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Introduction to Complex Orbital Momenta.

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Summary. — In this paper the orbital momentum j, until now considered as an integer discrete parameter in the radial Schrödinger wave equations, is allowed to take complex values. The purpose of such an enlargement is not purely academic but opens new possibilities in discussing the connection between potentials and scattering amplitudes. In particular it is shown that under reasonable assumptions, fulfilled by most field theoretical potentials, the scattering amplitude at some fixed energy determines the potential uniquely, when it exists. Moreover for special classes of potentials $V(x)$, which are analytically continuable into a function $V(z)$, $z=x+iy$, regular and suitable bounded in $x>0$, the scattering amplitude has the remarcable property of being continuable for arbitrary negative and large cosine of the scattering angle and therefore for arbitrary large real and positive transmitted momentum. The range of validity of the dispersion relations is therefore much enlarged.

1. – In the following we shall choose dimentionless variables, by putting $x=kr$, where r is the distance from the origin, k the wave number (fixed). We can write then Schrödinger's equation as follows:

$$\psi'' - \frac{\lambda^2-\frac{1}{4}}{x^2}\,\psi + \psi - U(x)\,\psi = 0\,. \tag{1.1}$$

Here λ is a generalized complex orbital momentum; when λ assumes positive half-integer values (hereafter referred to as the physical values) we shall write $\lambda = j+\frac{1}{2}$.

Eq. (1.1) is even in λ. For brevity we shall single out all quantities cor-

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responding to $U(x) = 0$ with the index ^{0}. We shall also suppose

$$\int_0^\infty x|U(x)|\,\mathrm{d}x < Q < \infty\,, \qquad |x^2 U(x)| < M = \text{const.} \tag{1.2}$$

Although much of the theory here developed could be retained under a weaker condition, some interesting results would not hold or would require too lengthy proofs. We list here a set of solutions of (1.1) with the appropriate boundary conditions:

$$\begin{cases} F(\lambda, \eta, x) & x \to \infty & \sim \cos\left(x - \eta - \dfrac{\pi}{4}\right) \\ G(\lambda, x) & x \to \infty & \sim \cos\left(x - \dfrac{\pi}{4}\right) \\ S(\lambda, x) & x \to \infty & \sim \sin\left(x - \dfrac{\pi}{4}\right) \\ \varphi(\lambda, x) & x \to 0 & \sim x^{\lambda + \frac{1}{2}}\,. \end{cases} \tag{1.3}$$

Clearly:

$$\begin{cases} F(\lambda, 0, x) = G(\lambda, x)\,; \qquad F\left(\lambda, \dfrac{\pi}{2}, x\right) = S(\lambda, x)\,, \\ F(\lambda, \eta, x) = \cos\eta\, G(\lambda, x) + \sin\eta\, S(\lambda, x)\,, \\ F(\lambda, \eta, x) = F(-\lambda, \eta, x)\,. \end{cases} \tag{1.4}$$

A slight generalized form of a theorem of Poincaré states that $F(\lambda, \eta, x)$, for fixed η and x, is an entire function of λ. If $U(x) = 0(x^{-2+2C})$, $C > 0$ (*) small x, then $\varphi(\lambda, x)$ is analytic in $R(\lambda) > -C$ if $C < 1$ and in $R(\lambda) > -1$ if $C > 1$. Therefore $\varphi(\lambda, x)$ and $\varphi(-\lambda, x)$ have a common domain of analyticity in the strip: $|R(\lambda)| < C$, (< 1).

For those values for which $\varphi(\lambda, x)$ exists we must have a relation of the kind:

$$\varphi(\lambda, x) = 2\lambda\left[C(\lambda) S(\lambda, x) - S(\lambda) C(\lambda, x)\right]\,, \tag{1.5}$$

$C(\lambda)$, $S(\lambda)$ depend on λ only and are analytic in $R(\lambda) > -C(-1)$ with the possible exception of $\lambda = 0$, where there may be a simple pole. We shall not

(*) In the following we shall use, when there is no danger of confusion, the same letter C for quite different constants in different formulas.

reproduce here the details of the proofs of all these statements which are mostly by-products of more careful estimates contained in the Appendix.

We now observe that the Wronskian of any two solutions of (1.1) is a constant. We have for instance from the small x limit:

$$\varphi(\lambda, x)\varphi'(-\lambda, x) - \varphi(-\lambda, x)\varphi'(\lambda, x) = -2\lambda = W[\varphi(\lambda, x), \varphi(-\lambda, x)] . \tag{1.6}$$

Similarly:

$$W[C(\lambda, x), S(\lambda, x)] = 1 . \tag{1.7}$$

Introducing (1.5) into (1.6) and using (1.7) we find:

$$C(\lambda)\, S(-\lambda) - S(\lambda)\, C(-\lambda) = \frac{1}{2\lambda} . \tag{1.8}$$

This identity is of paramount importance in our theory. From it and (1.5) we can write $C(\lambda, x)$ and $S(\lambda, x)$ in terms of $\varphi(\lambda, x)$ and $\varphi(-\lambda, x)$:

$$\begin{cases} C(\lambda, x) = C(\lambda)\varphi(-\lambda, x) + C(-\lambda)\varphi(\lambda, x) , \\ S(x, \lambda) = S(\lambda)\varphi(-\lambda, x) + S(-\lambda)\varphi(\lambda, x) . \end{cases} \tag{1.9}$$

From (1.9) if $R(\lambda) > 0$ we deduce:

$$C(\lambda) = \lim_{x\to 0} x^{\lambda-\frac{1}{2}}C(\lambda, x) \qquad \text{and similarly for } S(\lambda) .$$

Finally we list here the unperturbed functions:

$$\begin{cases} \varphi^0(\lambda, x) = x^{\frac{1}{2}}\Gamma(\lambda+1)2^\lambda J_\lambda(x) , \\ C^0(\lambda, x) = \sqrt{\frac{\pi}{2}}\, x^{\frac{1}{2}} \dfrac{1}{2\cos(\pi\lambda/2)}[J_\lambda(x) + J_{-\lambda}(x)] , \\ S^0(\lambda, x) = \sqrt{\frac{\pi}{2}}\, x^{\frac{1}{2}} \dfrac{1}{2\sin(\pi\lambda/2)}[J_\lambda(x) - J_{-\lambda}(x)] , \\ C^0(\lambda) \quad = \sqrt{\frac{1}{2\pi}}\, 2^\lambda \sin\dfrac{\pi\lambda}{2}\, \Gamma(\lambda) , \\ S^0(\lambda) \quad = \sqrt{\frac{1}{2\pi}}\, 2^\lambda \cos\dfrac{\pi\lambda}{2}\, \Gamma(\lambda) . \end{cases} \tag{1.10}$$

2. – If $\lambda = j + \frac{1}{2}$, $\varphi(\lambda, x)$ is the only solution which satisfies the boundary conditions required for the physical interpretation. The phase δ is defined

then from its asymptotic behaviour at large distances:

$$(2.1)\qquad \begin{cases} \varphi(\lambda, x) \simeq 2\lambda T(\lambda) \cos\left[x - \dfrac{\pi\lambda}{2} + \delta(\lambda)\right], \\ T^2(\lambda) \;\; = C^2(\lambda) + S^2(\lambda)\,. \end{cases}$$

