Field Theory of Unstable Particles*

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Using the example of a spinless boson field, the structure of the simplest Green's function is developed to provide a uniform theory of particles, stable and unstable. Some attention is given to the time decay law of unstable particles and it is emphasized that a full account of the relevant physical situation must be contained in its mathematical representation, leading to the conclusion that an essential failure of the exponential decay law marks the limit of applicability of the physical concept of unstable particle. There is a brief discussion of the $\pi$ and $K$ mesons.

Some attention\(^1\) has been directed recently to the field theoretic description of unstable particles. Since this question is conceived as a basic problem for field theory, the responses have been some special device or definition, which need not do justice to the physical situation. If, however, one regards the description of unstable particles to be fully contained in the framework of the general theory of Green's functions, it is only necessary to emphasize the relevant structure of these functions. That is the purpose of this note. What is in essence the same question, the propagation of excitations in many-particle systems where stable or long-lived "particles" can occur under exceptional circumstances, has already been discussed \(^2\) along these lines.

A relativistic field describes a localized excitation which produces a spectrum of energies or masses that ranges down to a lower limit characteristic of the field type. Thus for any electrically charged boson or lepton field this theoretical lower limit is the electron mass $m$, while for electrically neutral fields of these two varieties the mass spectrum in principle extends down to zero. These limits are set by the masses of the absolutely stable\(^2\) particles to which one is led by decay processes that respect essentially only the conservation of electric charge $Q$, in addition to the usual mechanical properties. With fermion fields carrying nucleonic charge $N$, however, the absolute conservation of the latter evidenced by the sta-

\* This work was supported in part by the U.S. Air Force. It was reported at the Ninth International Conference on High Energy Physics, Kiev, U.S.S.R., July 15–25, 1959.

\(^1\) See, for example, Matthews and Salam (1) and a subsequent work of these authors, together with a recent paper by Lévy (1).

\(^2\) At least on a sub-cosmological scale.
bility of the proton implies a mass lower limit that equals $M$, the proton mass, for fields with $Q = N$, becomes $M + m$ for electrically neutral fields, and is $M + 2m$ for $Q = -N$ (omitting the binding energy of hydrogen atom or ion!).

The propagation of such excitations, superimposed on the vacuum state, is described by the Green's functions as time-ordered field correlation functions. For a single spinless boson field $\phi(x)$, the simplest example is

$$G(x - x') = \delta((\phi(x)\phi(x'))_{+})$$

$$= \int \frac{(dk)}{(2\pi)^4} e^{ik(x-x')}G(k),$$

where, according to the elementary relativistic structure of its spectral form,

$$G(k) = \int \frac{dB[k^2]}{k^2 + k^2 - ie}$$

Here $B[k]$ is a real non-decreasing function that equals zero for sufficiently small $k^2$ and approaches unity as $k^2 \to \infty$. Another form appears on exploiting the latter property as a minimum formal characterization of a local field,

$$G(k^2) \sim \frac{1}{k^2}, \quad k^2 \to \infty.$$

We first remark that the complex variable function

$$G(z) = \int \frac{dB[k^2]}{k^2 - z}$$

can have no complex zeros, since

$$\int \frac{(k^2 - x) dB[k^2]}{(k^2 - x)^2 + y^2} + iy \int \frac{dB[k^2]}{(k^2 - x)^2 + y^2} = 0$$

implies $y = 0$. Accordingly the function

$$P(z) = G^{-1}(z) + z$$

is regular everywhere in the finite complex plane, with the exception of the positive real axis, a semi-infinite portion of which constitutes a branch line for this function corresponding to the branch line of $G(z)$ associated with the continuous spectrum. It is also possible to have isolated poles, which refer to real zeros of $G(z)$ appearing at points within the spectrum where $B[k]$ is constant. The function $z^{-1}P(z)$ vanishes at infinity in the cut plane and also has a pole at the origin, from which it follows that

$$z^{-1}P(z) = \frac{\lambda^2}{z} - \int_{x^2 > 0} \frac{ds[k^2]}{k^2 - z}$$
or

\[ S^{-1}(z) = \lambda^2 - z - z \int \frac{ds[\kappa^2]}{\kappa^2 - z}. \]

The real function \( s[\kappa^2] \) is shown to be nondecreasing on translating the property

\[ \frac{1}{2iy} [S(x + iy) - S(x - iy)] > 0 \]

into

\[ \frac{1}{2iy} [S^{-1}(x + iy) - S^{-1}(x - iy)] < 0. \]

In writing the residue of \( z^{-2}P(z) \) at the origin as \( \lambda^2 > 0 \) we have anticipated the expression of a physical requirement, that \( S(z) \) have no singularity for negative \( z \) or space-like \( k \). The function \( S^{-1}(z) \) begins at \( + \infty \), for \( z = -\infty \), and decreases monotonically along the real \( z \) axis, according to

\[ -\frac{d}{dz} S^{-1}(z) = 1 + \int \frac{\kappa^2 ds[\kappa^2]}{(\kappa^2 - z)^2}. \]

If \( S^{-1}(z) \) is to have no zero for \( z \leq 0 \), it is necessary and sufficient that \( S^{-1}(0) = \lambda^2 > 0 \). Alternatively, we recognize from the comparison of the two expressions for \( S(0) \) that

\[ \frac{1}{\lambda^2} = \int \frac{dB[\kappa^2]}{\kappa^2} > 0. \]

The second form thus obtained for \( S(k) \) is

\[ S(k) = \left[ k^2 + \lambda^2 - i\epsilon + k^2 \int \frac{ds[\kappa^2]}{k^2 + \kappa^2 - i\epsilon} \right]^{-1}, \]

the assumed existence of which requires the convergence at infinity of the integral \( \int (\kappa^2)^{-1} ds[\kappa^2] \). This convergence condition also assures the validity of the asymptotic form \( (k^2)^{-1} \) as \( k^2 \) approaches infinity either in a space-like or time-like direction. Should more be true and \( \int ds[\kappa^2] \) converge, which implies a finite total increase in \( s[\kappa^2] \) from an initial zero value, we could infer the more detailed asymptotic property

\[ S(k) \sim (k^2 + \mu_0^2)^{-1}, \]

where

\[ \mu_0^2 = \lambda^2 + s[\infty] > \lambda^2. \]

This includes \( k^2 = 0 \). The exceptional circumstances encountered for the electromagnetic field presumably do not occur with a spinless field.
An inspection of the spectral form for $G(k)$ shows that $\mu_0^2$ is also given by

$$\mu_0^2 = \int k^2 dB[k^2].$$

Under the conditions which permit the identification of $\mu_0$ it is possible to write

$$G(k) = \left[ k^2 + \mu_0^2 - i\epsilon - \int \frac{k^2 \, ds[k^2]}{k^2 + k^2 - i\epsilon} \right]^{-1}.$$

The constant $\mu_0$ can be interpreted as the mass parameter associated with the field independently of interactions, at least when these couplings are linear in the boson field.

One anticipates two possible spectral conditions from the general physical conviction that a continuous energy or mass spectrum never terminates nor has jump discontinuities. Either $B[k^2]$ is a continuous increasing function for all $k > \kappa_0$, or, there are one or more positive steps in $B[k^2]$ followed by a continuous increasing domain. In the first circumstance there is no discrete mass value, which is to say, no stable particle associated with the field, while the alternative corresponds to one or more such particles.

