L_1 \cdots L_r) &= \max [\alpha(i_1 \cdots i_r), \alpha(L_1 \cdots L_r)+1], \\
\beta_L(L_1 \cdots L_r) &= \max \left\{ \begin{array}{ll}
\beta(i_1 \cdots i_r) \quad & \text{for } \alpha(i_1 \cdots i_r) = \alpha(L_1 \cdots L_r) \\
\beta(L_1 \cdots L_r) \quad & \text{for } \alpha(i_1 \cdots i_r) = \alpha(L_1 \cdots L_r)+1 \end{array} \right.
\end{align*}
\]

(49)

\[
\begin{align*}
\beta(i_1 \cdots i_r) + \beta(L_1 \cdots L_r) = \max & \quad (L_1 \cdots L_r)+1 \\
\beta(i_1 \cdots i_r) + \beta(L_1 \cdots L_r) = \max & \quad (L_1 \cdots L_r)+1
\end{align*}
\]

(50)

The formula for \( \beta_L \), while giving a correct upper bound on the number of logarithms, is an overestimate, and will not be further discussed. (It may be noted, though, that \( \beta_L \) is still a non-negative integer.) The formula for \( \alpha_L \) may be rewritten:

\[
\begin{align*}
\alpha_L(L_1 \cdots L_r) &= \max [\alpha(L_1+u_1L, L_2+u_2L, \cdots, L_r+u_rL), \\
& \alpha(L_1 \cdots L_r)+1],
\end{align*}
\]
where the $u_1 \cdots u_r$ take only the finite set of values
$U_{i_1} \cdots, U_{i_1 \cdots i_r}$. According to statement C in the
Appendix, this formula is equivalent to (18), which was
to be proven.

The most interesting possible extension of this theo-
rem would be to introduce some sort of “positivity”
conditions on $f(P)$ which would enable us to set a lower
bound, as well as an upper bound, on the asymptotic
behavior of the integrals of $f(P)$. Our method of proof
is well suited to such a program, since we display ex-
plicitly in (29) the part of the domain of integration
giving each particular contribution to the asymptotic
behavior. (It is easy to verify in the example of Sec. II
that the covering intervals $J$ can be arranged so that
they don’t overlap.) It might then be possible to show
that certain theories are rigorously nonrenormalizable
in the Heisenberg representation.

It would also be very useful to refine the theorem so
that we are not forced to overestimate the powers $\beta (S)$
of $\ln \eta$. In order to include the possibility of negative
$\beta (S)$ it would be necessary to introduce powers of $\ln \eta$
into the definition (III-1).

Finally, it might be interesting to extend the theorem
to the case where the $\alpha$ depend on the individual vectors
$L_1 \cdots L_r$ and not just the manifold $\{L_1 \cdots L_r\}$. This is
very easy to do in the case where the subspace of integration
$I$ is one-dimensional, but has no obvious
physical application.

V. APPLICATION TO PERTURBATION THEORY

We shall now apply the general theorem proven above
to the determination of the convergence and asymptotic
properties of Green’s functions in covariant perturba-
tion theory. Our treatment will follow closely that of
the simple example discussed in Sec. II and at the end of
Sec. III.

Let us consider any particular Feynman diagram $\mathcal{G}$ in
an arbitrary local field theory. According to the usual rules there
is associated with each internal and external particle line
$j$ of $\mathcal{G}$ a bare propagator $\Delta_j (p_j, \sigma)$, where $p_j$ is the
momentum four-vector carried by line $j$, and $\sigma$ is a single
label representing all discrete variables such as spins,
polarizations, etc. The integrand $F$ corresponding to
diagram $\mathcal{G}$ is given as a simple product

$$ F = \gamma (\sigma) \prod_{j=1}^{M} \Delta_j (p_j, \sigma), \tag{1} $$

where $\gamma (\sigma)$ is the product of all vertex factors, such as
Dirac matrices, coupling constants, etc., and plays no
important role here. (Since all discrete indices are
subsumed under $\sigma$, the $\Delta_j$ and $\gamma$ are ordinary complex
functions, and can be multiplied without regard to their
order.) In theories with derivative coupling, we must
include in the $\Delta_j$ any factors of $p_j$ arising from deriva-
tive coupling vertices to which line $j$ may be attached.

