On Yukawa's Theory of the Beta-Disintegration.


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In conclusion, I wish to express my cordial thanks to Professor H. Yukawa for his kind interest in this work.

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Introduction.

In a recent paper by M. Taketani and the present author(1), a matrix formulation of the vector meson theory was described. Though this formalism is inadequate for the discussion of the general character (e.g., Lorentz invariance, etc.) of the theory, it proves to be very useful for the practical calculations. Especially, the spin summation of the meson can then be carried out automatically by using the orthogonality relations between the amplitudes of the meson wave function. In A, the theory was developed for the case when the electromagnetic interaction alone is involved. In the first part of the present paper, an extension to the general case when the mesons interact with the

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heavy and light particles is given. It is further shown that the
present formalism can be easily extended so as to include all the cases
of the mesons with spin 0 or 1 considered by Kemmer.

In the subsequent parts of this paper, a new derivation of the
beta-disintegration formula and the meson lifetime, resulting from each
of the four types of the meson theories, is given by using the above
formalism. The well-known discrepancy between the calculated and
experimental values of the meson lifetime is also discussed.

I. General Theory.

1. Vector Theory. In the vector theory of the meson developed
by Yukawa and others, the total Hamiltonian for the system consist-
ing of mesons, heavy particles and light particles was given by

\[ H = H_v + H_h + H_e + H'_e + H'_v + H'_h, \] (1)

with

\[ H_v = \int \frac{1}{4\pi} \hat{\nabla} \cdot \hat{F} \, dv + \frac{1}{\kappa} \hat{\nabla} \cdot \hat{F} \, dv, \] (2)

\[ H_h = \int \hat{\nabla} \cdot \hat{F} \, dv - \frac{i}{\hbar c} \rho_v \hat{\sigma} \cdot \hat{\nabla} \frac{\partial}{\partial \beta} \psi \, dv, \] (3)

\[ H_e = \int \hat{\nabla} \cdot \hat{F} \, dv - \frac{i}{\hbar c} \rho_e \hat{\sigma} \cdot \hat{\nabla} \frac{\partial}{\partial \beta} \psi \, dv, \] (4)

(2) An equivalent formulation based on Kemmer's theory (Proc. Roy. Soc. A 173 (1939), 91) has been given by A.H. Wilson (Proc. Camb. Phil. Soc. XXXVI (1940), 363), but this is not so convenient as ours owing to the singular character of the \( \beta \)-matrices.


(5) H. Yukawa, S. Sakata, M. Kobayasi and M. Taketani, Proc. Phys.-Math. Soc. Japan, 20 (1938), 720, which will be referred as IV. In the following, we shall use the same notations as in IV.
On Yukawa's Theory of the Beta-Disintegration.

\[ H_{\alpha} = \frac{4\pi}{\kappa} \int \left[ g_1 \bar{M}_\alpha M_\alpha + \bar{M}_\alpha M_\alpha + g_2 (\bar{S}S') + (\bar{S}'S) \right] dv, \quad (8) \]

\[ H_{\beta} = \frac{4\pi}{\kappa} \int \left[ g_1' \bar{M}_\beta M_\beta + g_2' \bar{S}'S' \right] dv, \quad (9) \]

Among these expressions, (2), (3) and (4) are the energies of the mesons, heavy particles and light particles in the absence of external fields, respectively, while the rest represents the interaction energies between these particles. \( \vec{F}, \vec{F}', \vec{F} \) and \( \vec{F}' \) are the field variables of the meson field satisfying commutation relations

\[ [\vec{F}_i(r', t), U_j(r, t)] = i4\pi\hbar c \delta^{ij} \delta(r' - r) \delta_{ij}, \quad (10)^\ast \]

\[ [\vec{F}_i(r', t), \tilde{U}_j(r, t)] = i4\pi\hbar c \delta^{ij} \delta(r' - r) \delta_{ij}, \quad (10)^\ast \]

and \( \Phi, \Phi', \Phi \) and \( \phi \) denote Dirac's wave functions for the proton, neutron, electron and neutrino, respectively. Further, the four vectors, \( (\vec{M}, M_0) \) and \( (\vec{M}', M'_0) \), and the six vectors, \( (\vec{S}, \vec{T}) \) and \( (\vec{S}', \vec{T}') \), are defined by the following equations:

\[ \vec{M} = \bar{\Phi} \Phi \sigma^a \psi, \quad \vec{M}_0 = -\bar{\Phi} \Phi \cdot \vec{r}, \quad (11) \]

\[ \vec{S} = \bar{\Phi} \Phi \sigma^a \sigma^b \psi, \quad \vec{S}' = -\bar{\Phi}' \Phi' \sigma^a \sigma^b \psi, \quad (12)^\ast \]

where \( \bar{\psi}, \psi, \bar{\psi}', \psi' \) and \( \sigma \) are the usual Dirac operators and the superscript \( (h) \) or \( (l) \) refers to the heavy particle or the light particle. The constants \( (g_1, g_2) \) and \( (g_1', g_2') \), which have each the dimension of an electric charge, determine the strength of the interactions of the mesons with the heavy and light particles, respectively.

Finally, \( m = -\hbar c, M_0, M, n \) and \( \mu \) denote the masses of the meson, proton, neutron, electron and neutrino, respectively.

Recently Taketani and the author have developed a matrix formulation of the vector meson theory for the case when the mesons interact only with an electromagnetic field. Introducing a six-component wave function

\[ \chi = \frac{1}{\sqrt{4\pi\hbar c}} \left( \begin{array}{c} \bar{\Phi} \\ \Phi \\ \bar{\Phi}' \\ \Phi' \\ \vec{r} \\ \vec{r}' \end{array} \right), \quad (13)^\ast \]

\[ [*] \quad [A, B] = AB - BA. \]

\[ [**] \quad A more general assumption involving the derivatives of the wave function for the light particle, which was made in IV in order to obtain the energy distribution of the beta-ray of Konopinski-Uhlenbeck type, leads to a very much too small value for the meson lifetime and is therefore, not considered here. \]

\[ (***) \quad \text{In order to avoid confusion, we write } \chi \text{ instead of } \Psi \text{ in } \Lambda. \text{ All other notations being the same as in } \lambda. \]
and six-rowed matrices

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 & 0 & -i \\
i & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\rho_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix},
\rho_2 = \begin{pmatrix}
0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\sigma_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -i \\
0 & 0 & 0 & 0 & i & 0 \\
0 & 0 & 0 & -i & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 \\
0 & i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\sigma_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

they rewrote (2) and (10) in the following forms:

\[
\tilde{H}_u = \int \chi \rho_2 \mathcal{H} \chi dv,
\]

\[
[\tilde{\chi}(\tilde{r}, t) \rho_2 \chi(\tilde{r}', t)] = -\delta_{\tilde{r} \tilde{r}'} \delta(\tilde{r} - \tilde{r}'),
\]

where the operator \( \mathcal{H} \) is defined by

\[
\rho_2 \mathcal{H} = \hbar c \left[ 1 - \frac{\text{grad div}}{2\varepsilon^2} + p_3 \left\{ \frac{\text{grad} \tilde{r}}{\varepsilon} - \frac{\text{grad div}}{2\varepsilon^3} \right\} \right],
\]

with

\[
\sigma = (\sigma_1, \sigma_2, \sigma_3).
\]