We define $\delta(\lambda)$ through (2.1) also when λ is generally complex. Comparing (2.1) with (1.5) we find:

$$(2.2)\qquad K(\lambda) = \frac{S(\lambda)}{C(\lambda)} = -\operatorname{ctg}\left[\frac{\pi\lambda}{2} - \delta(\lambda)\right].$$

The so defined $\delta(\lambda)$ will be hereafter referred to as the « interpolation » of the phase-shifts in the physical points. Not all analytic functions which interpolate the phase shifts can be generated by a potential. Some necessary conditions are:

a) $K(\lambda)$ is regular analytic in $R(\lambda) > 0$ with the exception of simple poles.

b) $K(\lambda)$ is an hermitian function, that is $K(\lambda) = [K(\lambda^*)]^* = K^\dagger(\lambda)$. (Clearly this follows from $C(\lambda)$, $S(\lambda)$ being also hermitian), if λ is real then $K(\lambda)$ is also real.

a) and *b*) are by no means sufficient. To see it let us evaluate the integral (see Appendix):

$$(2.3)\qquad P(\lambda) = \int_0^\infty |\varphi(\lambda, x)|^2 \frac{\mathrm{d}x}{x^2} > 0 \qquad R(\lambda) > 0\,.$$

(Notice that $P(\lambda)$ is *not* an analytic function of λ!).

The result can be stated as follows:

$$(2.4)\qquad \begin{cases} P(\lambda) = 2\lambda[C(\lambda)S'(\lambda) - C'(\lambda)S(\lambda)] = 2\lambda T^2(\lambda)\left[\dfrac{\pi}{2} - \delta'\right] > 0\,, & \lambda \text{ real} > 0, \\ P(\lambda) = \dfrac{2\lambda^2}{R(\lambda)I(\lambda)}\cdot|T^2(\lambda)|\sinh[\pi I(\lambda) - 2I(\delta)] > 0\,, & \lambda \text{ complex } K(\lambda) > 0. \end{cases}$$

The following inequalities can be therefore derived:

$$(2.5)\qquad \begin{cases} c) \quad \dfrac{\mathrm{d}\delta(\lambda)}{\mathrm{d}\lambda} < \dfrac{\pi}{2} & \lambda > 0 \text{ real}, \\ d) \quad I(\lambda)[\pi I(\lambda) - 2I(\delta)] > 0 & I(\lambda) \neq 0\,. \end{cases}$$

Consider now the function $W(L)=K(\lambda)$, $L=\lambda^2$. The half-plane $R(\lambda)>0$ is mapped into the whole L plane cut along the negative real axis (cut plane). $a)\dots d)$ can be then translated into properties of $W(L)$

a) $W(L)$ is regular analytic in the cut plane with the exception of simple poles.

b) $W(L)=W^+(L)$. In particular W is real if L is real >0.

c) if L real >0 then $\mathrm{d}W/\mathrm{d}L>0$ if the derivative is defined.

d) $W(L)$ is a following function in the sense of WIGNER, that is $I(W)$ and $I(L)$ have always the same sign, moreover W is real only if L is real and it can have poles or zeros only on the real axis.

All these properties agree in characterizing $W(L)$ as Wigner's function of L (1), slightly generalized in the sense that continuous distributions of singularities are admitted on the cut while Wigner's function $R(E)$ was meromorphyc. Cond. $a)\dots d)$ restrict considerably the class of interpolations but they are not yet sufficient since some limitation has still to be imposed on the growth of W or K for large λ. We shall derive them in the next part. We only observe that *c*) implies a limitation also on the physical phase shifts. Indeed let us integrate (2.5) *c*) between $\lambda=j+\frac{1}{2}$ and $\lambda=j+\frac{3}{2}$; we find:

$$\delta_{j+1}-\delta_j<\frac{\pi}{2}\,. \tag{2.6}$$

This is a weak but simple condition on the phase shifts in order that they may be produced by a potential.

3. – If λ is large we expect the perturbation created by $U(x)$ to decrease so that perturbed functions will eventually approach the unperturbed at ∞. Although this statement is in general true there is a number of cases in which it is grossly false so that great care has to be taken in deriving results along this line. For instance it is not generally true that $\delta(\lambda)$ vanishes in the limit of large λ. We shall see that this holds for real λ only, if no condition is imposed on $U(x)$, being valid along any ray in $\pi/2\geqslant\arg\lambda\geqslant-\pi/2$ for a very special class of potentials. We shall not bother here with a detailed explanation of how these results are derived. The general procedure is the following. We transform (1.1) into an equivalent Green's integral equation of the Volterra type in which the boundary conditions as already included. Suitable upper bounds are then placed upon the sum of the perturbative expansion using a Titchmarsh's lemma. In the Appendix we show the essential points in our procedure. It must be noted that sometimes the lack of simple formulae for

(1) E. WIGNER and J. NEUMANN: *Annals of Math.*, **59**, 418 (1954).

the Bessel functions of general complex order makes it impossible to derive some limit directly. It is then much better to derive the result, usually a condition on the growth of some analytic function in an angle, on the boundary of the domain and then to extend it in the inside using the Phragmen-Lindeloef theorem or its by-products. Using this technique it is possible to derive some more conditions on the interpolation which we state as follows:

$$\exp[-i\pi\lambda](\exp[2i\,\delta(\lambda)]-1)\to 0\,,\qquad -\frac{\pi}{2}<\arg\lambda\leqslant 0\,. \tag{3.1}$$

We can write it into the equivalent form

$$e):\quad \begin{cases} \lim\limits_{\lambda\to\infty}[K(\lambda)-K^0(\lambda)]=0\,, & \arg\lambda\neq 0,\\ \delta(\lambda)=O\left(\dfrac{1}{\lambda}\right). & \arg\lambda=0, \end{cases}$$

If $U(x)$ us specialized then we have stronger results.

For instance if $U(x)$ admits some bound of the kind: $(A>0,\ C>0,\ B>1)$:

$$U(x)<C\exp[-Ax^B]\,,\qquad x>x_0=\text{const.} \tag{3.2}$$

Then

$$|\ln(K(\lambda)-K^0(\lambda))|<-\left(1-\frac{1}{B}\right)2|\lambda|\ln|\lambda|+0(|\lambda|)\,.$$

This bound is essentially the one derived by Carter [2] for $\lambda=j+\frac{1}{2}$. If $B=1$, similar bounds of some use can be derived. A second way of restricting $U(x)$, of remarkable interest in connection with dispersion relations, is provided by the theorem:

« Let $U(x)$ admit an analytic continuation $U(z)$, $z=x+iy$, regular in the sector $\arg z<\sigma+\varepsilon<\pi/2$, and such that:

$$\int_0^{\infty e^{i\xi}}|z||dz||U(z)|<\infty\,,\qquad |\xi|\leqslant\sigma$$

is uniformly bounded within the sector, and moreover $U(z)=0(z^{-2+2C})$, $C>0$,

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([2]) D. S. Carter: *Thesis*, Princeton; N. Khuri: *Phys. Rev.* **107** 1148 (1957). We have had no opportunity of examining directly Carter's work and we know of his result only through Khuri's paper.