If there is no isolated singularity in $G(z)$ the function $G^{-1}(z)$ can have no zero for $0 < z < \kappa_0^2$, where $\kappa_0$ marks the threshold of the continuous spectrum. The condition for this property, which expresses the absence of a stable particle, is $G^{-1}(\kappa_0^2 - 0) > 0$, or

$$\lambda^2 > \kappa_0^2 \left[ 1 + \int_{\kappa_0^2}^{\infty} \frac{ds[k^2]}{k^2 - \kappa_0^2} \right].$$

Thus it is necessary, but not sufficient, for $\lambda$ to exceed $\kappa_0$ if no stable particle is to exist. This also follows from the observation that

$$\frac{1}{\lambda^2} = \int_{\kappa_0^2}^{\infty} \frac{dB[k^2]}{k^2} < \frac{1}{\kappa_0^2}.$$  

The sign of $G^{-1}(\kappa_0^2 - 0)$ must become negative if a discrete mass value is to occur in $G$. Indeed, the condition that $G^{-1}(z)$ vanish at $z = \mu^2 > 0$ is given by

$$\lambda^2 = \mu^2 \left[ 1 + \int \frac{ds[k^2]}{k^2 - \mu^2} \right],$$

$$< \kappa_0^2 \left[ 1 + \int \frac{ds[k^2]}{k^2 - \kappa_0^2} \right],$$

under the assumption that $s[k^2]$ is a continuous function. The inequality is obtained by remarking that the function of $\mu^2$ increases monotonically in the interval $0 < \mu^2 < \kappa_0^2$. A value of $\lambda^2$ that satisfies the inequality defines a unique $\mu^2$ in this segment.
To consider in more detail the situation of a stable particle we indicate explicitly the nature of the spectrum by writing

$$G(z) = \frac{B_0}{\mu^2 - z} + \int_{\kappa_0^2}^{\infty} d\kappa^2 \frac{B(\kappa^2)}{\kappa^2 - z},$$

where

$$0 < B_0 < 1,$$

and the continuous function $B(\kappa^2)$, which is positive for $\kappa^2 > \kappa_0^2$ and vanishes at the beginning of this domain, obeys

$$\int_{\kappa_0^2}^{\infty} d\kappa^2 B(\kappa^2) = 1 - B_0.$$

If we place

$$B(\kappa^2) = B_0 \beta(\kappa^2),$$

the latter property appears as

$$B_0 = \left[ 1 + \int_{\kappa_0^2}^{\infty} d\kappa^2 \beta(\kappa^2) \right]^{-1}.$$

We also have

$$\frac{1}{\lambda^2} = \frac{B_0}{\mu^2} + \int_{\kappa_0^2}^{\infty} d\kappa^2 \frac{B(\kappa^2)}{\kappa^2}$$

from which some inequalities can be inferred. Thus $\lambda^{-2} > B_0 \mu^{-2}$ and therefore

$$\mu^2 > B_0 \lambda^2,$$

while the property $\kappa^2 > \kappa_0^2 > \mu^2$ shows that $\lambda^{-2} < \mu^{-2}$, or

$$\lambda^2 > \mu^2 > B_0 \lambda^2.$$

A more precise version of the second inequality is

$$\frac{1}{\lambda^2} < \frac{B_0}{\mu^2} + \frac{1 - B_0}{\kappa_0^2},$$

one application of which is meaningful if $\lambda < \kappa_0$,

$$\left( \frac{\mu}{\lambda} \right)^2 > B_0 > \left( \frac{\mu}{\lambda} \right)^2 \frac{1 - (\lambda/\kappa_0)^2}{1 - (\mu/\kappa_0)^2}.$$

As $z$ varies from $\mu^2 + 0$ to $\kappa_0^2 - 0$ the function $G(z)$ increases monotonically from $-\infty$ to the limiting value

$$G(\kappa_0^2 - 0) = -\frac{B_0}{\kappa_0^2 - \mu^2} + \int_{\kappa_0^2}^{\infty} d\kappa^2 \frac{B(\kappa^2)}{\kappa^2 - \kappa_0^2}.$$
which must still be negative if \( g \) is to have no zero in the interval. Accordingly, under the restriction
\[
\int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{B(\kappa^2)}{\kappa^2 - \kappa_0^2} < \frac{B_0}{\kappa_0^2 - \mu^2},
\]
or
\[
(\kappa_0^2 - \mu^2) \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{\beta(\kappa^2)}{\kappa^2 - \kappa_0^2} < 1,
\]
the function \( g^{-1}(z) \) has no isolated singularity. This is made explicit by writing
\[
g^{-1}(z) = \lambda^2 - z - z \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{s(\kappa^2)}{\kappa^2 - \kappa_0^2},
\]
where the continuous positive function \( s(\kappa^2), \kappa > \kappa_0 \), vanishes for \( \kappa = \kappa_0 \). As one application of the inequality that characterizes this situation we note that
\[
\frac{B_0}{\kappa_0^2 - \mu^2} > \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{B(\kappa^2)}{\kappa^2 - \kappa_0^2} > \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{B(\kappa^2)}{\kappa^2} = \frac{1}{\lambda^2} - \frac{B_0}{\mu^2},
\]
which shows the necessity of the property
\[
B_0 > \left( \frac{\mu}{\lambda} \right)^2 \left[ 1 - \left( \frac{\mu}{\kappa_0} \right)^2 \right].
\]
A more stringent condition merges from the evaluation
\[
B_0^{-1} = -\frac{d}{dz} g^{-1}(z) \bigg|_{z=\mu^2} = 1 + \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{s(\kappa^2)}{\kappa^2 - \mu^2} + \mu^2 \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{s(\kappa^2)}{(\kappa^2 - \mu^2)^2},
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\]
together with
\[
\lambda^2 - \mu^2 \left[ 1 + \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{s(\kappa^2)}{\kappa^2 - \mu^2} \right],
\]
since
\[
B_0^{-1} - \frac{\lambda^2}{\mu^2} = \mu^2 \int_{\epsilon_0^2}^{\infty} d\kappa^2 \frac{s(\kappa^2)}{(\kappa^2 - \mu^2)^2} < \frac{\mu^2}{\kappa_0^2 - \mu^2} \left( \frac{\lambda^2}{\mu^2} - 1 \right),
\]
and therefore
\[
B_0 > \left( \frac{\mu}{\lambda} \right)^2 \frac{1}{1 - \left( \frac{\mu}{\kappa_0} \right)^2 \left( \frac{\mu}{\lambda} \right)^2}.\]
Related inequalities are

\[ \int_{k_0^2}^{\infty} \frac{dk^2}{(k^2 - \mu^2)^2} \frac{\kappa^2 s(\kappa^2)}{(k^2 - \mu^2)^2} = B_0^{-1} - 1 < \frac{\lambda^2 - \mu^2}{\kappa_0^2 - \mu^2} \]

and

\[ \int_{k_0^2}^{\infty} \frac{dk^2}{(k^2 - \mu^2)^2} B_0 s(\kappa^2) = 1 - B_0 < \frac{1 - (\mu/\lambda)^2}{1 - (\mu/\kappa_0)^2(\mu/\lambda)^2} \]

If the structure of \( \zeta(z) \) is such that

\[ (\kappa_0^2 - \mu^2) \int_{k_0^2}^{\infty} \frac{dk^2}{(k^2 - \mu^2)^2} \beta(\kappa^2) > 1, \]

this function has a zero and \( \zeta^{-1}(z) \) has a pole at \( z = \nu^2 \),

\[ \mu < \nu < \kappa_0. \]

Accordingly, we now have

\[ \zeta^{-1}(z) = \frac{\lambda^2 - z - \frac{s(\kappa^2)}{\nu^2 - z} - z \int_{k_0^2}^{\infty} \frac{dk^2}{(k^2 - \mu^2)^2}}{1 - \frac{s(\kappa^2)}{\nu^2 - \mu^2} + \int_{k_0^2}^{\infty} \frac{dk^2}{(k^2 - \mu^2)^2}}, \]

where the constant \( s \) and the continuous function \( s(\kappa^2) \), \( \kappa > \kappa_0 \), are positive, the latter approaching zero at the threshold of the continuous spectrum. It follows, conversely, from the monotonic decrease to \( -\infty \) as \( z \) increases to \( \nu^2 - 0 \), and the positive value at \( z = 0 \), that \( \zeta^{-1}(z) \) has a single zero at \( z = \nu^2 \), determined by

\[ \lambda^2 = \mu^2 \left[ 1 + \frac{s}{\nu^2 - \mu^2} + \int_{k_0^2}^{\infty} \frac{dk^2}{(k^2 - \mu^2)^2} \right]. \]

Again we calculate \( B_0 \) from

\[ B_0^{-1} = -\frac{d}{dz} \zeta^{-1}(z) \bigg|_{z=\nu^2} = \frac{\lambda^2}{\mu^2} + \mu^2 \left[ \frac{s}{(\nu^2 - \nu^2)^2} + \int_{k_0^2}^{\infty} \frac{dk^2}{(k^2 - \mu^2)^2} \right], \]

which implies a similar inequality for \( B_0 \), with \( \nu \) replacing \( \kappa_0 \),

\[ B_0 > (\mu/\lambda)^2 \frac{1 - (\mu/\nu)^2}{1 - (\mu/\lambda)^2(\mu/\nu)^2}. \]

