On the Structure of the Interaction of the Elementary Particles, I

---The Renormalizability of the Interactions---

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The applicability of the renormalization theory was investigated for the general types of the local interactions. It was found that the interaction having the coupling constant of the dimension $E^n$ could be renormalized if the conditions $\xi \leq 0$ were valid. It is remarkable that this condition is closely related to Heisenberg's classification of the interactions.

§ 1. Introduction

In the recent years, considerable progress in the theory of the elementary particles was achieved by the development of the renormalization theory. Especially, the quantum electrodynamics of the electron has had great success not only in circumventing the well known divergence difficulties but also in explaining the Lamb shift of the hydrogen atom and the subtle anomaly in the magnetic moment of the electron. In spite of this, as we have stressed in the previous paper, the renormalization theory should be regarded as an abstract formalism, behind which concrete structures of the elementary particles lie hidden. So long as the renormalization theory is successful, it is unnecessary to expose the detailed features of such hidden parameters, but as soon as the defects become obvious we must seriously look for them.

As is well known, the renormalization theory has encountered grave difficulties in the problems of the mesons. Not only do calculations still yield divergent results, but even if finite, we often obtain results which are quantitatively inconsistent with experimental data. For instance, none of the four types of the meson theory has yet been able to explain the anomalous magnetic moment of the nucleon.

Nevertheless, it would be too early to conclude that the renormalization theory failed in treating such problems. In fact, if we apply the philosophy of the renormalization to the utmost, any sorts of experiments can always be accounted for by introducing new kinds of interactions as the counter terms of the original interactions. For example, the magnetic anomaly of the nucleon can be explained by assuming the existence of the magnetic moment of Pauli type.

In the present stage of the theoretical advance, there is no criterion deciding what

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* A preliminary report of this work was made at the REKS meeting at Nagoya University, February 6, 1951 (See S. Sakata, H. Umezawa and S. Kamefuchi, Phys. Rev. 84 (1951), 154).
kinds of interactions are the primary interactions of the elementary particles, and what kinds of interactions are the consequences of the others. It is therefore quite legitimate at present to assume the existence of the Pauli moment as the primary interaction. However, if it were so, the great successes of the renormalization theory in the quantum electrodynamics would be also regarded as merely accidental, because we can not understand the reason why the Pauli moment of the electron vanishes identically.

Thus, it is required that the future theory of the elementary particles should account for the structure of the interactions, that is, the qualitative differences as well as the intimate relationships between various kinds of interactions. However, in the present stage, we must content ourselves with a more formal or phenomenological classification of interactions. Although the renormalization theory treats originally all kinds of interactions on the equal footing, it has become clear recently that they may be classified into two groups as follows: (a) those interactions which can be renormalized by assuming the coexistence of a finite number of interactions belonging to the same group, and (b) those interactions which require the further introduction of infinitely many interaction terms having successively higher derivatives as well as larger numbers of the field quantities. The electromagnetic interaction of the electron, scalar interaction of the scalar meson and so on are examples of the former group, while the electromagnetic interaction of the vector meson, the direct interaction between spinor particles and so on belong to the latter.

In the case when all the interactions realized in the nature belong to the first group, the renormalization theory will form a closed system in the framework of the present quantum field theory. On the contrary, in the case when there exist some interactions belonging to the second group, the renormalization theory will exceed the limit of its applicability. As the assembly of infinite number of interactions having successively higher derivatives is equivalent to a non-local interaction, we must then take into account the structure of the elementary particles consistently.

From these considerations it seems to be very important to classify all the interactions into two groups and to investigate whether the interactions of the second kind really exist in the nature or not. In this paper we shall treat the first subject, whereas the second subject will be discussed in the following paper. The renormalizability of the general types of the local interactions is studied in detail in § 2. As we shall show in § 3, this condition can be written in a very simple form

\[ \eta \leq 0, \]  

(1.1)

where \( \eta \) has such a physical meaning that the dimension of the coupling constant is written as \( (\text{length})^{\eta} \).

Now, in connection with this condition (1.1), which indicates the limit of applicability of the renormalization theory, it is of interest to recall Heisenberg's classification of interactions into the first \( (\eta = 0) \) and the second kind \( (\eta > 0) \). As is well known, his classification was also related to the limit of the applicability of the quantum field theory—though the previous formulation was adopted. In this respect, our results may be regarded

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* We use natural units throughout with \( \hbar = c = 1 \).
§ 2. Renormalizability of the interactions

In this section we shall investigate the conditions for the renormalizability of the most general type of interactions. Here, the interactions are said to be renormalizable if, by introducing a finite number of local interactions as the counter terms into the original interactions we can obtain a closed theory which does not contain any singularity.