$\varepsilon \to 0$, in the sector, then:

$$|K(\lambda) - K^0(\lambda)| = O(\exp[-2\sigma I(\lambda)]) . \tag{3.3}$$

Potentials satisfying the conditions of this theorem will be named σ-potentials those which do not 0-potentials. »

The proof of this theorem is based upon the WKB method. Indeed if one takes for granted the first WKB approximation then the above results already follow.

(3.2) and (3.3) are in a way complementary conditions on $U(x)$, indeed the first implies an upper bound on $\delta(\lambda)$ along $\arg\lambda = 0$ and nothing along $\arg\lambda = \pm\pi/2$, the second one works the opposite way. In the derivation of (3.3) it is essential to have $\sigma < \pi/2$, correspondingly it can be proved that a higher value of σ cannot improve the bound. Actually no potential attains:

$$|K(\lambda) - K^0(\lambda)| = O(\exp[-(\pi+\varepsilon)|\lambda|]) , \quad \varepsilon > 0, \qquad \arg\lambda = \pm\frac{\pi}{2} . \tag{3.4}$$

This can be seen by applying Carleman's theorem to both sides of (3.4) or using Carlson's theorem on the function $F^+(\lambda)$ defined in Appendix IV. One can include $\pi/2$ in our discussion using the weaker statement that if some potentials is a σ-potential, where σ is any angle $< \pi/2$, then:

$$|K(\lambda) - K^0(\lambda)| = O(\exp[-(\pi-\varepsilon)I(\lambda)]) , \qquad \varepsilon > 0$$

ε can be taken small at will.

Finally we report some limit:

$$\lim_{\lambda\to\infty}\frac{S(\lambda)}{S^0(\lambda)} = \lim_{\lambda\to\infty}\frac{C(\lambda)}{C^0(\lambda)} = 1 \qquad \arg\lambda \neq 0, \qquad R(\lambda) > 0. \tag{3.5}$$

4. – We shall give here a method of construction of $U(x)$ from the interpolation $K(\lambda)$. We sketch here the essential points only without giving detailed proofs of our statements. We wish to point out also that some more research will be needed in order to cast the theory into a fully satisfactory and rigorous form. We believe, however, that this is more properly a mathematician's task and that what we are going to show here is already enough for the physicist's needs.

The starting point is, as in Gel'fand and Levitan's theory [3-5] an integral

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([3]) I. M. Gel'fand and Levitan: *Amer. Math. Soc. Trans.*, Sec. 2.1.250 (1955).
([4]) R. Jost and W. Kohn: *Math. Phys. Medd.*, **27**, (9) (1953).
([5]) L. D. Faddeev: *Soviet Phys. Dokl. Transl.*, **3**, 747 (1959).

equation of the Volterra type which relates $\varphi(\lambda, x)$ to $\varphi^0(\lambda, x)$ through a kernel which does not depend on λ:

$$\varphi(\lambda, x) = \varphi^0(\lambda, x) + \int_0^x \frac{K(x, y)\,\varphi^0(\lambda, y)}{y^2}\,\mathrm{d}y\,. \tag{4.1}$$

One can justify (4.1) from two points of view. The first is that by virtue of the asymptotic expansion (see Appendix):

$$\varphi(\lambda, x) = x^{\lambda+\frac{1}{2}}(1 + D) \qquad \text{where} \qquad D < \frac{Qx^2}{C|\lambda|}\,. \tag{4.2}$$

we have the Mellin representation:

$$\varphi(\lambda, x) = x^{\lambda+\frac{1}{2}} + \int_0^x H(x, y)\,y^{\lambda+\frac{1}{2}}\,\frac{\mathrm{d}y}{y^2}\,, \tag{4.3}$$

and of course a similar for $\varphi^0(\lambda, x)$. It can be shown that the latter can be solved in $x^{\lambda+\frac{1}{2}}$ and if this function is fed into (4.3) then (4.1) follows. The second way will be apparent later. The y^{-2} factor is merely inserted for symmetry.

In considering the functions $\varphi(\lambda, x)$ or $\varphi^0(\lambda, x)$ the natural problem arises whether they can be considered a complete and orthogonal set in the interval $0 \dots \infty$. This problem has been considered since long by H. WEIL and several other mathematicians. The reader will find a fully satisfactory overall view on this subject in [6]. We shall report here the results found with the help of [6]. The $\varphi(\lambda, x)$ are eigenfunctions of the linear operator: $(R(\lambda) \geqslant 0!)$

$$\Omega_x = x^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1 - U(x)\right), \tag{4.4}$$

subjected to the condition of being of class L^2 in $0 \dots \infty$ if we adopt the norm:

$$(\varphi, \psi) = \int_0^\infty \varphi(x)\,\psi^*(x)\,\frac{\mathrm{d}x}{x^2}\,. \tag{4.5}$$

(It must be clear that under this norm all solutions of (1.1) are L^2 in any interval $H \dots \infty$). Using the standard terminology we are in the so called

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[6] E. C. TITCHMARSH: *Eigenfunction Expansions*, associated with second order differential equations I, II. (Oxford, 1946).

limit circle case. If no other condition is implied our set is clearly overcomplete and not orthogonal. We must find a subset which satisfies these conditions. This can be accomplished by imposing another boundary condition at ∞, which cannot be square integrability as in quantum mechanics, because it is already satisfied. The new condition is that all eigenfunctions should have the same asymptotic phase μ or, in other words, they must be multiples of $F(\lambda, \mu, x)$. Now there are only two types of choices of λ for this to happen:

A) $R(\lambda) = 0$, we refer to it as the continuum

B) $R(\lambda) > 0$, and $F(\lambda) = C(\lambda)\cos\mu + S(\lambda)\sin\mu = 0$.

In this case

$$F(\lambda, \mu, x) = F(-\lambda)\,\varphi(\lambda, x)$$

μ is here real and $0 < \mu < \pi$; for each value of μ we have a different expansion. It can be checked that all functions in *A*, *B*) satisfy the orthogonality relations:

$$(4.6)\qquad \begin{cases} \big(F(\lambda, \mu, x),\, F(\lambda', \mu, x)\big) = 4\pi\,\delta(\lambda' - \lambda'^2)\, F(\lambda)\, F(-\lambda)\lambda\,, \\ \qquad\qquad\qquad \text{if}\ \ R(\lambda) = 0,\quad R(\lambda') = 0\,, \\ \big(F(\lambda, \mu, x),\, F(\lambda_n, \mu, x)\big) = 0 \qquad F(\lambda_n) = 0, \qquad F(\lambda_m) = 0, \\ \big(F(\lambda_n, \mu, x),\, F(\lambda_m, \mu, x)\big) = -\,\delta_{nm}\,\dfrac{1}{2\lambda F(\lambda_n, \mu - (\pi/2))\, F'(\lambda_n, \mu)}\,. \end{cases}$$

