I. INTRODUCTION

We all know that water can come in different phases: as a solid, a liquid, or a gas. Above 100 °C, water becomes steam, and below 0 °C, it becomes ice. We also know that the boiling and freezing temperatures of water depend on the pressure. On top of the mountains south of Bern, for example, water boils at less than 90 °C because of the lower pressure. The pressure dependence of the boiling and freezing temperatures of water can be plotted in a phase diagram, as shown in Fig. 1.

Two points in this phase diagram catch one’s eye. The first one is the so-called triple point at 0 °C and 0.006-atm pressure. Below this pressure the liquid phase of water disappears, and ice directly evaporates when heated up. At the triple point, ice, water, and steam can coexist. So this is kind of interesting. But what I would like to convince you of is that another point in this phase diagram is really interesting, because key concepts of elementary particle physics are hidden in it. This is the so-called critical point at 374 °C and 220-atm pressure.

The critical point is the point at which the phase separation curve between water and steam ends. First, how can the separation curve between two obviously different things such as water and steam just end? Well, when we say that water and steam are obviously different we mean that they have drastically different properties such as densities. Figure 2 shows the density of water at 1-atm pressure as a function of temperature. The density jumps to less than \( \frac{1}{1000} \) of its value at the point where water becomes steam. This is called a first-order phase transition.

If we now raise the pressure, we find that the densities of water and steam at boiling temperature approach each other. The critical point is the point where they become equal, so that the distinction between water and steam seems to disappear. Now, if we measure exactly how the densities of water and steam approach each other as we move along the phase separation line, we find an interesting behavior: the density is not a smooth function of the temperature, but has a singularity (Fig. 3).

More precisely, close to the critical point the deviation of the density \( \rho \) from the critical density \( \rho_c \) is proportional to a power of the deviation of the temperature \( T \) from the critical temperature \( T_c \):

\[
|\rho - \rho_c| \sim |T - T_c|^\beta.
\]

Fig. 1. Phase diagram of water.

Fig. 2. Density of water at 1 atm pressure.

Fig. 3. Density of water along the phase separation curve in Fig. 1.
The so-called critical coefficient $\beta$ can be measured experimentally. One finds:

$$\beta = 0.33 \pm 0.01.$$  

This kind of phase transition, where $\rho$ does not jump but obeys a scaling relation such as (1) is called a second-order phase transition.

What I would like to do now is to first give an idea of what exactly happens at the critical point. I will explain things with the help of images from a computer simulation of a simple toy model, the so-called lattice gas model. As a test that this is the correct idea I will show that it indeed reproduces the power law (1), and that even the coefficient $\beta$ comes out right.

These are old ideas. The key idea is that of the so-called renormalization group. The renormalization group has important applications in particle physics, some of which I will mention in the end of the paper. My own results on the perturbative string theory can be found in my habilitation thesis.

II. THE LATTICE GAS MODEL

One might worry that one needs to know a lot about the structure of water molecules, hydrogen bonds, and so on, in order to explain why $\beta$ is about 0.33. But surprisingly, if one measures $\beta$ for other substances with waterlike phase diagrams like xenon or carbon dioxide, one always finds roughly 0.33 (see, e.g., Ref. 6). This value seems to be a universal property of ‘‘Van der Waals’’ gases, i.e., of gases with two basic properties: First, the molecules have a hard core, so that they cannot come arbitrarily close to each other. And second, there is a weak attractive force between the molecules.

Now consider a simple toy model of such a gas, the so-called lattice gas model. In the lattice gas model we imagine that three-dimensional space is a lattice as in Fig. 4. Only two dimensions are drawn for simplicity. At each lattice point there is either one molecule or none. So we equip each lattice point $i$ with an occupation number $n_i$ that is either 0 or 1. The rule that a lattice point is occupied by at most one molecule represents the hard core of the molecules.