On writing the expression for \( B_0^{-1} \) as

\[ B_0^{-1} = 1 + \frac{\nu^2 s}{(\nu^2 - \nu^2)^2} + \int_{k_0^2}^{\infty} \frac{dk^2 \kappa^2 s(\kappa^2)}{(k^2 - \mu^2)^2} \]

we learn that
\[ \frac{\nu^2}{(\nu^2 - \mu^2)^2} B_{08} < 1. \]

Under our present assumptions \( \Phi(\kappa_0^2 - 0) \) is positive, which is to say that the monotonic decrease of \( \Phi^{-1}(z) \) in the interval between \( \nu^2 + 0 \) and \( \kappa_0^2 - 0 \) does not change the sign of this function. Hence

\[ \lambda^2 > \kappa_0^2 \left[ 1 - \frac{s}{\nu^2} + \int_{\kappa_0^2}^{\nu^2} \frac{d\kappa^2}{\kappa^2 - \kappa_0^2} \right] \]

and, if \( \lambda^2 \) is eliminated by employing the equation that determines \( \mu^2 \), we get

\[ \frac{\nu^2}{(\kappa_0^2 - \nu^2)(\nu^2 - \mu^2)} > 1 + \int_{\kappa_0^2}^{\nu^2} \frac{d\kappa^2}{\kappa^2 - \mu^2} \left( \frac{\kappa^2 s(\kappa^2)}{(\kappa^2 - \mu^2)^2} \right) \]

\[ > 1 + \int_{\kappa_0^2}^{\nu^2} \frac{d\kappa^2}{\kappa^2 - \mu^2} \frac{\kappa^2 s(\kappa^2)}{(\kappa^2 - \mu^2)^2} = B_{08}^{-1} - \frac{\nu^2}{(\nu^2 - \mu^2)^2} s. \]

This shows the necessity of the condition

\[ \frac{\nu^2}{\kappa_0^2 - \mu^2} < \frac{\nu^2}{(\nu^2 - \mu^2)^2} B_{08} < 1 \]

if both \( \Phi(z) \) and \( \Phi^{-1}(z) \) contain a single pole. Other significant inequalities are implied by the lower limit for \( B_{08} \), together with

\[ B_0 + \frac{\nu^2}{(\nu^2 - \mu^2)^2} B_{08} < 1. \]

Thus

\[ B_0 < \frac{\nu^2 - \mu^2}{\kappa_0^2 - \mu^2}, \]

and on comparison with the lower bound for \( B_0 \), we find that

\[ \lambda > \mu(\kappa_0/\nu). \]

If the various parameters in \( \Phi^{-1}(z) \) were altered to satisfy the inequality

\[ \lambda^2 < \kappa_0^2 \left[ 1 - \frac{s}{\nu^2} + \int_{\kappa_0^2}^{\nu^2} \frac{d\kappa^2}{\kappa^2 - \kappa_0^2} \right], \]

this function would pass through zero in the interval between \( \nu^2 + 0 \) and \( \kappa_0^2 - 0 \), which implies a second pole for \( \Phi(z) \) at \( z = \mu^2, \mu < \nu < \mu' < \kappa_0 \). We have now sufficiently traced the unfolding of the Green’s function’s spectral structure to permit the visualization of a hypothetical situation where the field has \( n \) kinematically similar stable particles associated with it, of masses \( \mu_1, \ldots, \mu_n \), represented by poles of \( \Phi(z) \) at \( z = \mu_k^2, k = 1, \ldots, n \). There are also zeros of \( \Phi(z) \).
or poles of $G^{-1}(z)$ at $z = \nu_k^2$, where $\mu_1 < \nu_1 < \mu_2 \ldots \mu_{n-1} < \nu_{n-1} < \mu_n$, and there can, but need not necessarily appear an $n$th zero at $\nu_n$, located between $\mu_n$ and $\kappa_0$, the threshold of the continuum.

For the situation in which $G(z)$ exhibits $n = 1, 2, \ldots$ poles and $n - 1$ zeros, we define $F(z)$ such that

$$G^{-1}(z) = \lambda^2 - z - z \left[ \sum_{k=1}^{n-1} \frac{s_k}{\nu_k^2 - z} + \int_{\nu_k^2}^{\nu_n^2} d\kappa \frac{s(\kappa)}{\kappa^2 - z} \right] = \prod_{k=1}^{n} \frac{(\mu_k^2 - z)}{(\nu_k^2 - z)} F(z).$$

It will be observed that $F(z)$ approaches unity at infinity, has no zero or poles, and possesses a branch line extended from $\kappa_0^2$ to $\infty$. Accordingly, there is a branch of $\log F$ that vanishes at infinity and has no singularity other than the branch point at $\kappa_0^2$, which enables us to write

$$\log F(z) = \frac{1}{\pi} \int_{\kappa_0^2}^{\infty} d\kappa \frac{\varphi(\kappa)}{\kappa^2 - z}.$$ 

The real continuous function $\varphi(\kappa^2)$ approaches zero at infinity and must vanish at $\kappa = \kappa_0$, since $\log F(z)$ is regular at this point. The resulting expression for $G^{-1}(z)$ is

$$G^{-1}(z) = \prod_{k=1}^{n} \frac{(\mu_k^2 - z)}{(\nu_k^2 - z)} \exp \left[ \frac{1}{\pi} \int_{\kappa_0^2}^{\infty} d\kappa \frac{\varphi(\kappa)}{\kappa^2 - z} \right],$$

which must still be qualified by the restriction on $\varphi(\kappa^2)$ that is equivalent to the positiveness of $s(\kappa^2)$, $\kappa > \kappa_0$. That characteristic of the $s_k$ is already contained in this representation through the property $F(x) > 0$, $x < \kappa_0^2$, and the relationship of the $\mu_k$ and $\nu_k$. We let $z = x + iy$, $x > \kappa_0^2$, $y \to +0$, and compare the ratios of real and imaginary parts of $G^{-1}(z)$ for the two forms, using the evaluation

$$F(x + i0) = e^{i\varphi(x)} \exp \left[ \frac{1}{\pi} P \int_{\kappa_0^2}^{\infty} d\kappa \frac{\varphi(\kappa^2)}{\kappa^2 - x} \right],$$

which gives

$$\pi \kappa^2 s(\kappa^2) \cot \varphi(\kappa^2) = \kappa^2 \left[ 1 - \sum_{k=1}^{n-1} \frac{s_k}{\kappa^2 - \nu_k^2} + P \int_{\kappa_0^2}^{\infty} d\kappa \frac{s(\kappa^2)}{\kappa^2 - \kappa^2} \right] - \lambda^2.$$

The function on the right-hand side has a positive value at $\kappa = \kappa_0$, according to the nature of the spectrum, which implies that $\varphi(\kappa^2)$ is positive in a neighborhood of the continuum threshold. We conclude from the physical hypothesis of a non-zero $s(\kappa^2)$, $\kappa > \kappa_0$, that $\tan \varphi$ cannot vanish and therefore that the values of $\varphi$ lie entirely in the interval
The lower limit is attained at the threshold of the continuum and is asymptotically approached at infinity. Let us note here that

\[ \lambda^2 = \frac{\prod_{n=1}^{(\infty)} (\mu^2)}{\prod_{n=1}^{(\mu^2)}} \exp \left[ \frac{1}{\pi} \int_{\epsilon_0}^{\infty} d\kappa^2 \frac{\varphi(\kappa^2)}{\kappa^2} \right] > \mu_0^2. \]

If it should be true that \( s(\kappa^2) \) vanishes more rapidly than \((\kappa^2)^{-1}\) as \( \kappa^2 \) approaches infinity, so also would \( \varphi(\kappa^2) \), and one could make the identification

\[ \mu_0^2 = \sum_{n=1}^{(\infty)} \frac{\mu_n^2}{\epsilon_n^2} - \sum_{n=1}^{(\mu_n^2)} \frac{\nu_n^2}{\epsilon_n^2} + \frac{1}{\pi} \int_{\infty}^{\kappa_0^2} d\kappa^2 \varphi(\kappa^2). \]