Moreover there are infinitely many zeros of $F(\lambda)$, all real. We have correspondingly the expansion theorem:

$$(4.7)\qquad \delta(x - y) = \frac{1}{2i\pi}\int_P \mathrm{d}\lambda\, \frac{\varphi(\lambda, x)}{x}\, \frac{F(\lambda, \mu, y)}{y\, F(\lambda, \mu)}\,.$$

The path P is shown in Fig. 1. This path can be deformed into $R(\lambda) = 0$, and to loops which enclose the zeros of $F(\lambda, \mu)$. From $R(\lambda) = 0$ we have the *A* terms, from the loops the *B* terms. Some remarks are needed for (4.7); it is clearly a symbolical formula and statements of the kind: the Dirac function in (4.7) is independent of μ, and so must be the second term, are in general meaningless or false since $\delta(x - y)$ is a distribution in a space whose definition depends on μ. However, under some

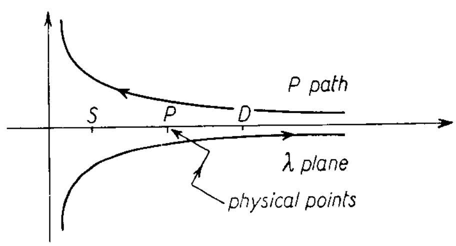

Fig. 1.

conditions, whose exact form will not be stated here, it is true that

$$(4.8)\qquad f(x)=\frac{1}{2\pi i}\int_P \mathrm{d}\lambda\, g(\lambda)\frac{F(\lambda,\mu,x)}{xF(\lambda,\mu)}\,,\qquad \text{if}\quad g(\lambda)=\int_0^\infty \mathrm{d}x\frac{1}{x}\,\varphi(\lambda,x)f(x)\,,$$

in particular (4.8) holds if $f(x)$ decreases fast enough. In (4.1) we have always $x>y$; we define $K(x,y)$ also when $y<x$ through the reversed equation:

$$(4.9)\qquad \varphi^0(\lambda,y)=\varphi(\lambda,x)-\int_0^y K(x,y)\,\varphi(\lambda,x)x^{-2}\,\mathrm{d}x\,.$$

Under these assumptions we find the following integral representation from (4.9), (4.1) and (4.7):

$$(4.10)\qquad K(x,y)=\frac{1}{2\pi i}\int_P \mathrm{d}\lambda\left[\frac{F^0(\lambda,\mu,x)}{F^0(\lambda,\mu)}\,\varphi(\lambda,x)-\frac{F(\lambda,\mu,x)}{F(\lambda,\mu)}\,\varphi^0(\lambda,y)\right].$$

From (4.10) one can verify the differential equation:

$$(4.11)\qquad \Omega_x K(x,y)=\Omega_y K(x,y)\,,$$

(4.1), (4.9) and (4.11) are consistent with (1.1) provided

$$(4.12)\qquad K(x,x)=\frac{x}{2}\int_0^x y\,U(y)\,\mathrm{d}y\,,\qquad K(x,y)=O(x)\,,\qquad O(y)\,,\qquad x,y\to 0.$$

Conversely (4.11), (4.12) can be used to define $K(x,y)$ as the solution of a certain differential partial equation with appropriate boundary conditions. The so defined $K(x,y)$, if fed into (4.1) generates a function which satisfies (1.1) with the appropriate boundary condition. This is a second way of justifying (4.1). By rewriting (4.10) as follows ($\mu=0$):

$$(4.13)\qquad K(x,y)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\mathrm{d}\lambda\left[\frac{C^0(-\lambda)}{C^0(\lambda)}-\frac{C(-\lambda)}{C(\lambda)}\right]\varphi(\lambda,x)\varphi^0(\lambda,y)\,+$$
$$+\frac{1}{2\pi i}\int_{\text{loops}}\mathrm{d}\lambda\frac{1}{2\lambda}\left[\frac{1}{C(\lambda)S(\lambda)}-\frac{1}{C^0(\lambda)S^0(\lambda)}\right]\varphi(\lambda,x)\varphi^0(\lambda,y)\,,$$

where the « loops » integral encloses the zeros of $C(\lambda)$ and $C^0(\lambda)$ only, avoiding those of $S(\lambda)$, $S^0(\lambda)$, and defining $Q(x, y)$ through the same formula where $\varphi(\lambda, x)$ has been replaced everywhere by $\varphi^0(\lambda, x)$ the integral equation follows:

$$K(x, y) = Q(x, y) + \int_0^x K(x, z)\, Q(x, y) \frac{dz}{z^2}\,. \tag{4.14}$$

If $K(x, y)$ is regarded as an unknown in (4.14) this equation is of the Fredholm type. It can be shown that the homogeneous counterpart of (4.14) has no non-trivial solutions and we assume therefore that (4.14) can be always solved under very large conditions. From $Q(x, y)$ we can deduce therefore $K(x, y)$ and $K(x, x)$ in particular. (4.12) yields then $U(x)$. Moreover, $Q(x, y)$ involves $\varphi^0(\lambda, x)$, which is an elementary function, and essentially the ratios $C(-\lambda)/C(\lambda)$, $S(-\lambda)/S(\lambda)$, having noticed from (1.8) that

$$\frac{C(-\lambda)}{C(\lambda)} - \frac{S(-\lambda)}{S(\lambda)} = -\frac{1}{2\lambda\, C(\lambda)\, S(\lambda)}\,.$$

The knowledge of these ratios implies that of $U(x)$. We are left with the task of connecting them to the interpolation $K(\lambda)$. To carry it out we observe that in virtue of $C(ia) = C(-ia)^*$ and $S(ia) = S(-ia)^*$ these ratios are simply exponential functions of arg $C(ia)$ and arg $S(ia)$. At the same time from (1.8) we know that:

$$K(\lambda) - K(-\lambda) = \frac{1}{2\lambda\, C(\lambda)\, C(-\lambda)}\,.$$

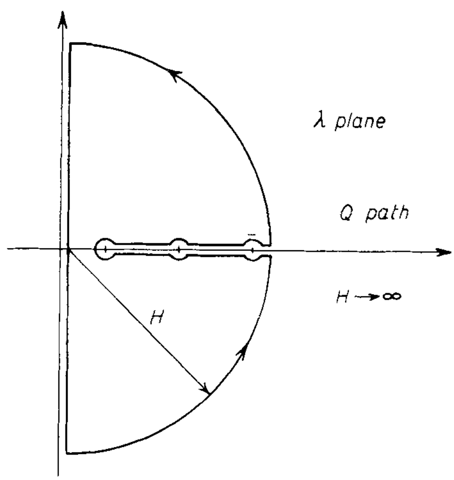

Fig. 2.