We define the energy of a given configuration of occupation numbers as

$$E = - \sum_{(ij)} n_i n_j.$$  

(3)

Here, $i$ and $j$ are two neighboring lattice points. So the sum is a sum over all links. Put differently, for each pair of neighboring molecules the energy is decreased by $-1$, e.g., the energy of the configuration in Fig. 4 is $-3$ since there are three links that are occupied at both ends. This definition of the energy represents the attractive force between the molecules, because the molecules try to stick together in order to minimize the energy.

We now imagine that the occupation numbers are subject to statistical fluctuations, with the total number of molecules

$$N = \sum_i n_i.$$  

(4)

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down. Let’s call them “plus” and “minus.” For a positive magnetic field the ground state is “plus.” For a negative magnetic field the ground state is “minus.”

Imagine the system is in the state “plus.” To reverse a single spin we need \(4 \cdot \frac{1}{2} \) units of energy since the spin has four neighbors and for each neighboring pair we must raise the energy from \(- \frac{1}{2}\) to \(+ \frac{1}{2}\) [see Fig. 5(a)]. Similarly, one needs \(L/2\) units of energy to create a “minus bubble” of length \(L\) [Fig. 5(b)]. One can say that bubbles have a surface tension in the sense that their energy is proportional to the circumference of the bubble, or to the surface of the bubble in the three-dimensional model.

We now imagine that the spins fluctuate. Let \(\{s_i\}\) be a given configuration of all \(s_i\). As is usual in statistical mechanics, we consider the ensemble of all possible spin configurations \(\{s_i\}\), weighted by the so-called “Boltzmann factor”

\[
\exp\left\{- \frac{E(\{s_i\})}{T} \right\},
\]

where \(T\) is the temperature. The Boltzmann factor can be thought of as a relative probability that is assigned to each spin configuration. Thus the expectation value of the spin at the point \(i\) is defined as

\[
\langle s_i \rangle = \frac{1}{Z} \sum_{\{s_i\}} s_i \exp\left\{- \frac{E(\{s_i\})}{T} \right\},
\]

where \(Z\) is a normalization constant,

\[
Z = \sum_{\{s_i\}} \exp\left\{- \frac{E(\{s_i\})}{T} \right\},
\]

that normalizes the probabilities such that their sum is unity. \(Z\) is called the partition sum.

The Boltzmann factor (8) suppresses spin configurations with high energy. More precisely, for small temperatures all configurations are very unlikely relative to the one with lowest energy. Figure 6 shows images of a computer simulation of the Ising model on a lattice of size 500 times 200 (Ref. 13). Positive spins are black and negative spins are white. We see two typical spin configurations at low temperature and zero magnetic field. The system is either in the state “plus” with only a few negative spins, or it is in the state “minus” with only a few positive spins. Those are our two phases. Because of their interpretation in the lattice gas model [see (6)] I shall call the upper phase “water” and the lower phase “steam.”

For very high temperatures, on the other hand, the Boltzmann factor is almost 1 for all energies. All spin configurations are therefore equally probable. So the typical configuration is a mix of randomly distributed spins (Fig. 7) that is neither water nor steam. There are no different phases.

We can now draw the phase diagram of the Ising model in Fig. 8. The temperature is plotted to the right and the magnetic field upward.

As we have seen, there is a phase separation curve at low temperature and zero magnetic field, \(B = 0\). The phase tran-
sition is first order in the sense that the spin expectation value $\langle s_i \rangle$ in (9)—and thus the density of molecules in the lattice gas model—jumps as a function of $B$. This is shown in Fig. 9. Starting from the state “water” at $B=0$ (the upper left-hand corner) we apply a small magnetic field $B$. In a time-dependent simulation of the model, sooner or later bubbles appear, despite the surface tension, that are large enough to spread and turn the system into “steam” (the lower right-hand corner) (for $B$ constant in time). One could say that the “water boils.”

We have seen that there is no difference between “water” and “steam” for high temperature. So we assume that the phase separation curve ends in a critical point at some critical temperature $T_c$. The analogy with the phase diagram of water (Fig. 1) is clear.