i) Suppose that there are \(m\) charged fields \(U^a(x)\) and \(n\) neutral fields \(V^\alpha(x)\) interacting with each other. They have generally several components, which will be denoted as \(U^a_\tau(x)\) or \(V^\alpha_\rho(x)\) when necessary. In the following, we shall adopt the interaction representation throughout. So that the field quantities satisfy the equations of motion for free fields*:

\[
\{\Box - (x^2)^2\} U^a = 0,
\]
\[
\{\Box - (x^3)^2\} V^\alpha = 0,
\]

(2.1)

with some subsidiary conditions, where \(x^2\) and \(x^3\) denote masses of respective fields. The commutation relations are given by

\[
[U^a_\tau(x), U^{a\ast}_\rho(x')]_\pm = i\delta_{\tau\rho}\delta_\mu\delta_{\nu\nu}\Delta^a_\mu(x-x'),
\]
\[
[V^\alpha_\rho(x), V^{\alpha\ast}_\sigma(x')]_\pm = i\delta_{\rho\sigma}\delta_{\mu\nu}\Delta^\alpha_\nu(x-x'),
\]

(2.2)*

where commutator or anti-commutator is taken according as the fields have integer or half-integer spins. \(\Delta^a_\mu(x)\) may be written generally in the form:

\[
\Delta^a_\mu(x) = \Gamma^a_\mu(x^\mu, i\partial_\mu)\partial (x),
\]

(2.3)

where \(\Gamma^a_\mu(x^\mu, i\partial_\mu)\) is a polynomial of \(\delta\)-th degree with respect to the derivative operators \(\partial_\mu = \partial / \partial x_\mu\).

The interaction of the system of fields can be described by the Tomonaga-equation

\[
i\frac{\partial \mathcal{F}(\sigma)}{\partial \sigma(x)} = H(x) \mathcal{F}(\sigma),
\]

(2.4)

where \(\mathcal{F}(\sigma)\) denotes the state vector on a surface \(\sigma\), while \(H\) is the interaction. By integrating this equation, Dyson\(^7\) has shown that the \(S\)-matrix, which combines the state vector on the surface in the remote past with that of the distant future is given by

\[
S = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \int dx_1 \cdots \int dx_n F[H(x_1), \cdots, H(x_n)].
\]

(2.5)

Now we assume that the most general type of the interactions is given by

\[
H = \sum_i H_i,
\]

(2.6)

with

\[
H_i = \sum_{\alpha} f_i I_{\alpha} \{ (\partial^{\alpha} U^{a\ast}_\alpha) \cdots (\partial^{\alpha} U^{a\ast}_{i\alpha}) \}^*.
\]

* The field quantities are to be understood as those expressed in irreducible representations. This must be borne in mind especially in the application of the conditions (I) and (II) given below.
\[ \times \left\{ \frac{\partial^\alpha U^a_{\beta\mu}}{\partial x_{\beta\mu}} \right\} \ldots \left( \partial^\alpha V^a_{\beta\mu} \right) \right\}, \]  
\[ (2.7) \]

where \( f_i \) denote the coupling constants and \( \partial^\alpha U^a \) stands for \( \frac{\partial^\alpha U^a}{\partial x_{\beta\mu}} \partial x_{\beta\mu} \ldots \partial x_{\beta\mu} \). As \( \lambda \) and \( \mu \) depend on \( \alpha \) and \( \nu \) on \( \beta \), they will be specified as \( \lambda^a, \mu^a, \nu^a \) when necessary. The detailed form of \( H \) is determined by various formal requirements such as Lorentz invariance, Hermitian property, charge symmetry, conservations of charge and spin, etc., but it need not be specified for the purpose of this paragraph. Though \( H \) contains in general the normal dependent terms, they may be neglected here altogether.

It would be very difficult or even impossible to derive such a general interaction as (2.6) and (2.7) from the canonical formalism, because it contains higher derivatives of the field quantities. Nevertheless, we assume here, from the stand point of the correspondence principle, that as a consequence of some procedure the S-matrix can be computed by (2.5) with \( H \) given by (2.6) and (2.7). In the following, it will be called the interaction in a broad sense.*

ii) By extending Dyson’s analysis of the S-matrix,1), we can now investigate the condition for the renormalizability of the interaction term (2.7). Let us first consider the contributions to the S-matrix from each of the field quantities involved in \( H_i \). If a virtual quantum of the \( \alpha \)-field passes between \( x' \) and \( x'' \), the matrix element will contain in the integrand a factor

\[ \langle P(\partial x^a_{\beta\mu} U^a_{\beta\mu}(x'), \partial x^a_{\beta\mu} U^a_{\beta\mu}(x'')) \rangle_s = i \frac{\partial^a_{\alpha\beta} \partial^a_{\mu\beta}}{\partial^a_{\alpha\mu}} \partial^a_{\nu\beta} (x' - x''), \]  
\[ (2.8) \]

where \( \partial^a_{\alpha\beta} (x' - x'') \) is Feynman’s propagation function for \( \alpha \)-field derived from \( \partial^a_{\alpha\beta} \). By using the representation in the momentum space,

\[ \partial^a_{\alpha\beta}(x) = \text{const.} \int d^4k \frac{\Gamma^a_{\alpha\beta}(x^a, k^a)}{k^a + (x^a)^2} \epsilon_{-\alpha\beta} \epsilon_{\mu\nu}. \]  
\[ (2.9) \]