We can calculate $\ln|C(\lambda)|$ and similarly $\ln|S(\lambda)|$ from the interpolation. However, owing to (3.5) arg $C(\lambda)/C^0(\lambda)$ and $\ln|C(\lambda)/C^0(\lambda)|$ if $\lambda = ia$ are in some sense conjugate functions if one forgets about the zeros of $C(\lambda)$ and $C^0(\lambda)$. We can still relate them by taking the real and imaginary part of Cauchy's formula:

$$\frac{1}{2\pi i}\int_Q \frac{1}{\lambda - ia - \varepsilon} \ln \frac{C(\lambda)}{C^0(\lambda)}\, d\lambda = \ln \frac{C(ia)}{C^0(ia)}\,, \qquad \varepsilon > 0 \text{ small.} \tag{4.15}$$

Q means here a path which avoids the branch points of the integrand, as shown in Fig. 2. The result expresses arg C/C_0 as an Hilbert transform of $\ln|C/C_0|$

with an infinite series of additional terms which depend on the location of the branch points, which in turn are perfectly known from $K(\lambda)$. This series converges because the zeros of $C(\lambda)$ and those of $C^0(\lambda)$ tend to be very close when λ is large and to cancel each other. Similarly one proceeds with $S(\lambda)$. We have completed the chain from $K(\lambda)$ up to $U(x)$. The only problem left now is the actual construction of $K(\lambda)$ from the values that it takes at $\lambda = j+\frac{1}{2}$. This problem is still unsolved although we are well on the way to do it. We shall discuss it in the next part.

5. – We shall give here some heuristic arguments on the interpolation problem. Our starting point is Carleman's theorem [5]:

$$(5.1)\quad \sum_n \left(\frac{1}{h_n} - \frac{h_n}{H^2}\right)\sin\theta_n - \sum_m \left(\frac{1}{k_m} - \frac{k_m}{H^2}\right)\sin\theta_m = \frac{1}{\pi H}\int\limits_{-\pi/2}^{\pi/2} \ln|L(He^{i\theta})|\cos\theta\,\mathrm{d}\theta + \\ + \frac{1}{2\pi}\int\limits_0^H \left(\frac{1}{a^2} - \frac{1}{H^2}\right)\ln|L(ia)\,L(-ia)|\,\mathrm{d}a + \frac{1}{2}\,R[L'(0)]\,,$$

$L(\lambda)$ is in this theorem a general analytic function of λ; regular in $R(\lambda) > 0$ with the exception of poles in $k_m \exp[i\theta_m]$ and having zeros at $h_n \exp[i\theta_n]$. Suppose now that we have two interpolations of the same set of phase shifts: $K(\lambda)$ and $H(\lambda)$. The difference $K(\lambda) - H(\lambda)$ must have zeros at least in $\lambda = j+\frac{1}{2}$ and poles at most at the poles of $K(\lambda)$ and $H(\lambda)$. (We cannot exclude that $K(\lambda)$ and $H(\lambda)$ coincide elsewhere and that either a new zero arises or two poles coalesce). The poles are distributed with the density 1. Supposing $L(\lambda) = K(\lambda) - H(\lambda)$ the contributions of zeros and poles in (5.1) are nearly opposite and of the order of $\ln H +$const. Their algebrical sum will tend to a constant limit. What can be said about the contribution of the boundary? For the sake of simplicity let us restrict ourselves to $H(\lambda) = K^0(\lambda)$ so that the problem now is to find a potential (in the following $V^0(x)$) which produces no scattering at a given energy. We have now some very useful estimate of the decrease of $L(\lambda)$ on the boundary. Quite generally $L(\lambda)$ will decrease in such a fashion as to make $Y(H)$ eventually negative and $J(H)$ decreasing. If, however, condition (3.2) is used we see that $Y(H)$ can be made arbitrarily large and negative if H is chosen sufficiently large, so that unless $K(\lambda) = K^0(\lambda)$ we face a contradiction in (5.1). Similarly, if $V^0(x)$ were a σ-potential $J(H)$ could be made arbitrarily large and negative and (5.1) would be again an absurdity. We see therefore that $V^0(x)$ must be a σ-potential which does not decrease faster than any exponential. The usual field theoretical potentials are therefore excluded. These results can be generalized and it can be shown that there is at most one σ-potential which yields a given set of phase shifts at a given energy. In some sense we find unicity in the inversion problem under rather broad conditions, that is, analiticity requirements in an arbitrarily small angle.

A specific example of $V^0(x)$ can be constructed as follows. We take $H^0(\lambda) = (K^0(\lambda) - 1)/(K^0(\lambda) + 1)$, $H^0(\lambda)$ is again a Wigner's function of λ^2 and it satisfies therefore all the same requirements of $K^0(\lambda)$. It has alternatively zeros and poles in the physical points. If $H(\lambda) = (K(\lambda) - 1)/(K(\lambda) + 1)$ then the ansatz $H(\lambda) = H^0(\lambda) \cdot [1 + (C/(\lambda^2 - \lambda_0^2))]$, where C and λ_0 are constants, still satisfies the correct properties and yields vanishing phases since both $H(\lambda)$ and $H^0(\lambda)$ take the same values at the physical points. λ_0 and C are not entirely arbitrary otherwise $H(\lambda)$ may be somewhere a non-increasing function. Since it is enough here to show the existence of at least two different interpolations we shall be satisfied with one example only which is provided by taking $H^0(\lambda) < 0$ and C small as readily checked. The general problem is still unsolved and we hope to tackle it in a forthcoming paper. In analogy to Bargman's potentials which are solvable for all energies but for a single partial wave it is possible to give potentials which are solvable for all values of the angular momentum at a fixed energy. Since these potentials are a rather academic case we shall deal with them in some future work.

6. – In this part we shall establish some results in the field of dispersion relations. As well-known these relations are statements of analiticity of the scattering amplitude as function of the energy and the transmitted momentum. Although here the energy is kept fixed it is still possible to derive for special classes of potentials enough properties as to guarantee for the existance of such relations. We restrict ourselves to σ-potentials ($\sigma \neq 0$) and moreover we impose some additional condition on $U(x)$ as follows:

$$|U(x)| < C \frac{\exp[-\alpha x]}{x}, \qquad x > x_0. \tag{6.1}$$

Correspondingly, the phase shift will decrease like $\exp[-\alpha'\lambda]$ along the real λ, where $\alpha' = \ln(1 + (\alpha^2/2)) < \alpha$.

Under these conditions we have the following asymptotic estimate:

$$\exp[2i\,\delta(\lambda)] - 1 \sim O(\exp[i(\pi - 2\sigma)\lambda - \alpha'\lambda]), \qquad -\frac{\pi}{2} \leqslant \arg\lambda \leqslant 0\,. \tag{6.2}$$

In these rough extimates we have neglected powers of λ which are not essential in our discussion. (6.2) is very useful if the Legendre expansion of the scattering amplitude:

$$f(z) = f(\cos\theta) = \frac{1}{2i}\sum_{j=0}^{\infty}(2j+1)(\exp[2i\delta_j] - 1)\,P_j(z)\,, \qquad z = \cos\theta, \tag{6.3}$$

is transformed into the integral:

$$f(z) = \frac{1}{2}\int \lambda \, d\lambda (\exp[2i\,\delta(\lambda)] - 1) \frac{1}{\cos \pi\lambda} P_{\lambda-\frac{1}{2}}(-z) . \tag{6.4}$$

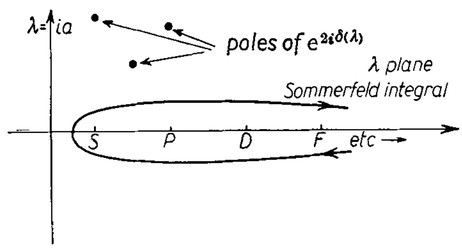

Fig. 3.