In the simplest example of this type, with one pole and no zero in \( G(z) \), we have

\[ G(z) = \frac{1}{\mu^2 - z} \exp \left[ - \frac{1}{\pi} \int_{\infty}^{\kappa_0^2} d\kappa^2 \frac{\varphi(\kappa^2)}{\kappa^2 - z} \right], \]

or

\[ G(z) = \frac{B_0}{k^2 + \mu^2 - \mu_0^2} \exp \left[ \left( k^2 + \mu^2 \right)^{-1} \frac{1}{\pi} \int_{\infty}^{\kappa_0^2} d\kappa^2 \frac{\varphi(\kappa^2)}{(\kappa^2 - \mu^2)(k^2 + \kappa^2 - \mu_0^2)} \right], \]

where it has been recognized that

\[ B_0 = \exp \left[ - \frac{1}{\pi} \int_{\infty}^{\kappa_0^2} d\kappa^2 \frac{\varphi(\kappa^2)}{\kappa^2 - \mu^2} \right]. \]

Other quantities that can be derived from this form are

\[ \beta(\kappa^2) = \frac{1}{\kappa^2 - \mu^2} \exp \left[ - (\kappa^2 - \mu^2)^{-1} \frac{1}{\pi} P \int_{\infty}^{\kappa_0^2} d\kappa'^2 \frac{\varphi(\kappa'^2)}{(\kappa'^2 - \mu^2)(\kappa'^2 - \mu^2)} \right] \times \frac{1}{\pi} \sin \varphi(\kappa^2) \]

and

\[ s(\kappa^2) = \left( 1 - \frac{\kappa^2}{\kappa_0^2} \right) \exp \left[ \frac{1}{\pi} P \int_{\kappa_0^2}^{\infty} d\kappa'^2 \frac{\varphi(\kappa'^2)}{\kappa'^2 - \kappa_0^2} \right] \frac{1}{\pi} \sin \varphi(\kappa^2). \]

The related function

\[ \sigma(\kappa^2) = B_0 \beta(\kappa^2), \]

which obeys
is introduced by eliminating $\lambda^2$ to make explicit the zero of $G^{-1}(z)$ at $z = \mu^2$:

$$G^{-1}(z) = (\mu^2 - z) \left[ 1 + \int_{k_0^2}^{\infty} dk^2 \frac{\kappa^2 \sigma(k^2)}{(k^2 - \mu^2)^2} \right]$$

and thus

$$G(k) = \frac{B_0}{k^2 + \mu^2 - i\varepsilon} \cdot \frac{1}{1 - (k^2 + \mu^2) \int_{k_0^2}^{\infty} dk^2 \frac{\kappa^2 \sigma(k^2)}{(k^2 - \mu^2)^2(k^2 + k^2 - i\varepsilon)}}.$$

The asymptotic form of this function shows directly that

$$\int_{k_0^2}^{\infty} dk^2 \frac{\kappa^2 \sigma(k^2)}{(k^2 - \mu^2)^2} = 1 - B_0 < 1.$$

A stronger inequality has already been found. It can be rederived by using the relation following from $G(0) = \lambda^{-2}$,

$$\left(\frac{\lambda}{\mu}\right)^2 B_0 = 1 - \mu^2 \int_{k_0^2}^{\infty} dk^2 \frac{\sigma(k^2)}{(k^2 - \mu^2)^2} - \mu^2 \int_{k_0^2}^{\infty} dk^2 \frac{\kappa^2 \sigma(k^2)}{(k^2 - \mu^2)^2},$$

to obtain

$$\int_{k_0^2}^{\infty} dk^2 \frac{\kappa^2 \sigma(k^2)}{(k^2 - \mu^2)^2} < 1 - (\mu/\lambda)^2 \left(1 - (\mu/\mu_0)^2(\mu/\lambda)^2\right).$$

When this $\sigma$ construction of $G$ is employed, $\varphi$ is computed from

$$\pi \frac{\kappa^2}{\kappa^2 - \mu^2} \sigma(\kappa^2) \cot \varphi(\kappa^2) = 1 + (\kappa^2 - \mu^2) \varphi \int_{k_0^2}^{\infty} dk^2 \frac{\kappa^2 \sigma(k^2)}{(k^2 - \mu^2)^2(k^2 - \kappa^2)},$$

and

$$\beta(\kappa^2) = \frac{1}{\kappa^2 - \mu^2}$$

$$\left[ 1 + (\kappa^2 - \mu^2) \varphi \int_{k_0^2}^{\infty} dk^2 \frac{\kappa^2 \sigma(k^2)}{(k^2 - \mu^2)^2(k^2 - \kappa^2)} \right]^2 + \left[ \frac{\kappa^2}{\kappa^2 - \mu^2} \sigma(\kappa^2) \right]^2.$$
If \( G(z) \) possesses an equal number \( n = 0, 1, \cdots \), of poles and zeros, we write

\[
G^{-1}(z) = \prod_{(n)} \left( \frac{\mu_n^2 - z}{\nu_n^2 - z} \right) (\kappa_0^2 - z)F(z).
\]

This \( F(z) \) approaches unity at infinity, and its singularities are comprised in a coincident pole and branch point at \( z = \kappa_0^2 \). Hence the log \( F(z) \) that vanishes at infinity has a branch line extended from \( \kappa_0^2 \) to \( \infty \),

\[
\log F(z) = \frac{1}{\pi} \int_{\kappa_0^2}^{\infty} dk^2 \frac{\varphi(k^2)}{k^2 - z},
\]

where the initial value of \( \varphi \) is chosen to reproduce the pole of \( F(z) \) at \( \kappa_0^2 \),

\[
\varphi(\kappa_0^2) = \pi.
\]

The equation to determine \( \varphi(k^2) \) has the same appearance as before,

\[
\frac{\pi \kappa^2 s(k^2)}{\cot \varphi(k^2)} = \kappa^2 \left[ 1 - \sum_{(n)} \frac{\delta \kappa}{\kappa - \nu_k^2} + P \int_{\kappa_0^2}^{\infty} dk'^2 \frac{\varphi(k'^2)}{k'^2 - k^2} \right] - \lambda^2,
\]

but now the right-hand side is negative at \( \kappa = \kappa_0 \). Accordingly,

\[
0 < \varphi \leq \pi,
\]

with the upper limit being attained at the continuum threshold and the lower limit approached asymptotically as \( k^2 \to \infty \). Note that

\[
\lambda^2 = \prod_{(n)} \left( \frac{\mu_n^2}{\nu_n^2} \right) \kappa_0^2 \exp \left[ \frac{1}{\pi} \int_{\kappa_0^2}^{\infty} dk^2 \frac{\varphi(k^2)}{k^2} \right] > \mu_1^2,
\]

with the exception of \( n = 0 \), where

\[
\lambda^2 = \kappa_0^2 \exp \left[ \frac{1}{\pi} \int_{\kappa_0^2}^{\infty} dk^2 \frac{\varphi(k^2)}{k^2} \right] > \kappa_0^2.
\]

When \( s(k^2) \) and \( \varphi(k^2) \) tend to zero with increasing \( k^2 \) more rapidly than \( (k^2)^{-1} \), we can make the identification

\[
\mu_0^2 = \sum_{(n)} (\mu_n^2 - \nu_k^2) + \kappa_0^2 + \frac{1}{\pi} \int_{\kappa_0^2}^{\infty} dk^2 \varphi(k^2).
\]

The simplest type of Green's function, with no zero or pole, is represented by

\[
G(z) = \frac{1}{\kappa_0^2 - z} \exp \left[ -\frac{1}{\pi} \int_{\kappa_0^2}^{\infty} dk^2 \frac{\varphi(k^2)}{k^2 - z} \right].
\]

For completeness we record the resulting functions \( B(k^2) \) and \( s(k^2) \), which characterize the spectral forms of \( G(k) \) and \( G^{-1}(k) \),
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\[ B(\kappa^2) = \frac{1}{\kappa^2 - \kappa_0^2} \exp \left[ -\frac{1}{\pi} P \int_{\kappa_0^2}^{\infty} d\kappa' \frac{\varphi(\kappa')}{\kappa^2 - \kappa'^2} \right] \cdot \frac{1}{\pi} \sin \varphi(\kappa^2), \]

\[ \kappa^2 s(\kappa^2) B(\kappa^2) = \left[ \frac{1}{\pi} \sin \varphi(\kappa^2) \right]^2. \]