As \( \Gamma^a_{\alpha\beta}(x^a, k^a) \) is a polynomial of \( \delta^a \)-th degree with respect to \( k^a \), (2.8) contributes to the matrix element in the momentum space a factor of the highest order \( k^a \delta^a + \delta^a - 2 \) for the large value of \( k^a \). In other words, \( \partial^a_{\alpha\beta} U^a \) in \( H_i \) contributes to the matrix element a factor of the highest order \( k^a \delta^a + \delta^a - 2 \). In the same way, we can see that the contribution from \( \partial^a_{\beta\mu} V^a \) is given by a factor of the order \( k^a \delta^a + \delta^a - 2 \).

For the convenience of the following discussion, we shall now introduce several quantities which characterize the interaction term \( H_i \):

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** The improved and more general formulation of this quantization method will be developed in the forthcoming paper (Y. Takahashi and H. Umezawa, Prog. Theor. Phys. in press).

** In (2.8) and (2.9) the surface dependent terms coming from the differentiations of \( \epsilon(x) \) have been neglected. Including these terms, however, does not make higher the degree of momentum and so the following arguments suffer no alteration. Usually, the surface dependent terms are of no direct physical meaning.
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\[ A_i = \sum_a (a_n^a + \cdots + a_k^a) + \sum_a (a_n^a + \cdots + a_k^a) + \sum_a (a_n^a + \cdots + a_k^a), \]  
\[ (2.10) \]

which is the total sum of the degrees of the derivation operators in \( H_n \)

\[ B_i = \sum_a b_o^{\alpha} + \sum_a b_o^{\mu} + \sum b_o^{\nu}, \]
\[ (2.11) \]

which is the total sum of \( b \)'s of field quantities appeared in \( H_n \)

\[ C_i = \sum_a (\lambda^{\alpha} + \mu^{\mu}) + \sum b, \]
\[ (2.12) \]

which is the number of the field quantities in \( H_i \) and finally

\[ K_i = 4 - A_i - B_i/2 - C_i. \]
\[ (2.13) \]

iii) In the following discussion, we shall denote, for the sake of simplicity, charged as well as neutral fields by the superscripts \( \alpha \)'s.

Now, let us consider a Feynman's graph having \( E \) external lines, \( I \) internal lines and \( n \) vertices. Among these we assume that \( E^a \) lines refer to the field \( \alpha \) and \( n_i \) vertices belong to the interaction term \( H_i \). Then we have

\[ \sum_a E^a = E, \]
\[ \sum_i n_i = n. \]
\[ (2.14) \]

Taking account of the fact that \( C_i \) lines start from the vertex \( H_i \) and one internal line runs between two vertices (or returns to its starting vertex), we find immediately

\[ 2I + \sum a E^a = \sum_i n_i C_i. \]
\[ (2.15) \]

By performing the integrations over internal momenta (i.e., contracting the appropriate operators), this graph could, in general, be brought into a form of a vertex, which resembles to a primary interaction giving rise to the corresponding transition. Since the free operators \( U^a \)'s (external lines) carry the derivation operators of the degree \( a^a \), the Feynman's graph for the interaction of such a general type should be specified as \( G(E^a, N^a) \), where \( N^a = \sum a^a \). Furthermore, as a result of the integration over internal momenta, \( G(E^a; N^a) \) splits into several independent graphs, denoted by \( G(E^a, N^a, M^a) \), which have further derivation operators on \( U^a \)'s with the total number \( M = 0, 1, 2, \ldots \). For each of these \( G(E^a, N^a, M^a) \) we may now introduce the characteristic constants in accordance with the definitions (2.10) to (2.13);

\[ A_h = \sum_a (N^a + M^a), \]
\[ B_h = \sum_a b^a E^a, \]
\[ (2.16) \]