This artifice is due to WATSON and it was used by SOMMERFELD (7) in some wave propagation problems. The integration path is shown in Fig. 3. It encloses the zeros of $\cos \pi\lambda$ and avoids the poles of the integrand in the upper quadrant. The important fact about it is that it may converge outside the customary Legendre ellipse. To prove it, one needs an asymptotic expansion of Legendre functions of large order:

$$P_{\lambda-\frac{1}{2}}(z) \sim \frac{1}{\sqrt{2\pi \sin\theta \cdot \lambda}} \exp[\pm i\lambda(\theta - \pi)] . \tag{6.5}$$

One must choose the sign which yields the larger result. From (6.5) and (6.2) one sees that it is possible to deform the path in the lower quadrant into $\lambda = ia$, $a < 0$, provided one has $(\theta = \theta_0 + i\theta_1)$

$$\theta_1 < \alpha' \qquad \text{and} \qquad \theta_0 > \pi - 2\sigma . \tag{6.6}$$

If the first condition is dropped the lower part of the deformed integral still makes sense but the original expansion diverges. We have to deal now with the upper part where we did not apply immediately the same artifice because of the poles in the integrand. From (6.2) we see that:

$$\exp[2i\,\delta(\lambda)] \to 1 \qquad \text{if} \qquad 0 < -\arg\lambda < \xi_0 ,$$

where

$$\operatorname{tg} \xi_0 = \frac{\alpha'}{\pi - 2\sigma} .$$

Correspondingly, the same limit holds in $0 < \arg\lambda < \xi_0$.

It follows that within this sector there is at most a finite number of poles

(7) A. SOMMERFELD: *Partielle Differential-Gleichungen der Physik* (Leipzig, 1947), p. 285.

of $\exp[2i\delta(\lambda)]$. The upper part of the path can be deformed along $\arg\lambda=\xi_0$, and the poles in the sector yield a finite sum of residues which are analytic functions (Legendre functions) of z, in the z plane cut along z real >1.

If we now try to shift the upper path along $\lambda=ia$, $a>0$ we find that, apart from the contribution of poles, (6.4) still makes sense provided $\theta_0>0$ which is included in (6.6). The infinitely many poles yield a converging series if the values of the integral along an arc of arbitrarily large radius A indented in $\lambda=iA$ and $\lambda=A\exp[i\xi_0]$, vanishes for $A\to\infty$. This can be achieved if the condition

$$\theta_1<\frac{\alpha'}{\pi-2\sigma}\left[4(\pi-\sigma)-\theta_0\right], \tag{6.7}$$

is satisfied. Now particularly interesting are the values $\theta=\pi+i\theta_1$ which correspond to z real <-1 or in the dispersion relation language to $\tau>2k$, τ being the transmitted momentum and k the wave number, $(x=kr)$. If σ can be taken arbitrarily near to $\pi/2$ then θ_1 may be arbitrarily large and correspondingly there is analyticity in z along the whole negative axis and, for arbitrary large real values of τ.

This result holds in particular for all those potentials which can be included into the formula

$$xU(x)=\int_{\alpha}^{\infty}\exp[-px]f(p)\,\mathrm{d}p\,, \tag{6.8}$$

Now from the previous literature (see [2,8]) one knows that $f(k,\tau)$ is, for τ fixed real, and an analytic function of k in the whole upper plane $I(k)>0$ (*) approaching $f_B(\tau)$ (Born approximation) for large k, if $\tau<2\alpha$. If the latter condition is not satisfied, a domain D arises where analyticity cannot be proved with the usual methods. This domain $D=D(\tau)$ however can be enclosed in a sufficiently large semicircle with center in $k=0$. Moreover, $f(k,\tau)$ still approaches $f_B(\tau)$ when k is large. We know that $f(k,\tau)$, if k is real, is analytic in a region which in terms of $z=1-(\tau^2/2k^2)$ is the sum of the interior of the ellipse of convergence and of the domain defined in (6.7) and the second of (6.6). In particular if k is kept fixed real and $\neq 0$, $f(k,\tau)$ is an analitic function of τ for real arbitrary positive values of τ and in a suitable neighbourhood of them. The same result can be proved with a limiting procedure when $k=0$. It is remarkable then that analyti-

(8) S. Gasiorowicz and H. P. Noyes: *Nuovo Cimento*, **10**, 78 (1958).

(*) Bound states will not be considered here. However these can be easily included into the theory.

city follows in the whole $I(k) > 0$ plane. Indeed one can take a semicircle enclosing $D(\tau)$, where $\tau < T =$ constant. Applying Cauchy's theorem to this semicircle we get

$$f(k, T) = \frac{1}{2\pi i}\int_{\frown} \frac{f(k, T)}{h-k}\,\mathrm{d}h\,. \tag{6.9}$$

If $\tau > 2\alpha$ this formula still defines a function, analytic in k and τ, the latter analyticity follows from the fact that all values on the boundary are analytic functions of τ. (6.9) defines a function also within D which must be the analytic continuation of the original function because it coincides with it outside D where is was defined. Take namely k outside $D(\tau)$ but inside the semicircle and split the semicircle into two closed loops, the first enclosing D but not k, the latter k but not $D(\tau)$. The contribution of the first is an analytic function of τ which vanishes if $\tau < 2\alpha$ and therefore always the last just yields the original function. The result follows. The above arguments are standard ideas from the theory of several complex variables. For special potentials one gets still stronger results. Indeed suppose that $|V(z)| < H/z^2$ within and along the boundary of $|\arg z| \leqslant \pi/2$. Let λ^0 be a pole of $\exp[2i\,\delta(\lambda)]$. From the discussion above we know that $R(\lambda^0) > 0$, $I(\lambda^0) > 0$. Moreover $\varphi(\lambda^0, z) \sim C\exp[iz]$ if $z \to \infty$. Also

$$\varphi''(\lambda^0, z) + \varphi(\lambda^0, z) - \frac{\lambda_0^2 - \frac{1}{4}}{z^2}\,\varphi(\lambda^0, z) - U(z)\,\varphi(\lambda^0, z) = 0\,. \tag{6.10}$$

Take now $z = iy$; y real. Then $\varphi \sim C\exp[-y]$ and

$$\ddot{\varphi} - \varphi - \frac{\lambda^{02} - \frac{1}{4}}{y^2}\,\varphi - U(iy)\,\varphi = 0\,, \tag{6.11}$$

the dots now refer to y derivatives. The conjugate reads

$$\ddot{\varphi}^* - \varphi^* - \frac{\lambda^{0*2} - \frac{1}{4}}{y^2}\,\varphi^* - U(-iy)\,\varphi^* = 0\,. \tag{6.12}$$