IV. CLOSE TO THE CRITICAL POINT

Suppose the Ising model is at the phase separation curve at low temperature $T$ in the state “water” (e.g., $T=0.9T_c$, $B=0$). We now raise the temperature, thus approaching the critical point along the separation curve. As $T$ grows, spin configurations with higher energy get less strongly suppressed by the Boltzmann factor (8). Thus the statistical fluctuations become more important and the typical configuration contains more and bigger bubbles. Figure 10 shows typical configurations for temperatures $T=0.9T_c$ (above), $T=0.98T_c$ (middle), and $T=T_c$ (below).

Let us denote the average size of a droplet by $\xi$. You can see clearly that $\xi$ grows with the temperature. At the critical temperature (the lower picture), $\xi$ seems to diverge. There you see black droplets inside white droplets inside black droplets and so on. You cannot really distinguish whether the picture shows black droplets in a white sea or white droplets in a black sea. The distinction between the phases disappears.

Let us now approach the critical temperature from above, again at zero magnetic field. Figure 11 shows typical configurations for temperatures $T=\infty$ (above), $T=1.08T_c$ (middle), and $T=1.017T_c$ (below). At infinite temperature the spins are randomly distributed. As the temperature drops, the spins start to feel the interaction with their neighbors through the Boltzmann factor. Therefore structures form. Let us now call $\xi$ the average size of the structures, the so-called correlation length. You see again that $\xi$ grows as we get closer to the critical temperature.

At the critical point it seems again as if the correlation length diverges. Structures of all magnitudes can be recognized in a typical configuration (Fig. 12, above). It is easy to imagine that the structures you see are only part of even larger structures that do not fit on the screen.

This divergence of the correlation length can also be measured experimentally in water and other substances near the critical point. One finds that the correlation length diverges as an inverse power of the deviation of the temperature from the critical temperature:

$$\xi \sim \frac{1}{|T-T_c|^\nu},$$

with a certain coefficient $\nu$. The relation (11) is similar to the relation (1). For the coefficient $\nu$ one measures:

$$\nu \approx 0.63 \pm 0.01.$$  

We will explain this behavior later, together with the behavior (1).

V. SCALE TRANSFORMATIONS

Another property of the critical point can also be seen in Fig. 12: scale invariance. By this, the following is meant:
Figure 12, below, shows a portion of another typical configuration of the Ising model near the critical point, but magnified by a factor of 2. This, of course, makes the picture more coarse grained. But merely from the distribution of the sizes of the structures you cannot determine well which of the pictures is the magnified one.

How does the Ising model behave under scale transformations when $T$ is near, but not at the critical temperature? Let us consider Fig. 11 again. It seems like the lower picture could be a magnified portion of the middle picture. So it seems as if, near the critical temperature, a scale transformation is equivalent to a change in temperature.

Since this is a key point, let us make more precise what we mean by a scale transformation. One can define it in two steps. The first step is to make a picture like Fig. 11, below, more coarse grained, e.g., one can combine $3 \times 3$ lattice cells into one large cell throughout the picture. When the majority of the nine small cells is black, the large cell is black; otherwise, it is white (Fig. 13).

The second step consists in down-sizing the picture by a factor of 3. We call both steps together a scale transformation by a factor of 3: it is as if we regarded the picture from 3 times as far away. The statement is that the resulting picture can be interpreted as a portion of a typical configuration of the Ising model at a different temperature $T'$. $T'$ is farther away from $T_c$ than the original temperature: the correlation length has decreased to $\frac{1}{3}$ of its size.

Now imagine an Ising model of infinite extent at a temperature $T$ very close to the critical temperature. Let us call $T$ the "actual," or "bare" temperature. We consider some small portion of the model. We now do a scale transformation as described above by a factor $e^\tau$. This results in an Ising model at a new temperature $T'$ that depends on $\tau$ (as opposed to $T$). Let us call this fictitious temperature $T'$ the "renormalized" temperature $T_{\text{ren}}(\tau)$. We have

$$T_{\text{ren}}(\tau=0) = T.$$  

(13)

This dependence of the renormalized temperature $T_{\text{ren}}(\tau)$ on a change of scale by $e^\tau$ is what we call the "renormal-
visualization group flow' of \( T_{\text{ren}}(\tau) \). The critical temperature \( T_c \), at which the system is scale invariant, is called a fixed point of this flow.