Further, they also showed that the wave function \( \chi \) satisfies in the absence of the external fields the equation

\[
i \hbar \frac{\partial \chi}{\partial t} - \mathcal{H} \chi = 0.
\]

(*) No confusion between these matrices and the Dirac operators will arise if it is kept in mind that the latter have superscripts.
This formulation can be easily extended to the general case when the interactions with the heavy and light particles are involved. To write the interaction energies (5) and (6) in terms of $\chi$, we introduce row-matrices $\tilde{\xi}$ and $\tilde{\xi}'$, which are defined by

$$\tilde{\xi} = \left( -\frac{g_1}{\kappa} \tilde{M}_0 \nabla_g - g_1 \tilde{T} - \frac{g_2}{\kappa} \tilde{S} \cdot \nabla \right), \quad (19)$$

$$\tilde{\xi}' = \left( -\frac{g_1'}{\kappa} \tilde{M}'_0 \nabla_g' - g_1' \tilde{T}' - \frac{g_2'}{\kappa} \tilde{S}' \cdot \nabla \right). \quad (20)$$

Using these notations, we obtain from (5) and (6) the expression

$$H' g = -\frac{4\pi \hbar c}{\kappa} \int \tilde{\xi} \chi \, dr + \text{comp. conj.}, \quad (21)$$

$$H' g' = -\frac{4\pi \hbar c}{\kappa} \int \tilde{\xi}' \chi \, dr + \text{comp. conj.}. \quad (22)$$

Now, we go to the momentum space of the mesons. By using the plane wave solutions of (18), we expand $\chi$ and $\tilde{\chi}$ as follows ((22) in $\Lambda$)

$$\chi = \sum_{\tilde{\lambda}, \lambda} \tilde{u}(\tilde{p}, \lambda, \varepsilon) u(\tilde{p}, \lambda, \varepsilon) \frac{e^{i \frac{\lambda}{\hbar}}}{\lambda}, \quad (23)$$

$$\tilde{\chi} = \sum_{\tilde{\lambda}, \lambda} \tilde{u}^*(\tilde{p}, \lambda, \varepsilon) \tilde{u}(\tilde{p}, \lambda, \varepsilon) \frac{e^{-i \frac{\lambda}{\hbar}}}{\lambda},$$

where $u(\tilde{p}, \lambda, \varepsilon)$, representing a column matrix with c-number components, is the solution of the equation

$$E_{\lambda, \varepsilon} = \left[ \frac{\tilde{p}^2}{2m_e c^2} + m_e c^2 + p_z^2 \right] \frac{1}{2m_e c^2} - \frac{(\sigma \tilde{p})^2}{m_e c^2} \right] \mu. \quad (24)$$

and where the constants $E$ and $\tilde{p}$ satisfy the relation

$$E = \pm 1 \p^2 + m_e c^2. \quad (25)$$

For a given momentum $\tilde{p}/c$, there are six independent solutions of (25), which we have distinguished by $\lambda$ and $\varepsilon$, $\lambda = 1, 0, -1$ and $\varepsilon = \frac{E}{|E|} = 1$, or $-1$ correspond to three polarization states and two charge states of the meson, respectively. The $u$'s have to be normalized so that (18) in $\Lambda$

$$\tilde{u}(\tilde{p}, \lambda, \varepsilon) \rho m(\tilde{p}, \lambda', \varepsilon') = \varepsilon \delta_{\lambda, \lambda'} \delta_{\varepsilon, \varepsilon'}. \quad (26)$$

from which we obtain the converse orthogonality relations

$$\sum_{\lambda} \sum_{\varepsilon} \tilde{u}(\tilde{p}, \lambda, \varepsilon) \rho m(\tilde{p}, \lambda, \varepsilon) = \delta_{\mu, \mu'}, \quad \sum_{\lambda} \sum_{\varepsilon} \tilde{u}(\tilde{p}, \lambda, \varepsilon) \rho m(\tilde{p}, \lambda, \varepsilon) = \delta_{\mu, \mu'}. \quad (27)$$

The explicit form of $u$ may be written as
where \( \vec{v}(p, 0) = \vec{v}(p, 1) \) and \( \vec{v}(p, -1) \) are unit vectors perpendicular to each other.

The \( a \)'s and \( a^\ast \)'s are \( q \)-numbers satisfying the commutation relations (23) in \( \Lambda \)

\[
[a(p, \lambda, \varepsilon), a^\ast(p', \lambda', \varepsilon')] = \varepsilon \delta_{\lambda\lambda'} \delta_{\varepsilon\varepsilon'}. \tag{29}
\]

Further, \( N^+(p, \lambda) = a^\ast(p, \lambda, 1) \) or \( N^-(p, \lambda) = a(p, \lambda, -1) \) can be interpreted as the number of the mesons with the charge \( e \) or \(-e\) in the state of energy \(|E|\), momentum \( \vec{p}/c \) or \(-\vec{p}/c \) and polarization \( \lambda \). Hence, \( a^\ast(p, \lambda, 1) \) and \( a(p, \lambda, 1) \) denote the operators which increase the number \( N^+(p, \lambda) \) and \( N^-(p, \lambda) \) by one respectively, whereas \( a(p, \lambda, 1) \) and \( a^\ast(p, \lambda, -1) \) denote those which decrease them by one respectively.

Inserting (23) in (15), (21) and (22), we obtain the expressions

\[
H_v = \sum_{\lambda, \varepsilon} \int \left| E \right| N^+(p, \lambda) + N^-(p, \lambda) + 1 \, \frac{4 \pi \hbar v}{\kappa} \sum_{\lambda, \varepsilon} a^\ast(p, \lambda, \varepsilon) \xi_v(p, \lambda, \varepsilon) e^{-i \mathcal{M} p \nu} dv + \text{comp. conj.}, \tag{30}
\]

\[
H'_v = \sum_{\lambda, \varepsilon} \int \left| E \right| N^+(p, \lambda) + N^-(p, \lambda) + 1 \, \frac{4 \pi \hbar v}{\kappa} \sum_{\lambda, \varepsilon} a(p, \lambda, \varepsilon) \xi'_v(p, \lambda, \varepsilon) e^{-i \mathcal{M} p \nu} dv + \text{comp. conj.}, \tag{31}
\]

with

\[
\xi_v(p, \lambda, \varepsilon) = \left( -i g_{\nu} M + \frac{\vec{p}_{\mu} \vec{\mu} + \vec{p}_{\nu} \vec{\nu}}{m_\alpha c^2} - ig_{\lambda} \vec{T} \times \vec{\mu} \right), \tag{32}
\]

\[
\xi'_v(p, \lambda, \varepsilon) = \left( -i g'_{\nu} M' + \frac{\vec{p}'_{\mu} \vec{\mu} + \vec{p}'_{\nu} \vec{\nu}}{m_\alpha c^2} - ig'_{\lambda} \vec{T}' \times \vec{\mu}' \right). \tag{33}
\]
From (31) or (32), we can immediately obtain the matrix elements for the emission and the absorption of a meson by a heavy or light particle.