The Green’s function $G$ corresponding to $\mathcal{G}$ is formed by
summing and integrating $F$ over all internal $\sigma$ and $p_j$;
it is then a function of the external variables.

Now, we are supposing in (1) that all energy-mo-
mentum conservation $\delta$ functions have already been
eliminated, leaving $N$ independent momentum four-

vectors, where of course $N < M$. Let us unite all compo-
nents of these independent four-vectors into a single vector $P$
in the $4N$-dimensional Euclidean space $R_{4N}$, so that each component $p_{\mu} (\mu = 0, 1, 2, 3)$ of each of the in-
ternal and external momenta can be written as a linear
combination of the components of $P$. To use vector
notation, we introduce for each $j$, $\mu$ a vector $V_{j \mu}$ in $R_{4N}$,
such that $p_{j \mu}$ is given by the scalar product

$$ p_{j \mu} = P \cdot V_{j \mu}, \tag{2} $$

and therefore

$$ F (P, \sigma) = \gamma (\sigma) \prod_{j=1}^{M} \Delta_j (P \cdot V_{j \sigma}). \tag{3} $$

[For an example, see Eq. (II-5).] If a vector $P$ is
orthogonal to all $V_{j \mu}$ for which $j$ is an internal (external)
line, then we shall say that $P$ lies in the external (internal)
subspaces $E (I)$; its components then involve
only external (internal) momenta. The subspaces $E$ and
$I$ are disjoint, and

$$ E + I = R_{4N}. \tag{4} $$

The Green’s function corresponding to graph $\mathcal{G}$ is now
given by the improper integral over the internal
momenta,

$$ G (P, \sigma, \alpha) = \sum_{\sigma \in \Sigma} \int_{-\infty}^{\infty} F (P + P', \sigma) d \alpha', \tag{5} $$

where $P \in E$. We will first study the asymptotic behavior
of $F$, and then apply the general asymptotic theorem to
learn what we want to know about $G$.

Unfortunately, as it stands the function $F (P, \sigma)$ does
not belong to the class $A_{4N}$, because of the special
circumstance that the propagators $\Delta_j (p_j, \sigma)$ depend
on the scalar product

$$ p_j^2 = p_j^0 \pm p_j^2 \pm p_j^3 = p_j^0, $$

which can vanish for nonzero $p_j$. In order to apply our
theorem, it is necessary to rotate the contour of integra-
tion for each energy integration in (5) in a well-
known manner from the real up to the imaginary axis.\footnote{R. J. Eden, Proc. Roy. Soc. (London) A210. 388 (1952).} A
general discussion of the difficulties encountered in
this step would be interesting, but beyond the scope of
this work; we shall simply assume henceforth that all
energy contours have been so deformed. Likewise, if the
integral $G$ is to be used as an insertion in a larger dia-
gram, we shall be interested in its behavior for imaginary
values of its energy arguments. We shall, therefore,
restrict ourselves throughout to consider all four-
vectors $p_j$ with imaginary fourth component $p_{j0} = \pm \gamma p_{j \mu}$,
and hence with positive-definite square
\[ p_j^2 = p_j^2 + p_j^2 + p_j^2 + p_j^2 \geq 0. \]

In this manner we circumvent, but do not solve, the special problems associated with the hyperbolic metric of space-time. [It will also be necessary in electrodynamics to introduce a small photon mass in order to avoid infrared divergences in (5).]