The construction of \( B(\kappa^2) \) in terms of \( s(\kappa^2) \) follows from the latter relation with the aid of

\[ \pi s(\kappa^2) \cot \varphi(\kappa^2) = 1 - \frac{\lambda^2}{\kappa^2} + P \int_{\kappa_0^2}^{\infty} d\kappa' \frac{s(\kappa'^2)}{\kappa^2 - \kappa'^2}; \]

the inverse evaluation is expressed by

\[ \pi B(\kappa^2) \cot \varphi(\kappa^2) = -P \int_{\kappa_0^2}^{\infty} d\kappa' \frac{B(\kappa'^2)}{\kappa^2 - \kappa'^2}. \]

It is worth noting that these forms, appropriate to purely continuous spectra, are flexible enough to include the more general possibilities of discrete spectra. This extension is attained most simply from the \( \varphi \) construction of \( \mathcal{G}(z) \) by permitting \( \varphi(\kappa^2) \) to be a discontinuous function. Thus, if we now designate the minimum value of \( \kappa \) as \( \kappa_0' \) and choose

\[ \varphi = \pi, \quad \kappa_0' \leq \kappa < \mu_1 \]
\[ \varphi = 0, \quad \mu_1 < \kappa < \nu_1 \]
\[ \varphi = \pi, \quad \nu_1 < \kappa < \mu_2 \]
\[ \vdots \]
\[ \varphi = \pi, \quad \nu_{n-1} < \kappa < \mu_n \]
\[ \varphi = 0, \quad \mu_n < \kappa \leq \kappa_0, \]

where \( \kappa_0 \) marks the beginning of the continuous spectrum, we get

\[ \exp \left[ -\frac{1}{\pi} \int_{\kappa_0'}^{\infty} d\kappa'^2 \frac{\varphi(\kappa'^2)}{\kappa^2 - \kappa'^2} \right] \]

\[ = \frac{\kappa_0'^2 - z \nu_1^2 - z \cdots \nu_{n-1}^2 - z \nu_n^2 - z}{\mu_1^2 - z \mu_2^2 - z \cdots \mu_n^2 - z} \exp \left[ -\frac{1}{\pi} \int_{\kappa_0^2}^{\infty} d\kappa^2 \frac{\varphi(\kappa^2)}{\kappa^2 - z} \right], \]

and there emerges

\[ \mathcal{G}(z) = \prod_{(n-1)} (\nu^2 - z) \cdot \prod_{(n)} (\mu^2 - z) \exp \left[ -\frac{1}{\pi} \int_{\kappa_0^2}^{\infty} d\kappa^2 \frac{\varphi(\kappa^2)}{\kappa^2 - z} \right], \]

the form of \( \mathcal{G}(z) \) appropriate to \( n \) poles and \( n - 1 \) zeros. Similarly, the Green's
function with $n$ zeros and poles appears on terminating the discontinuous sequence with

$$\varphi = \pi, \quad \nu_{n-1} < \kappa < \mu_n$$

$$\varphi = 0, \quad \mu_n < \kappa < \nu_n$$

$$\varphi = \pi, \quad \nu_n < \kappa \leq \kappa_0.$$ 

These remarks indicate that a Green's function containing poles and zeros can be regarded as the limit of one with purely continuous spectra in which regions of rapid variation of the phase $\varphi(\kappa^2)$ have become points of discontinuity. Evidently there are circumstances in which a sufficiently rapid variation is indistinguishable from a discontinuous one, which serves to remind us that a particle description of physical phenomena is an idealization, the validity of which depends upon the nature and accuracy of the description.

To study these situations of concentrated phase variation in more detail we first observe that if $\varphi$ is close to 0 or $\pi$, both $B(\kappa^2)$ and $s(\kappa^2)$ are correspondingly small, whereas one or the other of these functions acquires large values as $\varphi(\kappa^2)$ deviates from the extreme limits. If $\varphi$ diminishes from essentially the value $\pi$ to 0, with increasing $\kappa^2$, it is $B(\kappa^2)$ that changes rapidly, while $s(\kappa^2)$ is the strongly varying function when $\varphi$ increases from 0 to $\pi$. In either situation the center of the region of rapid variation can be placed at the point where $\varphi(\kappa^2) = \pi/2$. We shall designate such a center as $\mu$ if $\varphi$ is decreasing, and as $\nu$ for increasing $\varphi$. When $\kappa \sim \mu$ the equation to determine $\cot \varphi$ from $s(\kappa^2)$ reads, approximately,

$$\pi \mu^2 s(\mu^2) \cot \varphi(\kappa^2) = B^{-1}(\kappa^2 - \mu^2),$$

where $B$ is a positive constant, and therefore

$$\kappa \sim \mu: B(\kappa^2) = \frac{1}{\pi} B \frac{\pi \mu^2 B s(\mu^2)}{(\kappa^2 - \mu^2)^2 + (\pi \mu^2 B s(\mu^2))^2}.$$ 

If we now consider the limit $s(\mu^2) \to 0$, we get

$$\kappa \sim \mu: B(\kappa^2) = B\delta(\kappa^2 - \mu^2)$$

and the contribution from this neighborhood to $G(z)$ is the term $B/(\mu^2 - z)$, describing a pole at $\mu^2$. With $\kappa \sim \nu$, we can write an approximate equation to determine $\cot \varphi$ from $B(\kappa^2)$,

$$\pi B(\nu^2) \cot \varphi(\kappa^2) = -s^{-1}(\kappa^2 - \nu^2),$$

where the constant $s$ is positive. Then

$$\kappa \sim \nu: s(\kappa^2) = \frac{1}{\pi} s \frac{\pi s B(\nu^2)}{(\kappa^2 - \nu^2)^2 + (\pi s B(\nu^2))^2},$$
and in the limit $B(v^2) \to 0,$
\[ \kappa \sim v: s(\kappa^2) = s_0(\kappa^2 - v^2). \]

The resulting contribution to $G^{-1}(z)$ is $-2s/(v^2 - z).$

The physical situation we are primarily concerned with is one where $s(\kappa^2)$ is very small below a critical value of $\kappa,$ above which substantially larger values of $s(\kappa^2)$ occur. This is usually described by a decomposition of physical interactions into two categories, strong (S) and weak (W), which is roughly the content of the separation
\[ s(\kappa^2) = s_s(\kappa^2) + s_w(\kappa^2), \]
where
\[ s_s(\kappa^2) = 0, \quad \kappa \leq \kappa_{0s} \]
\[ s_w(\kappa^2) \ll s_s(\kappa^2), \quad \kappa > \kappa_{0s}, \]
and the positive continuous function $s_w(\kappa^2)$ approaches zero at $\kappa = \kappa_{0w} < \kappa_{0s}.$

We also have
\[ \lambda^2 = \lambda_s^2 + \lambda_w^2, \]
\[ \lambda_w \ll \lambda_s. \]

In the approximation that sets $s_w = \lambda_w = 0,$ the Green’s function has (for example) no zero, and one pole at the mass value $\mu_s,$ which is to say that
\[ \lambda_s^2 = \mu_s^2 [1 + \int_0^{\infty} \frac{d\kappa^2}{\kappa^2 - \mu_s^2} s_s(\kappa^2)]. \]

defines a value of $\mu_s < \kappa_{0s},$ and the description of some physical phenomena can be given in terms of a stable particle of mass $\mu_s.$ With a more accurate treatment, however, no stable particle exists, which is expressed by the inequality
\[ \lambda_s^2 + \lambda_w^2 > \frac{2}{\kappa_{0w}} \left[ 1 + \int_{\kappa_{0w}^2}^{\infty} \frac{d\kappa^2}{\kappa^2 - \kappa_{0w}^2} s_s(\kappa^2) + \int_{\kappa_{0s}^2}^{\infty} \frac{d\kappa^2}{\kappa^2 - \kappa_{0s}^2} s_s(\kappa^2) \right] \]

or
\[ (\mu_s^2 - \kappa_{0w}^2) \left[ 1 + \int_{\kappa_{0s}^2}^{\infty} d\kappa^2 \frac{s_s(\kappa^2)}{(\kappa^2 - \mu_s^2)(\kappa^2 - \kappa_{0w}^2)} \right] \]
\[ > \kappa_{0w} \int_{\kappa_{0w}^2}^{\infty} d\kappa^2 \frac{s_w(\kappa^2)}{\kappa^2 - \kappa_{0w}^2} - \lambda_w^2. \]

Evidently the smaller the magnitude of $s_w$ and $\lambda_w,$ the more closely is this condition satisfied by $\kappa_{0w} < \mu_s,$ which is the elementary criterion for instability.
It may be noted here that, under the conditions permitting the introduction of the interaction independent mass $\mu_0$, we have

$$\mu_0^2 = \lambda^2 + \int_{\kappa_0^2}^{\kappa} d\kappa' s(\kappa'^2),$$

which relation applies to the complete interaction scheme and to the strong interactions separately. Hence

$$0 = \lambda W^2 + \int_{\kappa_0^2}^{\kappa} d\kappa' s_W(\kappa'^2)$$

and the right-hand side of the above inequality is the positive quantity

$$\int_{\kappa_0^2}^{\kappa} d\kappa' \frac{\kappa'}{\kappa^2 - \kappa_0^2} s_W(\kappa'^2),$$

which shows that $\mu_0$ must exceed $\kappa_0$ by a corresponding finite amount for instability to appear.