In thermodynamics and in daily life one usually talks only about the bare temperature \( T \). On the other hand, in particle physics one usually talks about renormalized quantities (such as charges), since those are the ones that can be observed.

Next I would like to mention how the idea of the renormalization group flow can be used to explain the power laws (1) and (11) and to determine the coefficients \( \beta \) and \( \nu \).

VI. CRITICAL COEFFICIENTS

Let us quickly put these statements into formulas. We define the deviation of the temperature \( T_{\text{ren}} \) from the critical temperature as \( t \):

\[
 t(\tau) = T_{\text{ren}}(\tau) - T_c.
\]

Scale invariance at the critical point means that \( t=0 \) is a fixed point of the flow:

\[
 \frac{d}{d\tau} t(\tau) = 0 \quad \text{at} \quad t=0. \tag{15}
\]

When \( t \) is nonzero but small, the “flow velocity” \( dt/d\tau \) can be expanded in powers of \( t \):

\[
 \frac{d}{d\tau} t = \frac{1}{\nu} t + ct^2 + dt^3 + \ldots \tag{16}
\]

with coefficients \( \nu, c, \) and \( d \) that need to be determined. The right-hand side of the equation is the so-called “beta function” \( \beta(t) \). The flow of \( t \) is shown in Fig. 14. The arrows represent the flow velocity \( \beta(t) \). The renormalized temperature moves away from \( T_c \) as the scale is increased.

For very small \( t \), the terms \( t^2, t^3, \) etc., in the beta function in (16) can be neglected. Equation (16) then becomes a linear differential equation that can easily be solved:

\[
 t(\tau) = |T_{\text{ren}}(\tau) - T_c| \sim |T - T_c| e^{(1/\nu)\tau}. \tag{17}
\]

From this, a relation between the correlation length \( \xi \) and the actual temperature \( T \) can be derived: As discussed in the context of Fig. 11, the correlation length shrinks under a scale transformation by, say, a factor \( e^{\varepsilon} = 3 \) as follows:

\[
 \xi \rightarrow 3^{1/\nu} \xi. \tag{18}
\]

From (17) we see that such a scale transformation imitates the following change in the actual temperature:

\[
 |T - T_c| \rightarrow |T - T_c| 3^{1/\nu} \tag{19}
\]

(because the scale transformation and the change in temperature then lead to the same fictitious temperature \( T_{\text{ren}} \)). From (18) and (19) we get, near the critical point, the proportionality

\[
 \xi \sim \frac{1}{|T - T_c|^\nu}, \tag{20}
\]

because \( \xi \) then transforms as in (18) under the change of temperature (19). This is the relation (11), which is thereby explained by the renormalization group. But \( \nu \) still needs to be determined.

One now replaces the three-dimensional lattice again by a continuous space and also replaces the spin variable \( s_i \) by a continuous field \( \phi(x) \) that describes the average value of the spins in the vicinity of the point \( x \). In the lattice gas model, \( \phi \) is the density of the molecules from the critical density \( \rho_c = \frac{1}{2} \).

\[
 \phi(x) = \rho(x) - \rho_c. \tag{21}
\]

Next, one tries to describe the lattice gas model at the critical point and at large scales by an effective field theory for the scalar field \( \phi \), i.e., one replaces the partition sum in (10),

\[
 \sum_{\{s_i\}} \exp \left\{ - \frac{E(\{s_i\})}{T} \right\}, \tag{22}
\]

by a path integral

\[
 \int [d\phi(x)] \exp \{ - S[\phi(x)] \}, \tag{23}
\]

where we formally have to do one integral for each point \( x \) in space. \( S[\phi(x)] \) is a certain effective action for the field \( \phi(x) \) that I will come to later. Let me only mention now that the advantage of this continuum formulation is that it is much easier to do computations with it than with the lattice model. In particular, one can test whether what we have read off from Fig. 11 is really true: Near the critical temperature and at large scales, is a scale transformation by the factor \( e^{\varepsilon} \) really equivalent to a change of the temperature \( T \rightarrow T(\tau) \)?