We multiply now (6.11) by φ^* and subtract (6.12) multiplied by φ. The resulting equation is then integrated between $0 \ldots \infty$. We obtain

$$(\lambda^{02} - \lambda^{0*2})\int_0^\infty \frac{|\varphi|^2}{y^2}\,\mathrm{d}y = -2i\int_0^\infty I(U(iy))\,|\varphi|^2\,\mathrm{d}y\,. \tag{6.13}$$

If $I(U(z))$ is replaced by M/y^2 we obtain

$$R(\lambda^0)\,I(\lambda^0) < \frac{M}{2}\,. \tag{6.14}$$

This is a restriction on the position on poles: this result enables us to say that only a finite number of poles have $R(\lambda^0) > m+\frac{1}{2}$ where $m+\frac{1}{2}$ is some positive constant. In the Sommerfeld integral one can use $R(\lambda) = m+\frac{1}{2}$ as path, the contribution of the extra poles can be evaluated and yields a finite sum which diverges at most like some power of $-\cos\theta$ or τ when those quantities are large. The contribution of the $R(\lambda) = m+\frac{1}{2}$ path can be best evaluated by using (see App. V) asymptotic expansions of Legendre functions:

$$(6.15)\quad \begin{cases} f(\cos\theta) = \dfrac{1}{2\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty}\dfrac{\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})\cos\pi\lambda}\,\zeta\,d\zeta\,\{\exp[2i\,\delta(\lambda)]-1\}\cdot \\ \qquad\qquad\qquad\qquad \cdot\exp[i\zeta\theta_1+\zeta(\theta_0-\pi)]\cdot(-2\cos\theta)^m\,, \\ \lambda = m+i\zeta+\frac{1}{2}+\text{contr. extra poles.} \end{cases}$$

This contribution is therefore seen to be growing at most like some power of $-\cos\theta$ or τ when these variables are large in modulus (the cut $\cos\theta$ real > 1 being excluded). The Mandelstam representation can then be derived.

* * *

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Appendix I

In order to calculate $P(\lambda)$ we notice that $\varphi(\lambda, x)^* = \varphi(\lambda^*, x)$ and that :

$$(\text{A.1})\qquad \varphi(\lambda, x)\,\varphi''(\lambda^*, x) - \varphi''(\lambda, x)\,\varphi^*(\lambda, x) = \frac{1}{x^2}(\lambda^{*2}-\lambda^2)\,|\varphi(\lambda, x)|^2\,,$$

or if λ is real:

$$(\text{A.2})\qquad \varphi(\lambda, x)\frac{\partial}{\partial\lambda}\varphi''(\lambda.\,x) - \varphi''(\lambda, x)\frac{\partial}{\partial\lambda}\varphi(\lambda, x) = \frac{2\lambda}{x^2}\varphi(\lambda, x)^2\,.$$

Integration of these indentities between $0 \ldots \infty$ yields the desired formulas. The asymptotic behaviour of $\varphi(\lambda, x)$ for large x can be obtained through (15) and (1.3).

APPENDIX II

Most of the bounds used in this paper can be derived from suitable integral equations of the Volterra type and from the following lemma (5):

Lemma: Let $f(x) \geqslant 0$, $g(x) \geqslant 0$ and let $f(x)$ be continuous $g(x)$ integrable in $0 \leqslant x \leqslant X$. Let $f(x) < C + \int_0^x f(t)\, g(t)\, dt$, $0 \leqslant x \leqslant X$. Then $f(x) < C \exp\left(\int_0^x g(t) dt\right)$ $0 \leqslant x \leqslant X$.

This lemma will be used also in the interval $X \ldots \infty$ with obvious modifications. We list here the integral equations which we have used and the functions which in our derivation correspond to $f(x)$ and $g(x)$.

$$\text{(I)} \qquad \varphi(\lambda, x) = x^{\lambda+1} + \frac{1}{2\lambda}\int_0^x (U(y) - 1)\, \varphi(\lambda, y)\left[\frac{x^{\lambda+1}}{y^\lambda} - \frac{y^{\lambda+1}}{x^\lambda}\right] dy\,.$$

We have if $R(\lambda) > 0$: $(y < x)$

$$\left|\frac{x^{\lambda+1}}{y^\lambda} - \frac{y^{\lambda+1}}{x^\lambda}\right| < 2\left|\frac{y}{x}\right|^\lambda |x|\,;$$

we take $f(x) = |\varphi(\lambda, x) x^{-\lambda-1}|$ and $g(x) = |U(x) - 1|\, |x| (1/|\lambda|)$: the result is essentially (4.2). A similar bound holds for $\varphi'(\lambda, x)$

$$\text{(II)} \qquad C(\lambda, x) = \cos\left(x - \frac{\pi}{4}\right) + \int_x^\infty \left[\frac{\lambda^2 - \frac{1}{4}}{y^2} + U(y)\right] C(\lambda, y) \sin(y - x)\, dy\,;$$

here we replace the sines and cosines with 1 and take $C = 1$, $f(x) = |C(\lambda, x)|$, $g(x) = |((\lambda^2 - \frac{1}{4})/x^2) + U(x)|$. We obtain a result of the form:

$$|C(\lambda, x)| < C \exp\left[\frac{\lambda^2}{x}\right], \qquad \lambda \text{ large.}$$

Similarly we find a corresponding bound on $C'(\lambda, x)$ and on $S(\lambda, x)$ and $S'(\lambda, x)$. Introducing these bounds in the identity:

$$2\lambda\, C(\lambda) = C(\lambda, x)\, \varphi(\lambda, x) - C'(\lambda, x)\, \varphi(\lambda, x)$$

and taking $x = \lambda$ we find

$$\text{(A.3)} \qquad |C(\lambda)| < |\lambda^\lambda C^\lambda|\,,$$

where C is some constant. This bound, although very rough, can be sharpened with the help of Phragmen-Lindeloef's theorem (9):

$$\text{(III)} \qquad C(\lambda, x) = C^0(\lambda, x) + \int_x^\infty U(y)\, C(\lambda, y)[C^0(\lambda, x)\, S^0(\lambda, y) - C^0(\lambda, y)\, S^0(\lambda, x)]\, dy \,.$$

We consider this equation for $\lambda = ia$ (a real) only, since otherwise no reliable bound on the unperturbed solutions is available. We know from the Poisson integral representation of Bessel functions that:

$$C^0(\lambda, x)\,, \quad S^0(\lambda, x) < E\sqrt{\pi x}\,, \qquad \lambda = ia,$$

where $E > 1$ is a suitable constant.