With the above qualifications it is easy to see that the propagators \( \Delta_j(p_j,\sigma) \) have a very simple asymptotic behavior, given by
\[ \Delta_j(p_j,\sigma) = O((p_j^2)^{\alpha_j}), \]
for each fixed \( \sigma \), and for \( p_j \) tending to infinity along any direction. For example, if line \( j \) represents a spinless internal particle, \( \alpha_j = -2 \), since
\[ \Delta_j(p_j,\sigma) = 1/(p_j^2 + \mu^2), \]
while if it represents an internal particle of spin \( \frac{1}{2} \), \( \alpha_j = -1 \), because
\[ \Delta_j(p_j,\sigma) = S_e(p_j) = \frac{i p_j \gamma^a + m_j}{p_j^2 + m_j^2}. \]

Thus, from (3) and (7), we may now show that \( F(P,\sigma) \) belongs to the class \( A_{4N} \) (defined in Sec. III) with asymptotic coefficients, for any subspace \( S \subset R_{4N} \), given by
\[ \alpha(S) = \sum_j n_j \alpha_j, \]
(8)
\[ \beta(S) = 0, \]
(9)
the sum in (8) running over all \( j \) for which \( V_j \) is not orthogonal to the subspace \( S \).

Proof. Let us set \( P \) in (3) equal to
\[ P = L_1 \eta_1 \cdots \eta_m + L_2 \eta_2 \cdots \eta_{m+1} + \cdots + L_m \eta_m + C, \]
where \( L_1 \cdots L_m \) are independent vectors in \( R_{4N} \) (so \( m \leq 4N \)), \( C \) is a vector confined to a finite region \( W \), and \( \eta_1 \cdots \eta_m \) tend independently to infinity. Then from (7) we have for fixed \( \sigma \),
\[ \Delta_j(V_j, P,\sigma) = O((\eta_{j+1} \cdots \eta_m)^{\alpha_j}), \]
where \( l \) is determined by the condition that
\[ V_j \cdot L_1 = V_j \cdot L_2 = \cdots = V_j \cdot L_{l-1} = 0 \] but \( V_j \cdot L_l \neq 0. \) (12)
Therefore, from (3),
\[ F(P,\sigma) = O(\prod_{i=1}^{M} (\eta_{i+1} \cdots \eta_m)^{\alpha_j}), \]
where the product \( \prod_{i=1}^{M} \) includes only those \( j \) satisfying (12). Collecting powers of each \( \eta \), we have
\[ F(P,\sigma) = O(\eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_m^{\alpha_m}), \]
(14)
\[ \alpha_\eta = \sum_j \eta_\alpha, \]
(15)
where the sum \( \sum_j \) contains only those \( j \) for which \( (12) \) is satisfied for some \( l \leq r \), and hence just those \( j \) for which
\[ L_{i-1} \cdot V_j \neq 0 \] or \( L_r \cdot V_j \neq 0 \) or \( \cdots \) or \( L_r \cdot V_j \neq 0, \]
or in other words, over all \( j \) for which \( V_j \) is not orthogonal to the subspace \( \{ L_1 \cdots L_r \} \) spanned by \( L_1 \cdots L_r \). We therefore are entitled to write \( \alpha_\eta \) as a function only of the subspace \( \{ L_1 \cdots L_r \} \),
\[ \alpha_\eta = \alpha_\eta(\{ L_1 \cdots L_r \}), \]
(16)
and obtain (8) from (15).

We may now apply our general theorem. The first part tells us that the integral (5) converges if it converges superficially \( [i.e., \alpha(I) + \dim I < 0] \) and if all subintegrations converge superficially \( [i.e., \alpha(I) + \dim I' < 0 \text{ for } I' \subset I] \). According to the renormalization procedure suggested by Dyson and perfected by Salam, we must subtract from each \( F \) a series of counterterms, which have the effect of lowering these superficial divergences below zero. [For example, the last subtraction term, corresponding to the subspace \( j \) itself, is a polynomial of order \( \alpha(I) + \dim I \) in the external momenta.\] These subtractions have been proven equivalent to a renormalization of coupling constants, masses, and fields. Our theorem then verifies the conjecture used without proof by Dyson and Salam, that such subtractions actually do render all integrals convergent, to any finite order in perturbation theory.

In order to apply the second part of our theorem to learn about the asymptotic behavior of \( G \), and also to further understand its convergence properties, we shall now introduce a new concept, that of a subgraph \( G' \) of the graph \( G \).