* Such a decomposition of \( G(E^a, N^a) \) into \( G(E^a, N^a, M^a) \) is obtained when the use is made of the expansion of the integrand

\[ R(p, p^I) = R(p^0, p^I) + (p - p^0) \eta_{\alpha}\beta^\mu \Theta_{\mu\nu}^ R (p^0, p^I) + \frac{1}{2} (p - p^0) \eta_{\mu\nu}^R (p^0, p^I) + \cdots, \]

where \( p \) and \( p^I \) are the external and the internal momenta, respectively and \( p^0 \) is a constant free particle momentum.
\[ C_k = \sum_a e^a_a, \]
\[ K_k = 4 - A_k - B_k/2 - C_k. \]

As a first step to the study of renormalizability, let us consider the connected graphs with primitive divergence. It was shown in (ii) that \( \partial^a \hat{U}_i^a \) in \( H_i \) contributes to the matrix element in the momentum space a factor \( \hat{k}^{a+\hat{a}/2-1} \) for the large value of the momentum \( k \). Therefore, \( H_i \) will contribute the following power of momentum in the high energy region:

\[ A_i + \sum_a (\hat{a}^a/2-1) \lambda^a + \sum_a (\hat{a}^a/2-1) \mu^a + \sum_a (\hat{a}^a/2-1) \nu^a = A_i + 1/2 \cdot B_i - C_i. \tag{2.17} \]

As \( G(E^a, N^a, M^a) \) has \( n_i \) vertices which belong to \( H_i \), the highest power of the internal momentum in the matrix element is found to be

\[ N = \sum_i n_i (A_i + 1/2 \cdot B_i - C_i) - (A_k + 1/2 \cdot B_k - C_k). \tag{2.18} \]

Further contribution to the integrand of the S-matrix comes from the volume element of the momentum space. Among the \( 4I \) internal momenta \( 4(\sum_i n_i - 1) \) variables are eliminated by the conservation law of the energy-momentum, so that the contribution becomes in terms of the power of the internal momentum:

\[ N' = 4(I - \sum_i n_i + 1). \tag{2.19} \]

Thus the graph \( G(E^a, N^a, M^a) \) under consideration is divergent if

\[ N + N' \geq 0, \tag{2.20} \]

where it is to be understood that \( N + N' = 0, 1, 2, \ldots \) means logarithmic, linear, quadratic, \( \ldots \)-divergences, respectively. By using (2.15), (2.16), (2.18) and (2.19) we find

\[ N + N' = K_k - \sum_i n_i K_i. \tag{2.21} \]

So that the condition (2.20) becomes

\[ K_k \geq \sum_i n_i K_i. \tag{2.22} \]

It should be noted that the left hand side of this inequality is determined only by the properties of external lines.

From (2.22), we find the following remarkable facts:

a) If the condition

\[ K_i \geq 0 \quad \text{(for all } I) \quad \text{I} \]

is satisfied, the number of the primitive divergent graphs \( G(E^a, N^a, M^a) \) remains always

\* In general, \( I > \sum_i n_i - 1 \). We shall restrict ourselves, in the following, to the graphs for which \( I > \sum_i n_i - 1 \). The graphs for which \( I = \sum_i n_i - 1 \) always converge, since available \( \delta \)-functions eliminate all the internal momenta. The graphs consisting from 2-vertices only are examples of the latter case.
to be finite.* In fact, the possible values of $E^a, N^a, M^a$ for diverging graph are restricted by the relation

$$4 \geq \sum a (E^a + N^a + M^a),$$

and the order of divergence decreases according as the values of $E^a, N^a$ and $M^a$ increase. Furthermore, no matter how many interaction terms we may introduce, the number of primitive divergent graphs will not exceed a definite number $j$ so far as (I) is satisfied. The number $j$ depends only on the number of the fields participating in the interaction under consideration.

b) If there is at least one $H_i$ with $K_i < 0$, any $G(E^a, N^a, M^a)$ will always diverge for a sufficiently large value of $n_i$ (i.e., in the higher order approximation of the perturbation calculation). Moreover, there arises infinitely high degree of divergence in the graphs $G(E^a, N^a, M^a)$.* Accordingly, we have infinite number of primitive divergences.

iv) Now, we shall consider the question whether the divergences arising from diverging $G(E^a, N^a, M^a)$ can be removed by introducing the appropriate counter terms. The following two cases will be treated separately:

a) the case in which the condition (I) is satisfied;
b) the case in which the condition (I) is not satisfied at least by one of the interaction terms.

First, let us consider the case (a). In order to cancel the divergences, we must introduce new primary interaction terms $\sum H_{ii}$ as the counter terms which have the same forms as those of the terms obtained by contracting primitively diverging $G(E^a, N^a, M^a)$ terms and have coupling constants with infinite values. As the characteristic constants $K_{ii}$'s of the counter terms $H_{ii}$'s are respectively identical with $K_{ii}$'s of primitively diverging graphs $G(E^a, N^a, M^a)$, it is evident that $K_{ii}$'s also satisfy condition (I):

$$K_{ii} = K_{2} \geq \sum K_{ii} \geq 0.$$  

(2.24)

Therefore, according to the remarks given in the previous section, there appear no new primitive divergences even if we introduce the counter terms.*** Thus, as shown below, it is possible to renormalize all the divergences by suitably choosing $j$ coupling constants.