One finds that this is indeed the case, provided that one simultaneously allows for a dimension \( d_\phi \) of the field \( \phi \). This means that \( \phi \) is rescaled under scale transformations as follows:

\[
 \phi(x) \rightarrow \phi_{\text{ren}}(x) = e^{d_\phi \varepsilon} \phi(x). \tag{24}
\]

Combining (17), (21), and (24), one finds similarly as in deriving (20):

\[
 |\rho - \rho_c| \sim |T - T_c|^{\nu d_\phi}. \tag{25}
\]

This explains the experimentally observed behavior (1) with

\[
 \beta = \nu d_\phi. \tag{26}
\]

It remains to compute the coefficients \( \nu \) and \( d_\phi \). I cannot do this calculation here, and only mention that it turns out to be difficult in three dimensions but can be done relatively easily near four dimensions with the help of the effective field theory. One formally computes \( \nu \) and \( d_\phi \) in \( 4 - \varepsilon \) dimensions, where \( \varepsilon \) is small, and then tries to set \( \varepsilon = 1 \) in the result, which yields

\[
 \nu = \frac{1}{12} + \frac{7}{1440} \varepsilon + \frac{7}{12} \varepsilon^2 + \cdots, \tag{26}
\]

\[
 d_\phi = 1 - \frac{1}{12} \varepsilon + \frac{7}{12} \varepsilon^2 + \cdots. \tag{27}
\]

Therefore,

\[
 \beta = \nu d_\phi = \frac{1}{12} + \frac{7}{1440} \varepsilon + \frac{7}{12} \varepsilon^2 + \cdots. \tag{28}
\]

These are perturbation expansions in \( \varepsilon \). Such expansions are useful only when they converge. In this case we seem to be lucky: the coefficients quickly become smaller, and the series...
seems to converge even in the case $\epsilon=1$, corresponding to three dimensions (at least so it seems to this order in $\epsilon$). If we plug in $\epsilon=1$, we get

$$\nu \sim 0.63, \quad \beta \sim 0.34.$$  \hspace{1cm} (29)

This is in nice agreement with the experimental data (12) and (2).

VII. UNIVERSALITY

It remains to give a reason why xenon, carbon dioxide, water, and even the primitive lattice gas model all produce the same critical coefficients at the critical point. The renormalization group explains this as follows: The previously mentioned effective action for the field $\phi(x)$ is of the form

$$S[\phi(x)] = \int d^3x \sum_{r=1,2,3} \left( \frac{\partial \phi}{\partial x_i} \right)^2 - V(\phi),$$  \hspace{1cm} (30)

where the first term measures the density fluctuations of the gas, and the second term is a potential

$$V(\phi) = r \phi^2 + \sum_{k=4} \frac{g_k}{k!} \phi^k$$  \hspace{1cm} (31)

that has the form shown in Fig. 15 for small $\phi$ and $r<0$. $r$ is roughly $(T-T_c)$, and the coefficients $g_k$ depend on whether one describes H$_2$O, CO$_2$, Xe, or the lattice gas model (this action describes these systems only at the phase separation line).

One finds that all the coefficients $g_k$ flow under scale transformations. But they flow in such a way that all points in the space of coupling constants flow at large distances toward a single fixed line, as schematically shown in Fig. 16. In Fig. 14, only this line was drawn.