Definition. A set \( G' \) of internal and external lines \( j \) form a subgraph of \( G \) provided that there is no vertex in \( G \) to which is attached just one line of those in \( G' \). Clearly, a subgraph \( G' \) may be thought of as composed of a number of paths which begin and end in external lines or each other, but which never end abruptly within \( G \). Some examples are presented in Fig. 2.

We shall associate with each subspace \( S' \subset R_{4N} \) a subgraph \( G' \), consisting of all lines \( j \) such that \( V_j \) is not orthogonal to \( S' \). It is easy to see that this actually does define a subgraph obeying the above definition. [Proof: Suppose we have a vertex joining lines \( j_1, j_2, \ldots, j_r \). Then by momentum conservation we must have
\[ \pm V_{j_1} \pm V_{j_2} \pm \cdots \pm V_{j_r} = 0. \]
Thus if \( j_1, \ldots, j_{r-1} \) are not in \( G' \), so that \( V_{j_1} \cdots V_{j_{r-1}} \) are orthogonal to \( S' \), we must have \( V_{j_r} \) orthogonal to \( S' \) also, and hence \( j_r \) is not in \( G' \).] This correspondence allows a simple interpretation of the asymptotic powers \( \alpha(S') \).

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4 A detailed discussion of this point is given by N. N. Bogoliubov and D. W. Shirkov, Fortschr. Phys. 4, 438 (1956); they show that with proper use of "regulators" all integrations are rendered convergent.
The renormalization subtractions are designed to lower \( \mathcal{D}_I(\mathcal{G}') \) below zero for every purely internal subgraph \( \mathcal{G}' \) (including \( \mathcal{G} \) itself without its external lines, if \( \mathcal{G} \) is a "proper" diagram.) It is important that the number of differentiations needed to perform the subtraction associated with a subgraph \( \mathcal{G}' \) is not greater than \( \mathcal{D}_I(\mathcal{G}') \); this ensures that the renormalization counterterms introduced in the Lagrangian to account for these subtractions are themselves renormalizable interactions.

Now we are in a position to apply part (b) of our theorem. Equation (III-12) may be rewritten, for any \( S \subseteq E, \)

\[
\alpha_I(S) = \max_{\mathcal{G}' \subseteq S} \mathcal{D}_I(\mathcal{G}').
\]  

(20)

The "max" here is over all subgraphs \( \mathcal{G}' \) containing just that set \( \mathcal{E}_\alpha \) of external lines \( j \) for which \( \mathcal{V}_j \) is not orthogonal to \( S \). According to the interpretation of \( \alpha_I(S) \) in Sec. III, and of \( \mathcal{D}_I(\mathcal{G}') \) above, we can express (20) in the rule:

**If a set \( \mathcal{E}_\alpha \) of external lines of a graph \( \mathcal{G} \) have momenta going to infinity (i.e., for line \( j \) in \( \mathcal{E}_\alpha, p_j = \epsilon n_j \) where \( n_j \to \infty, \epsilon_j "almost any" set of fixed nonzero four-sector, then the integrated Green's function corresponding to \( \mathcal{G} \) will behave as \( O(n^{\alpha_I(\mathcal{G})}(\log n)^{\alpha_I(\mathcal{G})}) \), where \( \alpha_I(\mathcal{E}_\alpha) \) is the maximum of the dimensionality [given by (19)] of all subgraphs \( \mathcal{G}' \) of \( \mathcal{G} \) including the external lines \( \mathcal{E}_\alpha \) and no other external lines.** (A detailed example of the application of this rule is presented in Fig. 2.)

Strictly speaking, we should prove that the renormalization subtractions needed to lower all \( \alpha(I') + \dim I'(I' \subseteq I) \) underneath zero do not alter the value of \( \alpha_\mathcal{E} \).

The proof is tedious, and will be omitted. It is important to note, however, that we should restrict the subgraphs \( \mathcal{G}' \) above to those that do not contain any parts entirely disconnected from \( \mathcal{E}_\alpha \); for such parts are themselves purely internal subgraphs of \( \mathcal{G} \), the renormalization procedure invariably lowers their dimensionality below zero.