Here, we shall only sketch the outline of the renormalization procedure. In general, corresponding to a diverging graph there exists a diverging integral over the internal momenta $p_1, p_2, p_3, \ldots$, which, when employing the general method proposed by Salam,** is further subdivided into a set of "subintegrations" $p_1, p_2, p_3, \ldots; p_1 p_2, p_1 p_3, \ldots; p_1 p_2 p_3, \ldots$.

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* In his own theory of S-matrix, which starts with the consideration of unitarity and causality, Stueckelberg has also found the corresponding condition in quantum electrodynamics; A. Petermann and E. C. C. Stueckelberg, Phys. Rev. 82 (1951), 548; Helv. Phys. Acta 24 (1951), 317.

** The highest degree of divergence in the graph $G(E^a, N^a, M^a)$ is given by (2.21) unless there appear particular circumstances.

*** At first sight it seems unnecessary to examine the condition (2.24) for counter terms, since it may be expected that they play only the role of compensation of divergences and do not appear by themselves. But, this is incorrect. In general, there frequently occur the cases in which counter terms appear separately and directly contribute to the further divergences (for example, 'final Møller part' in the self-energy graph of meson).
As is evident from the above consideration, each diverging term arising from the subintegrations \( p_{11}, p_{12}, p_{13}, \ldots \) is always of the same operator form as any of the counter terms \( H_i \)'s. Hence, these divergences are all cancelled by the counter terms. Next, we shall consider the subintegrations \( p_i, p_{i1}, p_{i2}, p_{i3}, \ldots \). As the degree of divergence for a particular subintegration is, in this case, unaffected by subtractions corresponding to other subintegration and so there appear no new type of divergences after this manipulation, we see that divergences coming from the subintegrations \( p_i, p_{i1}, p_{i2}, p_{i3}, \ldots \) are again cancelled by the corresponding counter terms. Divergences due to the subintegrations \( p_1, p_2, p_3, \ldots \) are eliminated in a similar way, and so on. Proceeding successively in this way and finally removing the divergences of the final integration \( p_1, p_2, p_3, \ldots \), we can obtain the unique and finite result, being independent of the choice of the "basic variables" \( p_{11}, p_{12}, p_{13}, \ldots \).

The values of \( j \) coupling constants of the counter terms are determined as follows. Let \( f_{\nu} \) be the coupling constant of the counter term \( H_{\nu} \) corresponding to the diverging \( G(E^a, N^a, M^a) \) graphs \( (N^a + M^a : \text{fixed}) \). \( f_{\nu} \) is as usual expanded in the power of the coupling constants of \( H_{\nu} \)'s: \( f_{\nu} = \sum_{n} f_{\nu}^{(n)} \). The determination of \( f_{\nu} \) is thus reduced to those of \( f_{\nu}^{(n)} \)'s, which are carried through step by step from the lowest order in the following way: \( f_{\nu}^{(1)} \) is put equal to the negative of the sum of the divergent factors of lowest order \( G(E^a, N^a, M^a) \) \( (N^a + M^a : \text{fixed}) \) and in general \( f_{\nu}^{(n)} \) is put equal to the negative of the sum of the divergent factors of all diverging \( G(E^a, N^a, M^a) \) graphs \( (N^a + M^a : \text{fixed}) \) of the order \( (n) \), from which all the internal divergences (coming from all the subintegrations except the final one) have been already subtracted by suitably choosing the lower order part of the coupling constants \( f_{\nu}^{(m)} (m < n) \), in other words \( f_{\nu}^{(n)} \) is equal to the negative of the sum of the diverging factors coming from the final integration of \( G(E^a, N^a, M^a) \) graph.

In this way, by the consistent use of the counter terms** we can obtain the unique and absolutely convergent matrix elements for all scattering processes.

Next, we shall consider the case (b). In this case, as was mentioned in the previous section, any \( G(E^a, N^a, M^a) \) diverges if we take a sufficiently large value for \( n_i \). In order to remove these divergences in the same manner as in the case (a), it is necessary to introduce infinitely many counter terms with higher derivatives. Among these counter terms there are certainly terms with negative \( K'_i \), which give rise to further divergences, so that we have to introduce successively new counter terms one after another.

Such a serious situation would be avoided, if the infinitely many divergences could be eliminated simultaneously by introducing a finite number of counter terms. But, the following considerations exclude this possibility, too. In general, to cancel a diverging term the coupling constants of the counter terms must satisfy an algebraic equation. As we proceed to the higher order, new diverging terms, which are mutually independent, appear one after another, so that the number of equations to be satisfied increases unlimitedly.