The whole of the theory so far developed applies not only to the vector theory but also to the pseudovector theory considered by Kemmer, the only difference being the explicit forms of $M, M_0, S, T$, and $M', M_0', S', T'$. In the latter theory, these quantities must be replaced by the following expressions:

\[ M = \bar{\Phi} p_\alpha \gamma^\alpha, \quad M_0 = \bar{\Phi} p_\alpha^0 \gamma^\alpha, \]
\[ S = \bar{\Phi} p_\alpha \gamma^\alpha (h), \quad T = \bar{\Phi} p_\alpha \gamma^\alpha (h), \]
\[ M' = \bar{\Phi} p_\alpha' \gamma^\alpha, \quad M'_0 = \bar{\Phi} p_\alpha^0' \gamma^\alpha, \]
\[ S' = \bar{\Phi} p_\alpha' \gamma^\alpha (l), \quad T' = \bar{\Phi} p_\alpha' \gamma^\alpha (l). \]

2. Scalar Theory. The matrix formulation made in the preceding subsection can also be extended to the scalar meson theory studied by Yukawa and Sakata. In this theory, the total Hamiltonian is again given by (1), while (2), (5), (6), (8), and (9) must be replaced by the following expressions, respectively:

\[ H_s = \int \left[ \frac{1}{4e} U U' + \frac{1}{4} (\nabla U, \nabla U) + \frac{\kappa^2}{4e} U U' \right] dv, \]
\[ H'_s = \int \left[ f_1 (\tilde{U} U + U \tilde{U}) + \frac{\kappa^2}{4} ((\nabla \tilde{U}, M) + (\tilde{M}, \nabla \tilde{U}) \right. \]
\[ + 4\pi \epsilon \tilde{M} U' + 4\pi e \tilde{U} M \right] dv, \]
\[ H''_s = \int \left[ f_1 (\tilde{U} U + U \tilde{U}) + \frac{\kappa^2}{4} ((\nabla \tilde{U}, M) + (\tilde{M}, \nabla \tilde{U}) \right. \]
\[ + 4\pi \epsilon \tilde{M} U' + 4\pi e \tilde{U} M \right] dv, \]
\[ H'_0 = \frac{4\pi e^2}{\kappa^2} \int \tilde{M}, M_0 dv, \]
\[ H'_0 = \frac{4\pi \rho^2}{\kappa^2} \int (\tilde{M}, M_0 + M_0 \tilde{M}) dv, \]
\[ H'_0 = \frac{4\pi \rho^2}{\kappa^2} \int \tilde{M}, M_0 dv. \]

Here $U$ and $\tilde{U}$ are the scalar functions describing the meson field, from which $U'$ and $\tilde{U}'$ are derived by the equations

\[ \frac{d}{dt} U = \nabla^2 U + 4\pi f_1 \tilde{M}, \]
\[ \frac{d}{dt} \tilde{U} = \nabla^2 \tilde{U} + 4\pi f_1 M_0. \]
The commutation relations between these variables are given by

\[ [U(r', t), U^+(r'', t)] = i\hbar \delta(r' - r''), \]
\[ [\bar{U}(r', t), \bar{U}^+(r'', t)] = i\hbar \delta(r' - r''). \] (44)

The scalar quantities \( W \) and \( W' \) denote

\[ W = \phi^3 \phi, \]
\[ W' = \phi'^3 \phi'. \] (45, 46)

The constants \( (f_1, f_2) \) and \( (f'_1, f'_2) \), having each the dimension of an electric charge, determine the strength of the interaction of the scalar mesons with the heavy and light particles, respectively. All other notations have the same meaning as in the preceding subsection.

Now, if we define a two component wave function \( \chi \) by

\[ \chi = \begin{pmatrix} U & \bar{U} \end{pmatrix}, \] (47)

operators \( \rho_1, \rho_2, \rho_3 \) acting on this \( \chi \) by

\[ \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (48)

and row-matrices \( \tilde{\xi} \) and \( \tilde{\xi}' \) by

\[ \tilde{\xi} = (-f_1, \tilde{W} - f_2 (M, \text{grad}), -f_2 M_0), \]
\[ \tilde{\xi}' = (-f'_1, \tilde{W}' - f'_2 (M', \text{grad}), -f'_2 M'_0), \] (49, 50)

(37), (38), (39), and (44) become again the forms:

\[ H_u = \int \tilde{\chi} \rho_1 \hat{\mathcal{H}} \chi \, dv, \] (51)
\[ H'_{\chi'} = -\sqrt{4\pi \hbar c} \int \tilde{\xi} \chi \, dv + \text{comp. conj.}, \] (52)
\[ H'_{\chi'} = -\sqrt{4\pi \hbar c} \int \tilde{\xi}' \chi \, dv + \text{comp. conj.}, \] (53)
\[ \frac{\partial [\tilde{\chi}(r, t), \rho_2]_{\chi}}{\partial t} = -\delta_{\mu \nu} \delta(r - r'), \] (54)

with

\[ \rho_2 \hat{\mathcal{H}} = \hbar c \begin{pmatrix} 1 & -\text{grad div} \\ \frac{1}{2\kappa^2} & -\rho_3 \frac{\text{grad div}}{2\kappa^2} \end{pmatrix}. \] (55)

The transformation to the momentum space can also be performed
in the similar manner. We expand $\chi$ and $\tilde{\chi}$ in the plane waves as

$$
\chi = \sum_{\vec{p}, \varepsilon} a(\vec{p}, \varepsilon) u(\vec{p}, \varepsilon) e^{i\vec{p}\cdot\vec{r}/\hbar c},$
$$
\tilde{\chi} = \sum_{\vec{p}, \varepsilon} a^*(\vec{p}, \varepsilon) \tilde{u}(\vec{p}, \varepsilon) e^{-i\vec{p}\cdot\vec{r}/\hbar c},$

(56)

where $\vec{p}$ and $\varepsilon$ have the same meaning as before, while $u(\vec{p}, \varepsilon)$ is now

the solution of the equation

$$
E \vec{p} u = \left( \frac{\vec{p}^2}{2m_\mu c^2} + m_\mu c^2 + E \right) u, \quad (57)
$$

and satisfies the following orthogonality relations:

$$
\sum_{\varepsilon} \varepsilon \langle u(\vec{p}, \varepsilon) | p \rangle u(\vec{p}, \varepsilon') = \delta_{\varepsilon\varepsilon'}, \quad (58)
$$

The explicit form of $u$ may be written as

$$
u_1 = i\varepsilon \sqrt{\frac{m_\mu c^2}{2E}}, \quad (59)
$$

The $a$'s and $a^*$'s, satisfying the commutation relation

$$
[a(\vec{p}, \varepsilon), a^*(\vec{p}', \varepsilon')] = \varepsilon \delta_{\varepsilon\varepsilon'}, \quad (60)
$$

denote the operators similar to $a(\vec{p}, \lambda, \varepsilon)$ and $a^*(\vec{p}, \lambda, \varepsilon)$ in the pre

ceeding subsection.