We will consider not an individual Feynman diagram, but the whole sum of Feynman diagrams for a particular set of external lines up to some sufficiently large but finite order, we find a remarkable simplification. **When a set \( \mathcal{E}_\alpha \) of external lines have momenta tending to infinity, then the total Green's function has as its asymptotic power a quantity \( \alpha(\mathcal{E}_\alpha) \) which depends only on the numbers of lines in \( \mathcal{E}_\alpha \), and is given by**

\[
\alpha(\mathcal{E}_\alpha) = 4 - \frac{3}{4} f(\mathcal{E}_\alpha) - b(\mathcal{E}_\alpha) - \min(\frac{3}{2} f(\mathcal{S}') + b(\mathcal{S}')).
\]

(21)

Here \( b(\mathcal{S}) \), \( f(\mathcal{S}) \) are the number of spin 0 (or photon) or spin \( \frac{1}{2} \) lines in the set \( \mathcal{S} \). The minimum in (21) is taken over all sets \( \mathcal{S}' \) of lines such that the virtual transition \( \mathcal{E}_\alpha \leftrightarrow \mathcal{S}' \) is not forbidden by selection rules. (If we are concerned with connected or proper Green's functions, we may also stipulate that \( \mathcal{S}' \) must contain at least one or at least two lines.) For example, if \( \mathcal{E}_\alpha \) consists of a
pair of incoming nucleon and antinucleon lines, the "min" in (21) is reached for $\mathcal{G}'$ a pair of pions, so that $\alpha(E)$ is given by $4 - 3 = 1$. This is the maximum asymptotic power of any connected diagram or sum of diagrams for which a nucleon and an antinucleon external momenta tend to infinity, with the other external momenta fixed. (The diagram of Fig. 2 shows the realization of this maximum in this case.)

In order to verify this rule we need only note that every subgraph $\mathcal{G}'$ included in (20) has attached to it all lines in $\mathcal{G}$, together with a set $\mathcal{G}'$ of "bridges," consisting of internal and external lines belonging to $\mathcal{G}$ but not to $\mathcal{G}'$. We can therefore write in (19)

$$F(\mathcal{G}') = f(\mathcal{E}) + f(\mathcal{G})$$

and inserting (22) into (20) we obtain (21). Any possible set $\mathcal{G}'$ of bridges will occur if we go to high enough order, so that the maximum is always attained.

Our result cannot easily be extended to the logarithmic powers $\beta(E)$; it is known that these depend strongly on the structure and order of the graphs considered. Thus, although our proof shows that any Green's function, calculated to any finite order, belongs to a class $A_\infty$ with asymptotic powers $\alpha(E)$ given by (21), it is entirely possible that the logarithmic powers in the infinite sum add up in such a manner that the total Green's function does not have asymptotic power $\alpha(E)$, or perhaps does not even belong to a class $A_\infty$. However, in the present state of field theory we may hope that results based on perturbation theory may serve as a useful guide.

APPENDIX: PROJECTIONS OF SUBSPACES

In Sec. III we introduce the operation of projecting one subspace along another. As our use of this operation may perhaps be unfamiliar, we shall define it more precisely, and prove some simple statements used in the proof and interpretation of our theorem.

Let $I$ be a subspace of a vector space $R_\infty$. It is always possible to choose (not uniquely) another subspace $E \subset R_\infty$ such that $I \cap E$ is disjoint (and therefore independent) and such that $R_\infty = I + E$. With such a choice of $E$ the operator $\Lambda(I)$, the usual projection along $I$ on $E$, becomes well defined: For any vector $V \in R_\infty$ we write $V = L + L' + L''$, $L, L' \in I$, and set $\Lambda(I)V = L$. If a set of vectors $L_i$ span a subspace $S' \subset R_\infty$ then $\Lambda(I)S' = S$, where $S$ is the subspace spanned by the corresponding $L_i$.