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* In general \((n)\) implies, as it were, the degree of complexity of the construction of the graph.

** The subtraction method with the use of counter terms is easily shown to be equivalent to the conventional renormalization of mass and charge. This problem has been also considered by G. Takeda in the private communication and will be discussed in general in our forthcoming paper, Prog. Theor. Phys. in press.
It is obvious that such infinitely many simultaneous equations for finite number of variables have no solution except for trivial ones. Therefore it is impossible to remove all the divergences by introducing only a finite number of local interactions.

Summarizing the results obtained above, we may conclude as follows: When the interaction satisfies the condition (I), all the divergences may be eliminated by introducing a finite number of local interactions, that is, the renormalization theory succeeds to construct a consistent closed theory. On the contrary, if there is at least one interaction term for which this condition is not satisfied, the renormalization theory will break down completely.

§ 3. The "Universal Length" as the limit of the renormalization theory

Hitherto, the limit of validity of the quantum theory of fields was discussed by many authors. It has been generally believed that the main defects of the present theory came from the ignorance of the finite size of the elementary particles, and that the future theory would be established only by taking into account the existence of the so called "universal length" consistently. As far as we deal with free fields, it may be permitted to ignore the finite size of elementary particles, because the universal length finds expression solely in the form of their masses. But, as soon as the interactions are introduced, we shall be confronted in general with the problem of the structure of the elementary particles.

In order to discuss this problem in detail, Heisenberg had classified, for some years ago, the interactions into two groups:

i) the interaction of the first kind, which contains a dimensionless coupling constant $Z$,

ii) the interaction of the second kind, which contains a coupling constant of dimension (length)$^\gamma$ with $\gamma > 0$.

If $Z \ll 1$, the interaction of the first kind may always be regarded as a small perturbation, and, moreover, as it does not contain the universal length, we may certainly ignore the structure of the elementary particles. On the contrary, the interaction of the second kind can not be considered as a small perturbation at least in the high energy region, as can easily be shown by the dimensional consideration, the ratio of the $(n+1)$-th order approximation to the $n$-th order one in the perturbation calculation involves a factor $(l/\lambda)^n$ for small $\lambda$, where $l$ means a characteristic length contained in the coupling constant and $\lambda$ denotes the wave length. In such a case, the reaction of the self field becomes very effective and the structure of the elementary particles could not be ignored.

Nevertheless, these arguments had not been fully justified, before the renormalization theory was developed. Though the interaction of the electron with the electromagnetic fields belongs to the first kind, it leads to the well known divergence difficulties, as long as we adopt the previous formulation. But, the situation was entirely changed now. In fact, it has become clear recently, as was ingeniously shown by Dyson, that a non-singular quantum electrodynamics could be established without touching on the problem of the structure of the elementary particles.
Now, it seems to be very interesting to compare Heisenberg’s classification of the interactions \(l\) with that of ours performed in § 2. For this purpose, we shall first determine the dimension of the coupling constant \(f_i\) appeared in the general type of the interaction \(H_i\). As we have adopted the natural unit, \(f_i\) has a dimension of (length) \(^{-\sigma}\). If we assume that the normalization of the field quantities \(U^\alpha(x)\) is so performed that \(I_{ij}^\alpha\) in the right hand side of the commutation relations (2·2) becomes in the following form:

\[
I_{ij}^\alpha = \alpha_{ij}^{(0)} \delta^{\alpha} + \alpha_{ij}^{(1)} x^\alpha \delta^{\alpha - 1} + \cdots + \alpha_{ij}^{(b)} (x^\alpha)^{b}
\]

with \(\alpha_{ij}^{(0)}, \alpha_{ij}^{(1)}, \cdots\) being dimensionless constants, then

\[
\partial^{\alpha} U^\alpha \sim \lambda^{-[\sigma + \delta + 1]} \quad \text{for small } \lambda, \quad (3·2)
\]

where \(\lambda\) denotes a dimensionless constant. As \(H_i\) has a dimension of (length) \(^{-4}\), we can easily obtain from (2·7), (3·2)

\[
\eta_i = A_i + B_i/2 + C_i - 4 = -K_i. \quad (3·3)
\]

Observing this result, we see immediately that the classification made in § 2 coincides exactly with that of Heisenberg,\(^7\) if the interactions with \(\eta_i < 0\), which were not considered by him, are also classified into those of the first kind. By using (3·4), we may write (I) in the following form:

\[
\eta_i \leq 0 \quad \text{(for all } i), \quad (\text{II})
\]

which gives a physical basis for the validity of the renormalization theory.\(^*\)