Inserting (56) in (51), (52) and (53), we obtain

$$
H_u = \sum_{\varepsilon} |E| \cdot \{ N^-(\vec{p}) + N^+(\vec{p}) + 1 \}, \quad (61)
$$

$$
H' = -\sqrt{\frac{4\pi \hbar c}{\kappa}} \sum_{\vec{r}, \varepsilon} a(\vec{p}, \varepsilon) u(\vec{p}, \varepsilon) \int \bar{\xi}_\varepsilon(r) e^{-\hat{p}\cdot\hat{r}/\hbar c} d\varepsilon + \text{comp. conj.}, \quad (62)
$$

$$
H'_u = -\sqrt{\frac{4\pi \hbar c}{\kappa}} \sum_{\vec{r}, \varepsilon} a^*(\vec{p}, \varepsilon) \tilde{u}(\vec{p}, \varepsilon) \int \bar{\xi}'_\varepsilon(r) e^{-\hat{p}\cdot\hat{r}/\hbar c} d\varepsilon + \text{comp. conj.}, \quad (63)
$$

where $N^-(\vec{p})$, $\bar{\xi}_\varepsilon(r)$ and $\bar{\xi}'_\varepsilon(r)$ denote

$$
N^-(\vec{p}) = a^*(\vec{p}, 1) a(\vec{p}, 1), \quad N^+(\vec{p}) = a(\vec{p}, -1) a^*(\vec{p}, -1), \quad (64)
$$

$$
\bar{\xi}_\varepsilon(r) = (-f_1 W - f_2 \hat{M}_p, f_2 \hat{M}_0), \quad (65)
$$

and

$$
\bar{\xi}'_\varepsilon(r) = (-f_1' W - f_2' \hat{M}_p, f_2' \hat{M}_0), \quad (66)
$$
As before, (62) and (63) correspond to the emission or the absorption of a scalar meson by a heavy and light particle, respectively.

Furthermore, replacing $W$, $W'$, $M, M', M^0, M^0$ by

$$W = \phi_c \phi^* (h) \phi \mu,$$
$$W' = \phi_c \phi^* (l) \phi \mu,$$

the above formula can also be applied to the pseudoscalar meson theory.

II. The Theory of Beta-Decay.

As was first pointed out by Fermi\(^7\), the process of beta-disintegration can be considered as a transformation of a neutron in the nucleus into a proton with the simultaneous transition of the light particle from a neutrino state of negative energy to an electron state of positive energy. In Yukawa's theory, this process takes place in two different ways: (1) by direct interaction $H_{\psi \phi}$ between the heavy and light particles and (2) by the indirect interaction through the meson field which is pictured as

$$\text{Neutron} \rightarrow \text{Proton} + \text{Negative Meson},$$
$$\text{Negative Meson} + \text{Neutrino} \rightarrow \text{Electron},$$
or

$$\text{Neutrino} \rightarrow \text{Electron} + \text{Positive Meson},$$
$$\text{Positive Meson} + \text{Neutron} \rightarrow \text{Proton}.$$

The matrix element $H_{\beta}$ for the beta-decay process is therefore the sum of those for the direct and indirect interactions which we shall denote as $H_{\beta}^{(1)}$ and $H_{\beta}^{(2)}$, respectively.

The calculation of $H_{\beta}^{(1)}$ is very simple and gives immediately the expressions:

$$H_{\beta}^{(1)} = \frac{4\pi}{h^2} \int \left[ g_1 g_0 \tilde{M} \tilde{M}^0 + g_2 g_2' (\tilde{S}, S') \right] dv, \quad (\text{Vector Theory}) \quad (70a)$$

$$H_{\beta}^{(2)} = \frac{4\pi}{h^2} \int \left[ g_1 g_0 \tilde{M} \tilde{M}^0 + g_2 g_2' (\tilde{T}, T') \right] dv, \quad (\text{Pseudovector Theory}) \quad (70b)^*$$

$$H_{\beta}^{(3)} = \frac{4\pi f^2 f^2'}{h^2} \int \tilde{M} \tilde{M}^0 dv, \quad (\text{Scalar Theory}) \quad (70c)$$

\(^7\) E. Fermi, ZS. f. Phys. 88 (1934), 161.

\(^*\) In this case, the invariant quantity

$$-\frac{4\pi}{h^2} \int \left[ (\tilde{S} + \tilde{S}')(\tilde{T} + \tilde{T}')(\tilde{S} + \tilde{S}')(\tilde{T} + \tilde{T}')(\tilde{S}) \right] dv$$

must be added to the total Hamiltonian, in order to avoid the occurrence of the static interaction of the $\delta$-function type.
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\[ H^{\psi}_{\beta} = \frac{4\pi e^2}{\kappa^2} \int \tilde{M}_\alpha M_{\beta} \, dr. \]  
(Pseudoscalar Theory)  
(70d)

On the other hand, \( H^{\psi}_{\beta} \) is given by

\[ H^{\psi}_{\beta} = \sum \left\{ \frac{(F|H_\psi|I)(I|H_{\psi}^*|A)}{E_a - E_I} \right\} \left\{ \frac{(F|H_\psi^*|I)(I|H_{\psi}^*|A)}{E_a - E_{II}} \right\}, \]  
(71)

where \( A \) and \( F \) denote the initial and final states respectively and \( I \) and \( II \) the two intermediate states corresponding to the two schemes of the interactions as shown in (69). \( E_a, E_I \) and \( E_{II} \) are the energies of the states \( A, I \) and \( II \), respectively. The summation must be taken over all values of the momentum \( \vec{p'} \) and the states of polarization \( \lambda \) of the meson which is emitted in the intermediate state. The matrix elements \( (F|H_\psi|I), \) etc. can be easily obtained from the equations (31) and (32) or (62) and (63), and are given by

\[
(F|H_\psi|I) = \int \frac{4\pi \hbar c}{\kappa} \int \bar{\xi}_{\psi}(\vec{r}_I) \psi(\vec{r}_a, \lambda, -1) e^{i\vec{k'} \vec{p}/\hbar} d\vec{r}_I, \\
(H|H_\psi^*|II) = \int \frac{4\pi \hbar c}{\kappa} \int \bar{\xi}_{\psi}(\vec{r}_I) \psi(\vec{r}_a, \lambda, 1) e^{-i\vec{k'} \vec{p}/\hbar} d\vec{r}_I, \\
(F|H_{\psi}^*|I) = \int \bar{\xi}_{\psi}(\vec{r}_a) \psi(\vec{r}_a, \lambda, -1) e^{i\vec{k'} \vec{p}/\hbar} d\vec{r}_a, \\
(H|H_{\psi}^*|II) = \int \bar{\xi}_{\psi}(\vec{r}_a) \psi(\vec{r}_a, \lambda, 1) e^{-i\vec{k'} \vec{p}/\hbar} d\vec{r}_a.
\]  
(72)