In the region $\kappa \sim \mu < \kappa_0$, the equation to determine $\varphi(\kappa^2)$ is

$$\pi \kappa s_W(\kappa^2) \cot \varphi(\kappa^2) - \kappa^2 - \lambda^2 + \mu^2 \int_{\kappa_0^2}^{\kappa} d\kappa' \frac{s_W(\kappa'^2)}{\kappa'^2 - \kappa^2}$$

or, approximately,

$$\pi \mu^2 s_W(\mu^2) \cot \varphi(\mu^2) = B^{-1}(\mu^2 - \mu_0^2) - \lambda W^2 + \mu^2 \int_{\kappa_0^2}^{\kappa} d\kappa' \frac{s_W(\kappa'^2)}{\kappa'^2 - \mu^2}$$

$$+ (\kappa^2 - \mu^2) \mu^2 \int_{\kappa_0^2}^{\kappa} d\kappa' \frac{1}{\kappa'^2 - \mu^2} \frac{d}{d\kappa'^2} \left( \kappa'^2 s_W(\kappa'^2) \right).$$

The value of $\kappa$ at which $\varphi = \pi/2$ is not quite $\mu_0$, but is given by

$$\mu^2 = \mu_0^2 + B^{-1} \left[ \lambda W^2 - \mu^2 \int_{\kappa_0^2}^{\kappa} d\kappa' \frac{s_W(\kappa'^2)}{\kappa'^2 - \mu^2} \right]$$

and the constant $B$ in

$$\pi \mu^2 s_W(\mu^2) \cot \varphi(\mu^2) = B^{-1}(\mu^2 - \mu_0^2)$$

is obtained from

$$B^{-1} = B_0^{-1} + \mu^2 \int_{\kappa_0^2}^{\kappa} d\kappa' \frac{1}{\kappa'^2 - \mu^2} \frac{d}{d\kappa'^2} \left( \kappa'^2 s_W(\kappa'^2) \right).$$

If the equation that determines $\mu$ is further approximated by inserting $\mu_0$ for $\mu$
on the right-hand side, a slight increase in accuracy is produced by replacing
$B_0$ with $B$,

$$
\mu^2 = \mu_0^2 + B \left[ \lambda w^2 - \mu_0^2 \int_{x_0w^2}^{\infty} d\kappa \frac{\kappa^2}{\kappa^2 - \mu_0^2} \right].
$$

We see that the continuum in the neighborhood of mass value $\mu$ is characterized by

$$
\kappa \sim \mu: B(\kappa^2) = B \frac{1}{\pi} \frac{\gamma\mu}{(\kappa^2 - \mu^2)^2 + (\gamma\mu)^2},
$$

where

$$
\gamma = \pi \mu B_0 w(\mu^2),
$$

and this is equivalent to specifying a discrete mass only when the physical circumstances do not permit the constant $\gamma$ to be distinguished from zero.

We have now reached the point in our examination of the structure of a particular type of Green's function where an explicit consideration of time dependence is needed to complete the physical interpretation. It suffices for this purpose to use a mixed description in which the spatial momentum is specified, the time dependent Green's function $G(k,t)$ being derived from

$$
G(k,t) = \int \frac{dB[k^2]}{k^2 + \kappa^2 - k^2_0 - i\epsilon}
$$

as the frequency integral

$$
G(k,t) = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} e^{-ik^0t} G(k,t^0).
$$

On performing the $k^0$ integration we get

$$
G(k,t) = \int dB[k^2] \frac{i}{2E_\kappa} e^{-iE_\kappa|t|},
$$

with

$$
E_\kappa = (k^2 + \kappa^2)^{1/2}.
$$

We first consider the situation of a single discrete mass, for which

$$
G(k,t) = B_0 \frac{i}{2} \left[ \frac{1}{E_\kappa} e^{-iE_\kappa|t|} + \int_{\kappa^2}^{\infty} d\kappa \beta(\kappa^2) \frac{1}{E_\kappa} e^{-iE_\kappa|t|} \right].
$$

Our principal concern here is with the asymptotic form of this function for large $t$ (which is henceforth taken to be positive). The predominant contributions to the asymptotic value will come from the single term of definite frequency asso-
associated with the discrete mass, and from the immediate neighborhoods of the lowest order thresholds. At the mass value which is the threshold of a new physical process, some derivative of the function $\beta(\kappa^2)$ is singular. A first-order threshold is one where the first derivative is singular, as in

$$\kappa \geq \kappa_j: \frac{d}{d\kappa} \beta(\kappa^2) \sim \frac{\beta_j}{\kappa_j^3} \left[ \frac{1}{4\pi \kappa - \kappa_j} \right]^{1/2}.$$ 

We shall assume here that $\kappa_0$, the lowest of all thresholds, is of first order. Then

$$\int_{\kappa_0^2}^{\infty} d\kappa^2 \beta(\kappa^2) \frac{1}{E_\kappa} e^{-iE_\kappa t} = \frac{2}{i\hbar} \int_{\kappa_0^2}^{\infty} e^{-i\kappa \cdot x_1} d\beta(\kappa^2)$$

$$\sim \sum_j \frac{2}{i\hbar} e^{-i\kappa \cdot x_j} \int_0^\infty d(\kappa - \kappa_j) \beta_j \left[ \frac{1}{4\pi \kappa - \kappa_j} \right]^{1/2} e^{-i(\kappa_j/K_{\kappa_j})t(\kappa - \kappa_j)}$$

provided

$$\kappa j \gg E_{\kappa_j}/\kappa_j,$$

which is satisfied for all thresholds whenever

$$\kappa 0 \gg E_{\kappa_0}/\kappa_0,$$

and

$$G(k,t) \sim B_0 \frac{i}{2} \frac{1}{E_\mu} e^{-i\kappa \cdot \mu t} + \sum_j \beta_j \left[ \frac{E_{\kappa_j}}{i\kappa_j^2 t} \right]^{3/2} e^{-i\kappa_j t}.$$ 

The absolute magnitude of the threshold contributions, compared with the single discrete mass term, is less than $\beta(E_{\kappa_0}/\kappa_0^2)^{3/2}$, where

$$\beta = \sum_j \beta_j \frac{E_{\kappa_j}}{i\kappa_j^2}.$$ 

If $\beta$ does not exceed unity (it is certainly finite), this ratio is already small when the asymptotic evaluation applies, and for large $\beta$ it becomes so when

$$\kappa 0 \gg \beta^{1/2}(E_{\kappa_0}/\kappa_0).$$

We have now suitably qualified the statement that, eventually, the only significant contribution to the Green’s function comes from the term referring to the stable particle of mass $\mu$,

$$G(k,t) \sim -B_0 \frac{i}{2} \frac{1}{E_\mu} e^{-i\kappa \cdot \mu t} = B_0 \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}}{k^2 + \mu^2 - k_{00}^2 - i\epsilon}.$$ 

Without the constant $B_0$ this is the simple propagation function that displays the complete set of states accessible to a single particle of mass $\mu$. The constant
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$B_0$ appears here as the reminder of what is fundamental in the theory—the field, not the particle. Accordingly this constant is removed on transferring the description to the more immediate physical basis provided by the stable particles and their interactions. That, in a few words, is the general physical meaning of renormalization, as distinguished from the more prevalent notion in which it is equated to a formal subtraction device for the elimination of the divergences that appear in perturbation calculations.