In fact, we may easily understand from (II) the reason why the renormalization theory succeeds for the interactions of the first kind. It is certainly due to the fact that the structure of the elementary particles does not play an essential role in these cases. In order to get a closed theory, the coexistence of finite number of interactions belonging to the first kind is often required. As an example of such a case, we may mention the interaction between the nucleon and the scalar neutral meson \(V^\alpha(x)\) via scalar coupling which requires to introduce another interaction having the form \(f \int \xi^3 d\sigma (\eta_i = -1)\).\(^4\) It should, however, be noted, as was shown in § 2, that the introduction of the interaction of the second kind had never been required for the closure of the theory which involves merely the interaction of the first kind. This point is in striking contrast to the conclusion obtained by Heisenberg,\(^7\) who emphasized the inevitable appearance of the interaction of the second kind in company with that of the first kind, provided the field has a non-vanishing mass.\(^**\)

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\(^*\) It should be noted that before \(\eta_i\) is determined all the field quantities in \(H_i\) are to be so normalized that (3·1) is valid.

\(^**\) Heisenberg has pointed out that in the quantum electrodynamics there appears, for example, the term of the scattering of photon by photon, which belongs to the interaction of the second kind and is closely related to the applicability of the theory. However, this term is in itself of the form of non-local interaction and apparently contains interactions of the second kind as a part. In the renormalization theory it lies certainly within the limit of the applicability, because it is not required to introduce the primary interaction belonging to the second kind.
On the other hand, if there is an interaction of the second kind, it will be required that there exist simultaneously infinite number of interactions which belong also to those of the second kind. Such an assembly of interactions is equivalent to a non-local interaction corresponding to an extended model of the elementary particles. So that, we should then take into account the structure of the elementary particles from the beginning.

Thus, one of the most important problems to be solved in the future would be the question whether interactions of the second kind really exist in the nature or not. If it becomes clear that they do not exist at all, a non-singular and closed quantum field theory will be established by means of the renormalizations of masses and coupling constants, and the well known divergence difficulties will be hidden behind these procedures. On the contrary, if the existence of the interaction of the second kind is found, we can not remain in the framework of the renormalization theory. In order to answer to this question, we shall investigate, in the following paper, the types of interactions which may be realized for various sorts of elementary particles.

Appendix

A remark on the general type of interactions

In § 2, we discussed the renormalizability of the interactions by taking out each element $H_i$ of the most general type of interactions separately. However, the actual interactions are the linear combinations of these elements and may be written in the form which is obtained by inserting $D^a_i(\partial_\mu)_{\mu,\nu,\cdots}$, in place of $\partial_\mu^a U^a_i$, where $D^a_i(\partial_\mu)$'s denote rational functions of $\partial_\mu$ as well as $x^a$.

The renormalizability condition for such general interactions is also given by (I), provided the definition of $a_i^a$ is suitably changed. Usually $a_i^a$ is equal to the highest degree of $\partial_\mu$ involved in $D^a_i(\partial_\mu)$, but there are some exceptional cases in which this rule does not hold. In such cases, it is convenient to define $a_i^a$ as one half of the difference between the highest degrees of derivation operators operated on the $\Delta$-functions in $D^a_i(\partial_\mu) \times D^a_i(\partial_\mu') \Delta_\mu$ and of $\Delta_\mu'$. We shall call this $a_i^a$ the true degree of $D^a_i(\partial_\mu) U^a$.

As an example of these exceptional cases, we shall consider the following operator operating on the field $U_{\mu,\nu,\cdots}$ with integral spin and non-vanishing mass:

$$D(\partial_\mu) = \partial_{\mu;\nu} = \partial_{\mu;\nu} - \partial_\nu \partial_\mu,$$

which is the four dimensional rotation and satisfies the relation

$$\partial_{\mu;\nu} \partial_{\mu'} = \partial_\mu \partial_{\nu'} - \partial_{\nu'} \partial_\mu = 0.$$

Though the highest degree of this operator is 1, the true degree is found to be zero, because $a_i^a$ corresponding to this field contains a factor

$$R(\mu', \nu') = \left( \partial_{\mu';\nu'} - \frac{1}{x^a} \partial_{\mu'} \partial_{\nu'} \right),$$

so that $D^a_i(\partial_\mu) D^a_i(\partial_\mu)$ does not raise effectively the degree of the derivative of the delta
function.*