(In the cases of the scalar and pseudoscalar theory, \( u(\vec{p}, \lambda, \pm 1) \) must be replaced by \( u(\vec{p}, \lambda, \mp 1) \). Inserting (72) in (71), we obtain

\[ H^{\psi}_{\beta} = \frac{4\pi \hbar c}{\kappa} \int d\vec{r}_I d\vec{r}_a \sum_{\lambda} e^{i\vec{p}' \vec{p}/\hbar} \left\{ \frac{\bar{u}(\vec{p}, \lambda, -1) \xi_{\psi}(\vec{r}_a)}{W_0 - \sqrt{\vec{p}'^2 + m^2 c^4}} \right\} \left\{ \frac{\bar{u}(\vec{p}, \lambda, 1) \xi_{\psi}(\vec{r}_a)}{W_0 + \sqrt{\vec{p}'^2 + m^2 c^4}} \right\}, \]  
(73)

where \( W_0 \) is the energy difference between the states of the neutron and proton. Since \( W_0 \) is always small compared with \( m c^2 \) in the ordinary beta-decay, \( H^{\psi}_{\beta} \) becomes approximately

\[ H^{\psi}_{\beta} \approx -\frac{4\pi \hbar c}{\kappa} \int d\vec{r}_I d\vec{r}_a \sum_{\lambda} \frac{\bar{u}(\vec{p}, \lambda, -1) \xi_{\psi}(\vec{r}_a)}{\sqrt{\vec{p}'^2 + m^2 c^4}} X \]  
(74)

with

\[ X = \sum_{\lambda} e^{i\vec{p}' \vec{p}/\hbar} \left\{ \bar{u}(\vec{p}, \lambda, \pm 1) \right\} \left\{ \bar{u}(\vec{p}, \lambda, \mp 1) \right\} \xi_{\psi}(\vec{r}_a). \]  
(75)

(In the scalar theories, the summation over \( \lambda \) is unnecessary). The summations in (75) can be easily performed with the help of the converse orthogonality relations (27) or (58). The results are given by
\[ X = \frac{m_u c^2}{\sqrt{p^2 + m_u^2 c^4}} \tilde{\xi}_p(r_h) \left[ \frac{p^2}{2m_u^2 c^4} + 1 - \rho_3 \right] \left[ \frac{\sigma p}{2m_u^2 c^4} \right] \tilde{\xi}_p^*(r_l), \]

(Vector or Pseudovector Theory) \ (75a)

or

\[ X = \frac{m_u c^2}{\sqrt{p^2 + m_u^2 c^4}} \tilde{\xi}_p(r_h) \left[ \frac{p^2}{2m_u^2 c^4} + 1 - \rho_3 \right] \left[ \frac{\sigma p}{2m_u^2 c^4} \right] \tilde{\xi}_p^*(r_l). \]

(Scalar or Pseudoscalar Theory) \ (75b)

By using the explicit forms of \( \tilde{\xi}_p \) and \( \tilde{\xi}_p^* \), the above expressions become

\[ X = \frac{m_u c^2}{\sqrt{p^2 + m_u^2 c^4}} \left[ g_1 g_1' \frac{p^2}{m_u c^2} \tilde{M}_0 M_0 + (\tilde{M}, M') + \frac{(\tilde{M}, p)(M', p)}{m_u^2 c^4} \right] \]

\[ + g_1 g_1' \frac{p^2}{m_u^2 c^4} (S, S') \left[ \frac{(\tilde{S}, p)(S', p)}{m_u^2 c^2} \right] \]

\[ + \left( 1 + \frac{p^2}{m_u^2 c^4} \right) (\tilde{T}, T') - \frac{(\tilde{T}, p)(T', p)}{m_u^2 c^4}. \]

\[ \left\{ \text{Vector or Pseudovector Theory} \right\} \ (76a) \]

and

\[ X = \frac{m_u c^2}{\sqrt{p^2 + m_u^2 c^4}} \left[ f_1 f_1' \tilde{W} W' + f_2 f_2' \left( 1 + \frac{p^2}{m_u^2 c^4} \right) \tilde{M}_0 M_0' + \frac{(\tilde{M}, p)(M', p)}{m_u^2 c^4} \right] \]

\[ - if_1 f_1' \tilde{W}(M', p) + if_2 f_2' (\tilde{M}, p) \tilde{W}'. \]

(Scalar or Pseudoscalar Theory) \ (76b)

Inserting these expressions in (74), \( H^{\alpha \beta} \) reduces to

\[ H^{\alpha \beta} = 4 \pi \hbar c^2 \int \int d^3 r d^3 \ell \left[ g_1 g_1' \tilde{M}_0 M_0 \frac{\Delta_0}{\kappa^2} - (\tilde{M}, M') \frac{\Delta_0}{\kappa^2} \right] \]

\[ + g_1 g_1' \left( \tilde{S}, S' \right) \frac{\Delta_0}{\kappa^2} \left[ \frac{\Delta_0}{\kappa^2} - 1 \right] \]

\[ - (\tilde{T}, \text{grad}_\alpha(T', \text{grad}_\alpha)) \frac{\Delta_0}{\kappa^2} \]

\[ - g_1 g_2 \left( \tilde{S}, M', \text{grad}_\alpha \right) \frac{\tilde{M}_0}{\kappa} \frac{\Delta_0}{\kappa^2} \]

\[ - g_2 g_1 \left( \tilde{S}, M', \text{grad}_\alpha \right) \frac{\tilde{M}_0}{\kappa} \frac{\Delta_0}{\kappa^2} \]
(Vector or Pseudovector Theory) (77a)

\[ H^2_\beta = 4\pi \hbar^2 \varepsilon^3 \int \int d\nu_1 d\nu_2 \left[ - f_1 f'_2 \tilde{W} \tilde{W}' + f_2 f'_1 \left( \frac{\Delta\kappa}{\kappa^2} - 1 \right) \right. \\
+ \left( \frac{\tilde{M}_1}{\kappa} \right) (M', \text{grad}_h) + f_1 f'_2 \left( \frac{\tilde{M}_2}{\kappa} \right) (M', \text{grad}_h) \\
\left. - f_1 f'_2 \left( \frac{\tilde{M}}{\kappa} \right) \text{grad}_h \frac{W'}{\kappa} - f_2 f'_1 \left( \frac{\tilde{M}}{\kappa} \right) \text{grad}_h \frac{W}{\kappa} \right] e^{-\frac{\varepsilon^2}{\kappa^2} - \frac{\varepsilon^2}{\kappa^2}}. \]