Finally, we come to the time dependence of the Green’s function when there is no stable particle. From what has already been said it is clear that the true asymptotic form contains only the contributions of the first-order thresholds. There is, however, another significant contribution under the circumstances outlined in the discussion of weak and strong interactions. In the vicinity of mass value $\mu$, the function $B(\kappa^2)$ varies strongly and attains large values. The contribution of this neighborhood to $\mathcal{G}(k,t)$ is

\[
\frac{i}{2} \int_{\kappa - \mu} \kappa^2 B(\kappa^2) \frac{1}{E_\kappa} e^{-iE_k t} \\
\cong \frac{i}{2} B \frac{1}{E_\mu} e^{-iE_\mu t} \int_{-\infty}^{\infty} d(\kappa - \mu) \frac{1}{2\pi} \frac{\gamma}{(\kappa - \mu)^2 + (\frac{\gamma}{2})^2} e^{-i(\kappa/\mu) t (\kappa - \mu)} \\\n= \frac{i}{2} B \frac{1}{E_\mu} e^{-iE_\mu t} e^{-1/2(\kappa/\mu)^2 t},
\]

provided that

$$\gamma \ll \mu,$$

which evidently describes the exponential decay of an unstable particle of mass $\mu$ and proper mean life time $\tau$, given by

$$\frac{1}{\tau} = \gamma = -\frac{1}{\mu} \text{Im} B \mathcal{G}^{-1}(k) \mid_{\kappa^2 + \mu^2 = 0}.$$

Thus, a more complete asymptotic evaluation of $\mathcal{G}(k,t)$ for

$$\kappa_0 t \gg E_{\kappa_0}/\kappa_0$$

is given by

$$\mathcal{G}(k,t) \sim B \frac{t}{2} \left[ \frac{1}{E_\mu} e^{-iE_k t} e^{-1/2(\kappa/\mu)^2 t} + \sum_j \beta_j \frac{E_{\xi_j}}{E_{\xi_j}} \left( \frac{E_{\xi_j}}{\kappa_0^2 t} \right)^{3/2} e^{-iE_{\xi_j} t} \right].$$

The threshold summation includes both “weak thresholds”, $\kappa_j \geq \kappa_{0W}$, and “strong thresholds”, $\kappa_j \geq \kappa_{0S}$. The lowest threshold $\kappa_0$ may refer either to $\kappa_{0W}$

* More accurately, the first-order thresholds for the production of stable particles.
or \( k_0 \), depending upon the desired accuracy of the discussion. (A weak threshold at zero mass requires a separate treatment.) When the threshold asymptotic evaluation has become valid, or at the possibly greater time produced by the factor \( \beta^{2/3} \), the argument of the exponential decay factor is some large numerical multiple of

\[
\frac{\mu}{\bar{E}} \frac{E_{\ell 0}}{k_0} \frac{\gamma}{k_0} \sim \gamma.
\]

Hence, if \( \gamma \) is sufficiently small,

\[
\gamma \ll k_0,
\]

there is an extended time interval during which the contribution of the thresholds has become negligible and the Green's function is completely dominated by the term referring to the unstable particle,

\[
G(k, t) \sim B \frac{i}{2} \frac{1}{E} e^{-i\mathbf{k} \cdot \mathbf{r}} e^{-\frac{1}{2}(\mu / \mathbf{k} \cdot \mathbf{r})^2} = B \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}}{k^2 + (\mu - \frac{1}{2} \gamma)^2 - k^0}
\]

The exponential time decay of this term, in comparison with the algebraic decrease associated with the stable particle threshold contributions, indicates however that the latter are eventually the sole survivors of the asymptotic evaluation, as we have remarked. Does this mean that the intrinsic decay law of unstable particles ceases to be of exponential form after a sufficiently long time?

The answer to this question (which has occasionally been given by a categorical affirmative) illustrates the necessity of including suitable idealizations of all relevant aspects of the physical situation in its mathematical representation. We first remark that any physical process can be conceived as initiating in and terminating with stable particles. An example is the reaction

\[
p + p \rightarrow p + p + e^+ + e^- + 2\gamma + 2\nu + 2\bar{\nu}.
\]

The multipoint Green's function that gives an account of this process will be constructed, in part, from the simplest Green's functions, describing the propagation of the various stable particles into or out of the interaction region, and from a function that gives the details of the interactions among these particles. The nature of this interaction is specified more closely when we add the information that the above reaction has proceeded through the following stages:

\[
p + p \rightarrow K^+ + \sum^+ + n,
\]

\[
K^+ \rightarrow \bar{\nu} + (\mu^+ \rightarrow e^+ + 2\nu),
\]

\[
\sum^+ \rightarrow p + (\pi^0 \rightarrow 2\gamma),
\]

\[
n \rightarrow p + e^- + \bar{\nu}.
\]
Correspondingly, the function that characterizes the stable particle interactions will contain the Green's functions that refer to the propagation of the intermediate unstable particles, together with details of their localized interactions. But this is not all. The statement that the reaction has involved the particle $K^+$, for example, and not some kinematically equivalent combination of particles, must have a physical counterpart in an aspect of the measurement apparatus that serves to display the presence of the particle, and a mathematical counterpart, in an operation on the relevant Green's function, that conveys the effect of the selective measurement. The only valid basis for asserting that a particular unstable particle has been involved is through an energy-momentum or mass determination of sufficient accuracy to rule out all other possibilities. This need not be a disturbing measurement in any other respect, however.

The mathematical representation of this mass filter is supplied by the modified Green's function

$$M_S(k,t) = \int dB[k^2] \frac{i}{2E^k} e^{-ik_xt} M(\kappa),$$

where $M(\kappa)$ is roughly characterized by the properties

$$M(\kappa) = 1, \quad |\kappa - \mu| \lesssim \Delta \mu$$

$$M(\kappa) = 0, \quad |\kappa - \mu| \gtrsim \Delta \mu$$

and $\Delta \mu$, measuring the precision of the mass determination, is restricted by

$$\gamma \ll \Delta \mu \ll \mu.$$

Thus the threshold contributions, referring to recognizably distinct physical processes, are removed from the Green's function and

$$M_S(k,t) \approx \frac{i}{2} B \frac{1}{E^\mu} e^{-ik_xt} g \left( \frac{\mu}{E^\mu} t \right),$$

where

$$g(s) = \int_{-\infty}^{\infty} d(\kappa - \mu) \frac{1}{2\pi} \frac{\gamma}{(\kappa - \mu)^2 + \left(\frac{1}{2}\gamma\right)^2} e^{-i(\kappa - \mu)s} M(\kappa)$$

$$= e^{-i(1/2)\gamma s} - \int_{-\infty}^{\infty} d(\kappa - \mu) \frac{1}{2\pi} \frac{\gamma}{(\kappa - \mu)^2 + \left(\frac{1}{2}\gamma\right)^2} (1 - M(\kappa)) e^{-i(\kappa - \mu)s}.$$ 

Since $1 - M(\kappa)$ does not differ from zero until $|\kappa - \mu| \gg \gamma$, the denominator of the latter integral can be approximated by $(\kappa - \mu)^2$. On performing a partial integration that becomes asymptotically valid when

$$\Delta \mu s \gg 1,$$
we obtain
\begin{align*}
g(s) & \sim e^{-\gamma s} - \frac{\gamma}{\Delta \mu} \frac{1}{\Delta \mu \delta} \frac{i}{2\pi} \int_{-\infty}^{\infty} d(\kappa - \mu) e^{-i(\kappa - \mu)s} \frac{d}{d\kappa} M(\mu),
\end{align*}
which contains a further approximation that is justified by the localized nature of \(dM/d\kappa\). The second term of \(g(s)\) is evidently quite negligible, at least for moderate values of \(\gamma s\). The question before us, however, concerns the behavior of \(g(s)\) after very many lifetimes have elapsed.