These situations are, in general, due to the following facts: For a given tensor field 
\( U_{\mu_1 \mu_2 \cdots \mu_n} (x) \) with rank \( n \) and non-vanishing mass, we can introduce the following \( n \) quantities 
\( U^{(1)}_{[\mu_1 \nu_1], \nu_2} \mu_3 \cdots \mu_n, U^{(2)}_{[\mu_1 \nu_1] [\nu_2 \nu_3], \nu_3} \mu_4 \cdots \mu_n, \ldots, U^{(n)}_{[\mu_1 \nu_1] \cdots [\mu_n \nu_n], \nu_n} \) defined by

\[
U^{(1)}_{[\mu_1 \nu_1] \mu_2 \cdots \mu_n} = \frac{\partial}{\partial x_{\mu_1}} U_{\mu_2 \cdots \mu_n}, \quad U^{(2)}_{[\mu_1 \nu_1] [\nu_2 \nu_3], \mu_4 \cdots \mu_n} = \frac{\partial}{\partial x_{\nu_3}} U_{\mu_1 \nu_2 \cdots \mu_n} \cdots \frac{\partial}{\partial x_{\mu_2}} U_{\mu_1 \nu_2 \cdots \mu_n} \cdots \frac{\partial}{\partial x_{\mu_1}} U_{\mu_1 \nu_2 \cdots \mu_n}.
\]

As these quantities \( U^{(n)} \)'s satisfy the same wave equations and subsidiary conditions as those of the original field \( U \), the degree of the highest derivatives in their commutation relations are equal to those of \( U \). We obtain, therefore, that \( a^n = 0 \) for \( \partial_{\mu_1 \cdots \mu_n} \).

In terms of the spinor notation, including the fields of both integer as well as half integer spin, these circumstances are also stated as follows: Let \( \varphi^{(a)}_{\alpha_1 \cdots \alpha_n} (x) \) and \( \chi^{(a)}_{\alpha_1 \cdots \alpha_n} (x) \) be the field quantities with non-vanishing mass. They satisfy the wave equations:

\[
\partial_{\alpha_1} \varphi^{(a)}_{\alpha_1} \cdots = i \chi^{(a)}_{\alpha_1},
\]

where \( \partial_{\alpha_n} \) is the derivation operator in the spinor notation. From \( \varphi^{(a)} \) and \( \chi^{(a)} \), the field quantities \( \varphi^{(a)} \) and \( \chi^{(a)} \) are introduced in the following way,

\[
\varphi^{(1)}_{\alpha_1 \cdots \alpha_n} = \frac{i}{x} \partial_{\alpha_1} \varphi^{(a)}_{\alpha_1 \cdots \alpha_n},
\]

\[
\varphi^{(a)}_{\alpha_1 \cdots \alpha_n} = \frac{i}{x} \partial_{\alpha_1} \varphi^{(a-1)}_{\alpha_1 \cdots \alpha_n \alpha_{n+1}},
\]

\[
\chi^{(a)}_{\alpha_1 \cdots \alpha_n} = \frac{i}{x} \partial_{\alpha_1} \chi^{(a)}_{\alpha_1 \cdots \alpha_n}.
\]

Then, \( \varphi^{(a)} \) and \( \chi^{(a)} \) describes the fields satisfying the same wave equations as those of \( \varphi^{(a)} \) and \( \chi^{(a)} \) and the degree of the highest derivatives in the commutation relations of \( \varphi^{(a)} \) and \( \chi^{(a)} \) are the same as those of \( \varphi^{(a)} \) and \( \chi^{(a)} \). Therefore, the true degree of the operator \( \partial^{a} \varphi^{(a)} \) is found to be zero.

In conclusion, it should be noted that the above-mentioned arguments concerning the true degree give an example which shows that the special structure of the interactions could weaken the divergence of the interaction.

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* This situation illustrates the well known fact that in the case of vector field the degree of highest power of the momentum in the Fourier amplitude of \( F_{\mu_\nu} (\equiv \partial_\mu U_\nu - \partial_\nu U_\mu) \) is same as that of \( U_\mu \).
Note added in proof:

The characteristic difference between interactions of the first kind and those of the second kind may also be seen in the energy dependency of the cross section. In the former case the power of external energies in the leading term at the high energy region is left unchanged whatever order of approximation we may take into account. In the latter case, however, the degree of external energies will be increased unlimitedly when the higher order effects are considered. These circumstances are easily understood either from the dimensional consideration, or for the former case, from the procedure of the renormalization. Hence, we may say as follows: The damping effects of the field reaction do not give any appreciable effect in the theory having interaction of the 1st kind only as long as the coupling constants are very small, while they lead to drastic effects in the theory containing interactions of the 2nd kind even if the coupling constants are very small. The conclusion for the former theory, though already well known by the results of the lowest order approximation, is guaranteed for the first time after the inherent divergences have been removed by the renormalization method. In this connection, it should also be interesting to notice a possibility that the strong singularity in the latter theory may be altogether compensated by the correspondingly strong damping effects in a way recently suggested by Feynman, Hu (Phys. Rev. 80 (1950), 1109) and Thirring (Z. f. Naturf. 6a (1951), 462).
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