(Scalar or Pseudoscalar Theory) (77b)

Further, by making use of the relations

\[ 4\pi \hbar^2 \varepsilon^3 \int e^{-\frac{\varepsilon^2}{\kappa^2} - \frac{\varepsilon^2}{\kappa^2}} = \frac{e^{-\frac{\varepsilon^2}{\kappa^2} - \frac{\varepsilon^2}{\kappa^2}}}{|r_n - r_{l^1}|}, \]

and

\[ \Delta \left( \frac{e^{-\varepsilon r}}{r} \right) = \kappa^2 \frac{e^{-\varepsilon r}}{r} - 4\pi \delta(r), \]

we obtain

\[ H^2_\beta = \int \int d\nu_1 d\nu_2 \left[ g_1 g'_2 \left( \tilde{M}_1 M_0 - \left( \tilde{M}_1 M'_0 \right) \right) \right. \\
+ g_2 g'_1 \left( \tilde{S}_1 S'_0 - \left( \tilde{S}_1 S'_0 \right) \right) \left. \frac{\kappa^2}{\kappa} \right] e^{-\frac{\varepsilon^2}{\kappa^2} - \frac{\varepsilon^2}{\kappa^2}}. \]

and

\[ H^2_\beta = \int \int d\nu_1 d\nu_2 \left[ - f_1 f'_2 \tilde{W} \tilde{W}' + f_2 f'_1 \left( \frac{\Delta\kappa}{\kappa^2} - 1 \right) \right. \\
+ \left( \frac{\tilde{M}_1}{\kappa} \right) (M', \text{grad}_h) + f_1 f'_2 \left( \frac{\tilde{M}_2}{\kappa} \right) (M', \text{grad}_h) \\
\left. - f_1 f'_2 \left( \frac{\tilde{M}}{\kappa} \right) \text{grad}_h \frac{W'}{\kappa} - f_2 f'_1 \left( \frac{\tilde{M}}{\kappa} \right) \text{grad}_h \frac{W}{\kappa} \right] e^{-\frac{\varepsilon^2}{\kappa^2} - \frac{\varepsilon^2}{\kappa^2}}. \]

From (70) and (78), we obtain the total matrix elements for each of the four types of the meson theory.
Where we have ignored all the terms involving \( \sigma(h) \) or \( \sigma(h) \), because the velocity of the heavy particle in the nucleus may be considered as small compared with that of light.

The expression (79a) is identical with the result obtained in IV from the unquantized theory (8).

Now, the wave lengths of the light particles are always large compared with \( 1/\kappa \), so that the integration with respect to the light particle cannot be carried out by ignoring the space dependence of the light particle wave functions. In this approximation (79) becomes

\[
H=E = \int \frac{d\nu}{d\nabla} \left[ -g_{\alpha\beta} \left( \tilde{M}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] + \frac{2}{3} g_{\alpha\beta} (\tilde{H}, T') \right] (\text{Vector Theory})
\]

\[
H = \int \frac{d\nu}{d\nabla} \left[ -f_{\alpha\beta} \left( \tilde{H}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] + \frac{1}{3} g_{\alpha\beta} (\tilde{H}, T') \right] (\text{Pseudovector Theory})
\]

\[
H = \int \frac{d\nu}{d\nabla} \left[ -f_{\alpha\beta} \left( \tilde{H}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] (\text{Scalar Theory})
\]

\[
H = \int \frac{d\nu}{d\nabla} \left[ -f_{\alpha\beta} \left( \tilde{H}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] (\text{Pseudoscalar Theory})
\]

where we have ignored all the terms involving \( \rho_{\alpha\beta} \) or \( \rho_{\alpha\beta} \), because the velocity of the heavy particle in the nucleus may be considered as small compared with that of light.

The expression (79a) is identical with the result obtained in IV from the unquantized theory (8).

Now, the wave lengths of the light particles are always large compared with \( 1/\kappa \), so that the integration with respect to the light particle can be carried out by ignoring the space dependence of the light particle wave functions. In this approximation (79) becomes

\[
H = \int \frac{d\nu}{d\nabla} \left[ -f_{\alpha\beta} \left( \tilde{H}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] + \frac{1}{3} g_{\alpha\beta} (\tilde{H}, T') \right] (\text{Vector Theory})
\]

\[
H = \int \frac{d\nu}{d\nabla} \left[ -f_{\alpha\beta} \left( \tilde{H}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] (\text{Pseudovector Theory})
\]

\[
H = \int \frac{d\nu}{d\nabla} \left[ -f_{\alpha\beta} \left( \tilde{H}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] (\text{Scalar Theory})
\]

\[
H = \int \frac{d\nu}{d\nabla} \left[ -f_{\alpha\beta} \left( \tilde{H}, \frac{\tilde{M}}{\tilde{M}} \right) \kappa^2 \frac{\partial}{\partial \nabla} \right] (\text{Pseudoscalar Theory})
\]

(8) H.A. Bethe and L.W. Nordheim, Phys. Rev. 57 (1940), 908.

(*) The factor \( \frac{2}{3} \) in the second term was omitted by mistake in IV.
These expressions are identical with the original Fermi interactions, the only differences being that the constants are now determined by the mesic charges of the heavy and light particles.

The calculation of the beta-decay probability proceeds then on the same line as in Fermi’s theory. Neglecting the Coulomb force between the nucleus and the electron, we find for the probability per unit time that the electron with the energy between \( \varepsilon \) and \( (\varepsilon + d\varepsilon) \) is emitted

\[
W(\varepsilon) d\varepsilon = \frac{1}{T_1} \left| \frac{M_1}{T_2} + \frac{3}{2} \frac{M_2}{T_2} \right|^2 (\varepsilon - \varepsilon_0)^{\varepsilon - \varepsilon_0} d\varepsilon, \quad \text{(Vector Theory)} \quad (81a)
\]

\[
W(\varepsilon) d\varepsilon = \frac{3}{4} \frac{M_1}{T_2} (\varepsilon - \varepsilon_0)^{\varepsilon - \varepsilon_0} d\varepsilon, \quad \text{(Pseudovector Theory)} \quad (81b)
\]

\[
W(\varepsilon) d\varepsilon = \frac{1}{T_1} \varepsilon (\varepsilon - 1)(\varepsilon - \varepsilon_0)^{\varepsilon - \varepsilon_0} d\varepsilon, \quad \text{(Scalar Theory)} \quad (81c)
\]

\[
W(\varepsilon) d\varepsilon = \frac{3}{4} \frac{M_1}{T_2} (\varepsilon - \varepsilon_0)^{\varepsilon - \varepsilon_0} d\varepsilon, \quad \text{(Pseudoscalar Theory)} \quad (81d)
\]

with

\[
\begin{align*}
1 &= \frac{1}{T_1} \frac{\hbar}{m_e} \left( \frac{m_e}{m_n} \right)^2 \left( \frac{g_1 g_2}{\hbar c} \right)^2, \\
1 &= \frac{1}{T_2} \frac{32 \hbar}{3 \pi \hbar} \frac{m_e c^2}{m_n} \left( \frac{g_1 g_2}{\hbar c} \right)^2, \\
1 &= \frac{1}{T_2} \frac{3 \pi}{3 \pi \hbar} \left( \frac{m_e c^2}{m_n} \right)^2 \left( \frac{g_1 g_2}{\hbar c} \right)^2 + \left( \frac{e^2}{\hbar c} \right)^2, \\
1 &= \frac{1}{T_1} \frac{\hbar}{m_e} \left( \frac{m_e}{m_n} \right)^2 \left( \frac{g_1 g_2}{\hbar c} \right)^2, \\
1 &= \frac{1}{T_1} \frac{3 \pi}{3 \pi \hbar} \left( \frac{e^2}{\hbar c} \right)^2,
\end{align*}
\]