We must now introduce more detailed assumptions about \(M(\kappa)\) in order to provide analytical equivalents of realizable experimental procedure. The major hypothesis is the boundedness of all derivatives of \(M(\kappa)\). If this property failed at some value of \(\kappa\), that mass value would be specifically distinguished, in contrast with the limited selectivity of any finite act of measurement. We shall also assume, for simplicity, that the shape of \(M(\kappa)\) in the neighborhood of \(\mu + \Delta \mu = \mu_2\) is the mirror image of that near \(\mu - \Delta \mu = \mu_1\). Then, confining the integration to the latter neighborhood, we have
\begin{align*}
g(s) & \sim e^{-\gamma s} + \frac{\gamma}{\Delta \mu} \frac{\sin \Delta \mu \delta}{\Delta \mu \delta} \frac{1}{\pi} \int d(\kappa - \mu) e^{-i(\kappa - \mu)s} \frac{d}{d\kappa} M(\kappa).
\end{align*}
According to the boundedness hypothesis for all derivatives of \(M(\kappa)\), the latter integral decreases asymptotically more rapidly than any power of \((1/s)\). The asymptotic behavior is essentially exponential, of the form \(\exp \{-f(\Gamma s)\}\), where \(\Gamma \gg \gamma\) measures the boundary width of \(M(\kappa)\), or physically, the latitude in the precise specification of the mass interval accepted by the filter. Thus during the extended interval
\begin{align*}
\Gamma^{-1} \ll s \lesssim \gamma^{-1},
\end{align*}
the second term of \(g(s)\) is not only small in comparison with the exponential, but is also decreasing much more rapidly. Whether or not this remains true for all time depends upon the precise character of the function that gives the exponential dependence on \(\Gamma s\). If \(f(\Gamma s)\) is ultimately some finite numerical multiple of \(\Gamma s\), the first term of \(g(s)\) will dominate for all \(s\), whereas should the growth of \(f(\Gamma s)\) with increasing \(s\) be less than linear, the roles of the two terms in \(g(s)\) will eventually reverse.

The situation can be studied with the aid of the following class of functions, which satisfy the requirements that have been imposed on \(M(\kappa)\):
\[
\kappa \sim \mu_1; \quad \frac{d}{d\kappa} M(\kappa) = \frac{c}{\Gamma} \exp \left\{ -\frac{\lambda}{2\sigma} \left[ \frac{\Gamma^2}{\Gamma^2 - (\kappa - \mu_1)^2} \right]\right\}, \quad |\kappa - \mu_1| \leq \Gamma
\]
\[
= 0, \quad |\kappa - \mu_1| \geq \Gamma.
\]
Here $\lambda$ and $a$ are positive numbers and

$$e^{-1} = \int_1^1 dx \, e^{-(\lambda/2a)(1-x^2)^{-a}}.$$ 

The required asymptotic form can be obtained by suitably deforming the integration path into the complex plane, and

$$\int_1^1 dx \, e^{-(\lambda/2a)(1-x^2)^{-a}} e^{-ix\gamma} \sim 2 \left( \frac{1}{\lambda} \right)^{1/2} \left( \frac{\lambda}{\sigma} \right)^{1-((1/2)a)}$$

$$\exp \left[ - \left( \cos \frac{\pi \alpha}{2} \lambda \left( \frac{\sigma}{\lambda} \right)^a \right) \sin \left[ \frac{\pi \alpha}{4} + \sigma - \left( \sin \frac{\pi \alpha}{2} \right) \frac{\lambda}{2a} \left( \frac{\sigma}{\lambda} \right)^a \right] \right],$$

in which

$$\alpha = \frac{a}{1 + a} < 1.$$ 

Thus the magnitude of the $dM/dk$ integral is exponentially dependent upon the power $(\Gamma s)^a$ and $\alpha$ is certainly less than unity, the unit limiting value being unattainable in virtue of the factor $\cos (\pi \alpha/2)$ in the exponential. This implies that, after a sufficiently large number of periods have elapsed,

$$\gamma s \sim (\Gamma/\gamma)^a \gg 1,$$

the decaying exponential term, $e^{-(1/2)/\gamma s}$, ceases to dominate $g(s)$. But, precisely when this will occur, and the form of $g(s)$ that ensues, is unavoidably dependent upon the details of the measurement process and is not intrinsic to the unstable particle. We conclude that with the failure of the simple exponential decay law we have reached, not merely the point at which some approximation ceases to be valid, but rather the limit of physical meaningfulness of the very concept of unstable particle.

As it stands, this discussion of a simple boson Green’s function applies only to the Hermitian field associated with the unstable particle $\pi^0$ (where the “weak” decay mechanism is electromagnetic in nature). An extension to a two-component Hermitian field is needed to discuss the particles $\pi^\pm, K^\pm$, and $K_{1,2}$. The Green's function

$$G(x - x')_{ab} = i < \phi_a(x)\phi_b(x') > = \int \frac{(dk)}{(2\pi)^4} e^{ik(x-x')} G(k)_{ab}$$

is a symmetrical function of the indices $x,a$ and $x', b$. The space-time coordinates can appear only in the translational, rotational invariant combination $(x - x')^2$ (an otherwise conceivable dependence upon the algebraic sign of $x^0 - x'^0$, in the interior of the light cone, is excluded by the requirement of a lowest energy
state) and therefore $S(x - x')_{ab}$ is necessarily symmetrical in the indices $a$ and $b$. When these indices refer to the two axes of an electric charge space, the structure of the matrix $S_{ab}$ is completely restricted by the conservation of electric charge

$$[q, S] = 0, \quad q = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

for there is only one type of two-dimensional symmetrical matrix that commutes with the antisymmetrical charge matrix:

$$S(k)_{ab} = \delta_{ab} S(k).$$

Thus the problem is reduced to that of a single component field, and this situation is not changed by introducing the non-Hermitian fields that diagonalize $q$. The value of the electric charge does not affect the properties contained in the Green's function and masses and lifetimes are identical for $\pi^+$ and $\pi^-$, $K^+$ and $K^-$. It is a different matter when the two-fold multiplicity refers to a hypercharge space for this property is not conserved by the weak interactions, and the designations $K^Z(Y = +1)$ and $\overline{K}^Z(Y = -1)$ are meaningful only for sufficiently short times. We accept the conservation law implied by invariance under space and charge reflection and conclude that $S$ is invariant under charge reflection, since it is explicitly invariant to the reflection of the space coordinates. The two Hermitian field components are distinguished by the response to charge reflection, as it operates in the hypercharge space, and $S_{ab}$ must be a diagonal matrix. However, there is no necessary connection between the diagonal elements $S_1$ and $S_2$, except in the approximation that ignores weak interactions where they are equal. Thus from $S_1$ and $S_2$ we infer two distinct sets of masses and lifetimes, which characterize the unstable particles $K_1$ and $K_2$. The unstable particle contribution to $S(k,t)_a$ is

$$S(k,t)_a \sim \frac{i}{2} \frac{B_1}{E_\mu} e^{-i\epsilon_{\mu}\gamma_t} \exp \left( -\frac{1}{2} \frac{\mu_1}{E_\mu} \gamma_t \right)$$

or, in view of the small difference anticipated between $B_1$ and $B_2$, $\mu_1$ and $\mu_2$,

$$S(k,t)_a \sim \frac{i}{2} \frac{B_1}{E_\mu} e^{-i\epsilon_{\mu}\gamma_t} \left( \delta_{a1} \exp \left[ -\frac{i}{2} (\delta \mu - i\gamma_1)s \right] + \delta_{a2} \exp \left[ -\frac{i}{2} (\delta \mu - i\gamma_2)s \right] \right).$$

Here

$$\delta \mu = \mu_1 - \mu_2,$$

while $\mu$ and $s = t(\mu/E_\mu)$ refer to the average mass. If $\delta \mu \ll \Delta \mu$, the mass interval that is accepted in selecting electrically neutral $K$-particles, the asymptotic
structure of $y_{ab}$ is left intact and we encounter a kind of mass interferometer. We need not repeat the discussion, based previously on more elementary considerations, of the experimental possibilities thus implied. If $\delta\mu$ is so large that a mass separation of $K_1$ and $K_2$ becomes feasible, no significant interference effects can occur.

The analogous treatment for fermion fields is sufficiently different in detail that it will be deferred to another publication.

Note added in proof: It should have been mentioned in the text that these considerations are incomplete in one important respect. The full dynamical effect of the electromagnetic field is not represented since a more elaborate spectral form is required for the Green's functions of charge-bearing fields in the physical radiation gauge. This refinement will not alter our physical conclusions, however.