The last equation, $\Lambda(I)S' = S$, is usually taken in this paper as a condition on $S'$, with $I$ and $S$ fixed disjoint subspaces of $R_\infty$. As such a condition it is actually independent of the choice of $E$, as shown below by statement (A); we should, properly speaking, refer to $\Lambda(I)$ as the projection onto the space $R_\infty/I$.

(A) If $I$ and $S$ are disjoint subspaces, and we define $\Lambda(I)$ by choosing $E$ to be any subspace such that $I$ and $E$ are disjoint, $R_\infty = I + E$, and $S \subseteq E$, then $\Lambda(I)S' = S$ if and only if for every set of vectors $L_1, \ldots, L_r$ spanning $S$ there exists a set of vectors $L'_1, \ldots, L'_r \in I$ such that

$$S' = \{L_1 + L'_1, \ldots, L_r + L'_r\}$$

Equivalently, $\Lambda(I)S' = S$ if and only if there exists some set of vectors $L_1, \ldots, L_r$ spanning $S$ and $L'_1, \ldots, L'_r \in I$ satisfying (1).

[Proof: If $S'$ is given by (1), with $L'_1, \ldots, L'_r \in I$, and thus $L_1, \ldots, L_r$, then by definition $\Lambda(I)S' = \{L_1, \ldots, L_r\}$. On the other hand, we can always write any subspace $S'$ as in (1) where $L'_1, \ldots, L'_r \in I$ and if $\Lambda(I)S' = S$ we have $S = \{L_1, \ldots, L_r\}$. Clearly we can take any new set of $L_1, \ldots, L_r$ spanning $S$ and preserving (1), with a new set of $L'_1, \ldots, L'_r$.]

(B) If $\Lambda(I)S' = S$, with $I$ and $S$ disjoint, then $S' \subseteq S + I$.

$$\dim S \leq \dim S' \leq \dim I + \dim S.$$  (21)

(C) If $I$ is a vector not in $S$, then $\Lambda(I)(S') = S$ if and only if every pair $S'' = S + \{L_i\}$ or there exist numbers $u_1, \ldots, u_p$ with

$$S'' = \{L_1 + u_1L_i, L_2 + u_2L_i, \ldots, L_r + u_pL_i\}$$

where $S = \{L_1, \ldots, L_r\}$.

(These two alternatives represent the possibilities \(\dim S'' = \dim S'' + 1 = \dim S\), respectively.)

(D) If $I$ and $S$ are disjoint subspaces then $S' \subseteq S + I$ if and only if there is some subspace $S'' \subseteq S$ such that $\Lambda(I)S'' = S'$.

[Proof: The subspace $S''$ is just $\{L'_1, \ldots, L'_r\}$.]

(E) If $S_1, S_2, S_3$ are disjoint subspaces, and $\Lambda(S_2)S_3' = S$, then $\Lambda(S_2)S_3'' = S'$ if and only if $\Lambda(S_1 + S_2)S_3'' = S$.  

[Proof: We can write $S' = \{L_1 + L'_1, \ldots, L_r + L'_r\}$ where $S = \{L_1, \ldots, L_r\}, L_1', \ldots, L_r' \in S_3$. Then $\Lambda(S_2)S_3'' = S$ means that $S'' = \{L_1 + L'_1 + L''_1, \ldots, L_r + L'_r + L''_r\}$, where $L_1', \ldots, L_r' \in S_3$. Also, $\Lambda(S_1 + S_2)S_3''$ means that $S'' = \{L_1 + L'_1 + L''_1, \ldots, L_r + L'_r + L''_r\}$ where $L_1'', \ldots, L_r'' \in S_3$. The most general $L_i'' \in S_3$ may be written $L_i'' = L_i' + L''_i$, so these statements are equivalent.]}

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Note to be added in proof:—For the sake of mathematical rigor, the definition in Equation (III-1) of the class $A_\infty$ requires a slight modification. The coefficients $\beta$ should be taken as functions of the individual vectors $L_1, L_2, \ldots$ and not only of the subspaces $(L_1, L_2, \ldots)$. The proof in Sec. IV that if $\Lambda(A, E)$ then $\Lambda(A, E)$ is correct, with no changes (except minor notational ones) required.