(82)

and

\[
M_1 = \int \mathcal{P} \mathcal{Q} d\tau, \quad \mathcal{M}_1 = \int \mathcal{P} \mathcal{Q} d\tau,
\]

(83)

The energy distribution (81) is exactly the same as that obtained from Fermi’s theory. The terms involving \( M_1 \) give rise to transitions of the Fermi type, whereas those involving \( \mathcal{M}_2 \) to transitions of the Gamow-Teller type.†

III. The Lifetime of the Meson.

It was first suggested by Yukawa that the meson can disintegrate spontaneously into an electron and a neutrino even in vacuum. According to the theory described in I, the probability per unit time of the disintegration of a negative meson, for example, by emitting an electron in the direction within the solid angle \( d\Omega \) is given

\[
dw = \frac{2\pi}{\hbar} \sum |(F|H_{\nu}'|A)|^2 \frac{\hat{p}_e^2 \hat{p}_1 d\Omega}{(2\pi \hbar c)^3 dE_F},
\]

where \( \hat{p}_e/c \) denotes the momentum of the electron and where \( E_F \) is the total energy of the final state. Further, \( \sum \) means the summations over both spin directions of the electron and the neutrino. If the meson is initially at rest, (48) reduces to

\[
dw_0 = \frac{m_u c^5}{32\pi^2 \hbar^4} \sum |(F|H_{\nu}'|A)|^2 d\Omega.
\]

The matrix elements can be easily obtained from (22) or (63):

\[
(F|H_{\nu}'|A) = -\sqrt{\frac{2\pi \hbar c}{\kappa}} \frac{\hat{p}_e}{\sigma^\mu \sigma\nu u_e(p_e)},
\]

(Scalar Theory) (86)

with

\[
\langle \hat{p}_i | \bar{\epsilon}_{\sigma,\mu} | \hat{p}_f \rangle = \frac{(\epsilon q_\alpha \rho_{\alpha}^{\mu} u_e (\hat{p}_f)) \sigma^\nu (\hat{p}_f) \rho_{\nu}^{\sigma} u_\nu (\hat{p}_f)}{\sqrt{\frac{2\pi \hbar c}{\kappa}}},
\]

(Pseudovector Theory) (87a)

\[
\langle \hat{p}_i | \bar{\epsilon}_{\sigma,\mu} | \hat{p}_f \rangle = \frac{(\epsilon q_\alpha \rho_{\alpha}^{\mu} u_e (\hat{p}_f)) \sigma^\nu (\hat{p}_f) \rho_{\nu}^{\sigma} u_\nu (\hat{p}_f)}{\sqrt{\frac{2\pi \hbar c}{\kappa}}},
\]

(Pseudoscalar Theory) (87b)

\[
\langle \hat{p}_i | \bar{\epsilon}_{\sigma,\mu} | \hat{p}_f \rangle = \frac{(\epsilon q_\alpha \rho_{\alpha}^{\mu} u_e (\hat{p}_f)) \sigma^\nu (\hat{p}_f) \rho_{\nu}^{\sigma} u_\nu (\hat{p}_f)}{\sqrt{\frac{2\pi \hbar c}{\kappa}}},
\]

(Pseudoscalar Theory) (87c)

\[
\langle \hat{p}_i | \bar{\epsilon}_{\sigma,\mu} | \hat{p}_f \rangle = \frac{(\epsilon q_\alpha \rho_{\alpha}^{\mu} u_e (\hat{p}_f)) \sigma^\nu (\hat{p}_f) \rho_{\nu}^{\sigma} u_\nu (\hat{p}_f)}{\sqrt{\frac{2\pi \hbar c}{\kappa}}},
\]

(Pseudoscalar Theory) (87d)

Here, \( u_e \) and \( u_\nu \) denote the Dirac amplitudes of the electron and the neutrino, respectively.

By using the explicit forms (28) or (59) of \( u \), and with the equations (87), (86) becomes

\[
(F|H_{\nu}'|A) = -\sqrt{\frac{2\pi \hbar c}{\kappa}} \frac{\hat{p}_e}{\sigma^\mu \sigma\nu u_e (\hat{p}_e)} |g_1 \rho_1^{\mu} (\sigma^\nu \epsilon) + i g_2 \rho_2^{\mu} (\sigma^\nu \epsilon) | u_e (\hat{p}_e),
\]

(Vector Theory) (88a)

\[
(F|H_{\nu}'|A) = -\sqrt{\frac{2\pi \hbar c}{\kappa}} \frac{\hat{p}_e}{\sigma^\mu \sigma\nu u_e (\hat{p}_e)} |g_1 \rho_1^{\mu} (\sigma^\nu \epsilon) - ig_2 \rho_2^{\mu} (\sigma^\nu \epsilon) | u_e (\hat{p}_e),
\]

(Pseudovector Theory) (88b)

\[
(F|H_{\nu}'|A) = -\sqrt{\frac{2\pi \hbar c}{\kappa}} \frac{\hat{p}_e}{\sigma^\mu \sigma\nu u_e (\hat{p}_e)} |g_1 \rho_1^{\mu} - f_2 | u_e (\hat{p}_e),
\]

(Pseudoscalar Theory) (88c)
(Scalar Theory) \((88c)\)

\[
(F | H_\pi | A) = -\sqrt{\frac{2\pi \hbar c}{\kappa}} n_\pi(p_\pi) \{ i \bar{\rho}_+ \rho_+^\dagger - i \bar{\rho}_- \rho_-^\dagger \} n_\pi(p_\pi),
\]

(Pseudoscalar Theory) \((88d)\)

where the unit vector \(\hat{e}\) denotes the direction of the polarization of the vector meson.