On this line there is a fixed point. It obviously corresponds to the critical point. For example, in $4-\epsilon$ dimensions it lies, to lowest order in $\epsilon$, at

$$r = -\frac{1}{2}\epsilon, \quad \sigma = \frac{\epsilon}{4\pi^2},$$  \hspace{1cm} (32)

while the $g_k$ with $k>4$ are of order $\epsilon^2$. The only relevant parameter, i.e., the only parameter that is observable at large distances, is the parameter along the fixed line ($T-T_c$). The initial (bare) values of all other parameters $g_k$ are irrelevant in the sense that they flow to the same points at large distances. Water, hydrogen, xenon, and the lattice gas model all have the same critical coefficients because at large scales they are all described by the same one-parameter family of effective potentials—those that correspond to the fixed line.

VIII. ELEMENTARY PARTICLE THEORY

I have said that water and steam at the critical point are described by a three-dimensional field theory of a scalar field. Of course, there are other fields in nature. An example is the electromagnetic field which is—among other things— responsible for the attractive force between an electron and a positron. This attractive force is proportional to the square of the elementary charge $e$ and inversely proportional to the square of the distance (Fig. 17).

Out of the elementary charge $e$, Planck’s constant $\hbar$, and the speed of light $c$, a dimensionless constant can be formed, the so-called fine structure constant $\alpha$:

$$\alpha = \frac{e^2}{\hbar c} \sim \frac{1}{137}. $$  \hspace{1cm} (33)

The interaction of the electromagnetic field with electrons and positrons is described by a four-dimensional field theory called quantum electrodynamics. It turns out that $\alpha$ flows under scale transformations by the factor $e^\tau$, similar to the temperature discussed above. One can compute the following flow equation:

$$\frac{d}{d\tau} \alpha \sim -\alpha^2$$  \hspace{1cm} (34)

with a certain constant of proportionality. Here, the limit $\tau \rightarrow \infty$ corresponds to large scales. The corresponding flow diagram is shown qualitatively in Fig. 18. The arrows indicate how $\alpha$ changes under a change of scale. One sees that $\alpha$ decreases at large scales and increases at small scales.

![Fig. 17. Force between electrons and positrons.](image)
This effect can actually be measured. When electrons and positrons are collided with high energies so that they come close together, one finds that they scatter as if a was already about 1 at scales of $10^{-17}$ m.

Now, there are not only electric charges in nature but also color charges. Quarks exist in three different “colors”: red, green, and blue. There are also “color forces” between colored objects, with a corresponding interaction strength $\alpha_S$, which is of order 1 at scales of the order of $10^{-15}$ m (this is roughly the size of a nucleus).

The interaction of the corresponding “gluon” fields with quarks is described by another field theory, quantum chromodynamics. It turns out that $\alpha_S$ also flows under scale transformations. When $\alpha_S$ is small it obeys a differential equation of the form

$$\frac{d}{d\tau} \alpha_S \sim + \alpha_S^2.$$  

It is similar to the one for $\alpha$, but the constant of proportionality is different and, in particular, has a reversed sign. Therefore, the arrows in the flow diagram (Fig. 19) point in the other direction. You see that $\alpha_S$ grows at large scales and decreases at small scales.

This effect can also be measured. Experimentally, it seems as if $\alpha_S$ is only about $10^{-19}$ m. Quarks seem to feel the color forces only weakly at small distances. This is called “asymptotic freedom.”

**IX. STRING THEORY**

There are many other interesting applications of the renormalization group in elementary particle theory, but I want to conclude by mentioning its importance in the most ambitious branch of particle theory: in string theory. String theory is ambitious in that it promises to unify conventional elementary particle theory with Einstein’s theory of gravity.

What is string theory? I can only show two pictures here. Usually it is assumed that elementary particles are pointlike, in the sense that their classical trajectories are described by a vector $x(\sigma)$ that depends on a real parameter $\sigma$ (Fig. 20). In string theory one assumes that $x$ depends on two real parameters $\sigma_1$ and $\sigma_2$ which thereby parametrize a surface—the so-called “world-sheet” of the string (Fig. 21).