By inserting these expressions in \((85)\), and by performing the spin summations of the light particles with the help of the spur technique and the integration with respect to the direction of the emission, we find the reciprocal lifetime for a meson at rest:

\[
\frac{1}{\tau_0} = \frac{m_c^2}{2\hbar} \left(\frac{2}{3} \frac{g_{21}^2}{\hbar c} + \frac{1}{3} \frac{g_{22}^2}{\hbar c}\right), \quad \text{(Vector Meson)} \quad (89a)
\]

\[
\frac{1}{\tau_0} = \frac{m_c^2}{2\hbar} \left(\frac{2}{3} \frac{g_{21}^2}{\hbar c} + \frac{1}{3} \frac{g_{22}^2}{\hbar c}\right), \quad \text{(Pseudovector Meson)} \quad (89b)
\]

\[
\frac{1}{\tau_0} = \frac{m_c^2}{2\hbar} \left(\frac{f_{11}^2}{\hbar c} \frac{f_{22}^2}{\hbar c} \frac{1}{m_n} \right), \quad \text{(Scalar Meson)} \quad (89c)
\]

\[
\frac{1}{\tau_0} = \frac{m_c^2}{2\hbar} \left(\frac{f_{11}^2}{\hbar c} \frac{f_{22}^2}{\hbar c} \frac{1}{m_n} \right), \quad \text{(Pseudoscalar Meson)} \quad (89d)
\]

By the well-known formula of the special relativity, the lifetime for a meson moving with a velocity \(\beta c\) is then given by

\[
\tau = \frac{\tau_0}{\sqrt{1 - \beta^2}}.
\]

It was first stressed by Nordheim\(^{1}\) that in the vector theory there is a serious quantitative discrepancy between the calculated and experimental values of the lifetime of the meson. This can be seen as follows. Inserting \((82)\) in \((89a)\), we obtain the relation

\[
\frac{1}{\tau_0} = \frac{\pi \left(\frac{m_n}{m} \right)^{\frac{5}{2}} \frac{2}{3} \left(\frac{\hbar c}{g_{21}}\right)}{16 \left(\frac{\hbar c}{g_{22}}\right) T_1} + \frac{1}{4} \left(\frac{\hbar c}{g_{22}}\right) T_2.
\]

(91)

The constants \(T_1\) and \(T_2\) can be determined by comparing the beta-decay formula \((81a)\) with the lifetimes of the light elements. If we assume that the matrix elements \(\tilde{M}_1\) and \(\tilde{M}_2\) have their maximum possible values, these constants come out to be of the order of \(10^7\) sec. Further, the values of the constants \(g_1\) and \(g_2\) determined from the magnitude of the nuclear forces are \(g_1^2 \hbar c \approx 0.1\) and \(g_2^2 \hbar c \approx 0.1\). Using these values of the constants and with the value \(m_n \approx 170 m\), we find from \((91)\)

\[
\tau_0 \approx 3.7 \times 10^{-6} \text{ sec}.
\]

On the other hand, measurements of the meson decay in cosmic rays yield a lifetime of about \(1 - 5 \times 10^{-6}\) sec. Thus, the theoretical estimate...
is too short by a factor $10^{-2}$ in comparison with the experimental value.

In the pseudovector or scalar meson theory, we meet also the same difficulty.

*It is, however, to be noticeable that such a difficulty disappears in the pseudoscalar meson theory, since there is no relation like (91). In this case, we can determine the value of \( f_1 \) from the meson decay and that of \( f_2 \) from the beta-decay, separately. In spite of this advantage, we must, however, abandon this possibility, since the nuclear forces derived from this theory do not agree with the experience.*

In order to avoid this discrepancy, Möller, Rosenfeld and Rozenthal\(^{10}\) assumed that there exist two kinds of mesons and that while one of them has a lifetime of the order of $10^{-6}$ sec in accord with experiments, the lifetime of the other is so short that it can explain the sufficiently rapid nuclear beta-decay\(^*\).

On the other hand, the introduction of two types of mesons has also be required from the discussion of the problem of nuclear forces made by Möller and Rosenfeld\(^{11}\). According to them, a satisfactory field theory of nuclear forces must be such as not to give rise to any static potential of the dipole type. This condition together with the requirement that the forces between a proton and a neutron should lead to the correct positions of the $^3S$ ground level and excited $^1S$ level of the deuteron, can not be satisfied by any theory involving only one of the four possible types of fields. However, if we try to compose two types of mesons, there is one possibility, viz. a mixture of a vector meson field and a pseudoscalar meson field, satisfying the condition\(^{12}\):

\[
\frac{f_1^2}{g_1^2} = \frac{g_2^2}{f_2^2}.
\] (92)

Now, it is to be remarked that their arguments concerning the nuclear forces can also be extended to the static interaction between a heavy particle and a light particle or between two light particles. From the same requirements as before, we obtain the similar condition

\[
\frac{f_1^2}{g_1^2} = \frac{g_2^2}{f_2^2}.
\] (93)

In spite of the disturbance brought about by this relation, we may still have a sufficiently long lifetime for the pseudoscalar meson, since it depends only upon the value of \( f_1 \). *This result seems to be in favour of the assumption made by Möller and others.*

\(^{10}\) C. Möller, L. Rosenfeld and S. Rozenthal, Nature, 144 (1939), 761.

\(^*\) An alternative way of removing this discrepancy proposed by the present author (Phys. Rev. 58 (1940), 576) will be discussed in a separate paper.

\(^{11}\) Möller and Rosenfeld, Det Kgl. Danske Vidensk. Selsk. XVII, 8 (1940).

\(^{12}\) Möller, Phys. Rev. 58 (1940), 1118.
Conduction Electrons and Thermal Expansion

In conclusion, the author wishes to express his deep gratitude to Professor H. Yukawa for his kind interest in this work.

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(Added in Proof: Recently it has been shown by Rarita and Schwinger (Phys. Rev. 59 (1941), 436) that the nuclear forces derived from the pseudoscalar meson theory agree in sign and spin dependence with the sign and magnitude of the singlet-triplet difference and quadrupole moment of the deuteron system, if we take the influence of the non-central force into account correctly. In this connection it may also be remarked that the experimentally observed burst frequency of the meson is in good agreement with the calculation based on the pseudoscalar theory (R. F. Christy and S. Kusaka, Phys. Rev. 59 (1941), 436; See also J. R. Oppenheimer, Phys. Rev. 59 (1941), 462.)

Contributions of the Conduction Electrons in a Metal to the Thermal Expansion.

By Zirō Mikura.

(Read Nov. 2, 1940.)

Abstract.

The approximate proportionality between the thermal-expansion coefficient and the atomic heat of a metal in the moderate temperature region is well-known. However, it seems that this relation should be changed at those extremely low temperatures, where the contribution of the conduction electrons to the atomic heat is relatively large. It is the purpose of this paper to construct a formula for thermal expansion to cover this temperature region. The result is that in this region the thermal-expansion coefficient is no longer proportional to the atomic heat, and that it consists of two parts, corresponding to the Debye and the electronic terms of the latter respectively. The electronic term of the thermal-expansion coefficient is generally not proportional to that of the atomic heat. The former, however, as well as the latter, is proportional to the absolute temperature at low temperatures. From the new formula a possible explanation of the small expansibility of "Invar" has been suggested.

§1. Introduction.

The well-known theoretical relation between the thermal-expansion