It has proven to be very useful to reinterpret the coordinates $x(\sigma_1, \sigma_2)$ as fields that live on the world-sheet. The motion of the string is then described by a two-dimensional field theory. It can be shown that the corresponding action is, in the simplest case, of the form

$$\int d^2\sigma \sum_{\alpha,i} \left( \frac{\partial x^i}{\partial \sigma_\alpha} \right)^2,$$

where both $\alpha \in \{1,2\}$ and the index $i$ of space must be summed over. This action is similar to (30) without potential—but the coordinates $x^i$ are now the fields.

This action can be generalized in many ways. World-sheets that live in a curved space rather than in a flat space, e.g., are described by replacing

$$\sum_{\alpha,i} \left( \frac{\partial x^i}{\partial \sigma_\alpha} \right)^2 \rightarrow \sum_{\alpha,\alpha',i} \frac{\partial x^i}{\partial \sigma_\alpha} \frac{\partial x^i}{\partial \sigma_{\alpha'}} g_{ij}(x),$$

where $g_{ij}(x)$ is the metric of the curved space.

Just like the potential in (30) introduces a self-interaction of the field $\phi$, $g_{ij}(x)$ introduces a generally complicated interaction between the fields $x^i$. And just like the coupling constants $g_k$ in Fig. 16 flow under scale transformations, one...
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\[ \frac{d}{d\tau} g_{ij}(x) \sim R_{ij}(x) \]  

(38)

plus corrections, where \( R_{ij}(x) \) is the curvature tensor of space at the point \( x \). In particular, at a fixed point of the renormalization group flow the metric obeys Einstein's gravitational field equations \( R_{ij} = 0 \) (in vacuum) in lowest approximation. In this sense, string theory automatically contains Einstein's theory of gravity.\(^{11}\)

Those two-dimensional field theories can be generalized such that, in addition to a metric, they contain electromagnetic and other fields. The fixed point equations then become the classical equations of motion of these interacting fields.

There are many such fixed points. Their corresponding properties directly translate into predictions for elementary particle physics that can be compared with observation, such as the number of families of quarks and leptons, the observable gauge group, the number of Higgs fields, etc. An important part of (perturbative) string theory consists of finding the fixed points of the renormalization group flow in the general two-dimensional field theory and computing their properties.

To do this, one often uses methods that are related to those that one uses for the computation of the critical coefficients \( \beta \sim 0.33 \) and \( \nu \sim 0.63 \) for the phase transition between water and steam: the methods of the renormalization group and of “conformal,” i.e., scale invariant field theory.\(^{12}\) The reason for this relation between string theory and the theory of the phase transition is that water and steam at the critical point provide a prototype of a nontrivial scale invariant field theory, which is what I meant when I said in the beginning that the critical point is the most interesting point in the phase diagram of water (Fig. 1).\(^{13}\)

\(^{a}\)Work supported by Schweizerischer Nationalfonds.

\(^{b}\)Based on a review lecture for physicists and nonphysicists as part of the requirements for “Habilitation” at Bern University.


\(^{3}\)A more recent discussion can be found in: J. Zinn-Justin, **Quantum Field Theory and Critical Phenomena** (Oxford U.P., New York, 1989), Chap. 25.


\(^{9}\)For an introduction to string theory, see: M. B. Green, J. H. Schwarz, and E. Witten, Superstring Theory (Cambridge U.P., New York, 1987).


\(^{13}\)I thank H. Bebie for letting me use his program. Another simulation program for the Ising model is available at the internet address http://penguin.phy.bnl.gov/www/xtoys/xtoys.html.

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**THE ART OF WAR AND THE ART OF SCIENCE**

The best military historians in fact do recognize the difficulty in stating rules of generalship. They do not speak of a science of war, but rather of a pattern of military behavior that cannot be taught or stated precisely but that somehow or other sometimes helps in winning battles. This is called the art of war. In the same spirit I think that one should not hope for a science of science, the formulation of any definite rules about how scientists do or ought to behave, but only aim at a description of the sort of behavior that historically has led to scientific progress—an art of science.