The bound holds uniformly in x, λ since E does not depend on them. We take $C^0(\lambda, x) = f(x)E\sqrt{\pi x}$, $g(x) = 2xU(x)\,\pi E^2$ and since $\int_0^\infty x|U(x)| = Q < \infty$, we have:

$$\text{(A.4)} \qquad C(\lambda, x) < E\sqrt{\pi x}\, \exp[2\pi E^2 Q] = E'\sqrt{\pi}\, x\,,$$

which is of the same type. The same results follows for $S(\lambda, x)$,

$$\text{(IV)} \quad \varphi(\lambda, x) = \varphi^0(\lambda, x) + \frac{1}{2\lambda}\int_0^x U(y)\,\varphi(\lambda, y)[\varphi^0(\lambda, x)\varphi^0(-\lambda, y) - \varphi^0(\lambda, -x)\,\varphi^0(\lambda, y)]\, dy.$$

As in (III) we suppose $\lambda = ia$. We have the preliminary inequality:

$$\varphi^0(\lambda, x) < F\sqrt{\lambda\pi x} \qquad F \text{ const.}$$

we take then $\varphi(\lambda, x) = f(x)F\sqrt{\lambda\pi x}$ and $g(x) = 2xU(x)\pi F^2$ and as before we derive a bound of the kind:

$$\text{(A.5)} \qquad \varphi(\lambda, x) < F'\sqrt{\lambda\pi x} \qquad F' \text{ is a constant.}$$

From (A.5) and the formula:

$$\text{(A.6)} \qquad C^0(\lambda) + \frac{1}{2\lambda}\int_0^\infty U(x)\, C^0(\lambda, x)\,\varphi(\lambda, x)\, dx = C(\lambda)\,.$$

we can prove (3.5) along $R(\lambda) = 0$, $a \to \infty$. Indeed if we split the interval of integration in $0 \dots \sqrt{a}$ and $\sqrt{a} \dots \infty$ we find the following bound using (A.5):

$$\int_{\sqrt{a}}^\infty U(x)\, C^0(\lambda, x)\,\varphi(\lambda, x)\, dx < \pi E F'\sqrt{|\lambda|}\int_{\sqrt{|\lambda|}}^\infty x\,|U(x)|\, dx\,.$$

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(9) R. P. BOAS: *Entire Functions* (New York, 1955).

This part is $o(\sqrt{\lambda})$ and negligible $[C^0(\lambda)$ is $o(\sqrt{\lambda^{-1}})]$. In $0 \dots \sqrt{a}$ we expand $C^0(\lambda, x)$ following (1.9) and use (4.2). If $0 < x < \sqrt{a}$ from (4.2) we deduce:

$$\varphi(\lambda, x) = x^{\lambda+1}(1 + O(1)) \qquad \lambda \text{ large.}$$

Two terms arise from (A.6):

$$C^0(\lambda)\int_0^{\sqrt{a}} xU(x)(1 + O(1)) ,$$

and

$$C^0(-\lambda)\int_0^{\sqrt{\lambda}} xU(x)x^{2\lambda}(1 + O(1))\,\mathrm{d}x .$$

Both are negligible. The limit follows. We have shown here the simplest obtainable bounds. A more refined estimate can be derived using the WKB method but we deliver it to the next Appendix.

Appendix III

The WKB method gives fairly simple estimates of the wave function when λ is large but it is difficult to estimate the error when λ and x are simultaneously complex. Fortunately it is enough for our purposes to take $\lambda = ia$ and x complex. In this case, provided a is large enough, there are no turning points if $U(x)$ is a σ-potential. Indeed it is clear that in this case $U(z)$ admits the Cauchy's rapresentation:

$$zU(z) = \frac{1}{2\pi i}\int_{\infty \exp[-i\sigma']}^{\infty \exp[i\sigma']} \frac{z'U(z')}{z - z'}\,\mathrm{d}z' \qquad \sigma' = \sigma + \varepsilon.$$

The integration path is stretched along the two half-lines $+\infty \exp[-i\sigma']\dots 0$ and $0 \dots \infty \exp[i\sigma']$. Now within $|\arg z| \leqslant \sigma$ we have $|z - z'| \geqslant z \sin\varepsilon$ and clearly:

$$|zU(z)| < \frac{M}{|z|\sin\varepsilon} \qquad \text{or} \qquad U(z) < \frac{M}{z^2 \sin\varepsilon} ,$$

were $M = (1/\pi)\int_0^{\infty \exp[i\sigma]} |z||\mathrm{d}zU||(z)|$. Similarly

$$U^{(n)}|z| < \frac{M_n}{z^{2+n}}; \qquad M_n \text{ const.}$$

In order to avoid turning points the function $a^2+z^2+z^2U(z)$ must have no zeros in the sector B $|\arg z|<\sigma$. Suppose now that there is Z such that $a^2+Z^2U(Z)=-Z^2$. We have $Z^2U(Z)<(M/\sin\varepsilon)=M'$. Now $Z^2=-a^2+pe^{i\xi}$, where $p<M'$ and if a is taken large enough

$$z\sim\pm\left(ia+\frac{p\exp[i\xi]}{2ia}\right)\quad\text{and}\quad\arg z\sim\pm\frac{\pi}{2}.$$

Hence Z will eventually fall out of the sector Σ which will be free of zeros. If we define $s^2(a,z)=a^2+z^2+z^2U(z)$ one can prove with similar arguments that there exists, for sufficiently large a, a constant C such that:

(A.7) $$|s(z)|>Ca \qquad (C \text{ ind. of } a \text{ and } z).$$

It follows that if we defines $\xi(z)$ through the equations:

$$\frac{\mathrm{d}\xi}{\mathrm{d}z}=\frac{S(a,z)}{z},\qquad \lim_{z\to a}(\xi(z)-a\ln z)=0\qquad z\text{ in }\Sigma,$$

$\xi(z)$ is a single valued regular analytic function of z in Σ. In particular

$$\xi^0(z)=a\ln z-a\ln(a+\sqrt{a^2+z^2})+\sqrt{a^2+z^2}-a+a\ln 2a\,.$$

From (A.7) one can prove then

(A.8) $$|\xi(z)-\xi^0(z)|<\frac{H}{a}\,,$$

where H is independent of z and a. All these inequalities require explicitly $|\sigma|<\pi/z$ equality beeing excluded. Let us define also $w(\xi)=\sqrt{s(z)}\cdot\psi(z)$, where $\psi(z)$ is any solution of (1.1). (1.1) transforms then into:

(A.9) $$\frac{\mathrm{d}^2w}{\mathrm{d}\xi^2}+w=-J(z)\,w\,,$$

where

$$J(z)=\frac{1}{4s^2}\left\{\frac{5}{4}\left[z\frac{\mathrm{d}}{\mathrm{d}z}s^2(z)\right]^2+s(z)^2\left[z^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}s^2(z)+zs^2(z)\right]\right\}.$$

Putting then $X(z)=\sqrt{s(z)/az}\,\varphi(\lambda,z)$, we readily derive the integral equation:

(A.10) $$X(z)=\exp[i\xi(z)]-\int_0^z\sin[\xi(z)-\xi(t)]\,s(t)\,X(t)\,s(t)\frac{\mathrm{d}t}{t}\,.$$

Take now $X_1(z)=\exp[-i\xi]X(z)$. $X_1(z)$ satisfies:

(A.11) $$X_1(z)=1-\int_R\frac{1-\exp[2i(\xi(z)-\xi(t))]}{2i}\,J(t)\,X_1(t)\,s(t)\frac{\mathrm{d}t}{t}\,.$$

We choose a path R of integration such that along it $I(\xi^0(z)) < I(\xi^0(t))$. It can be proved from the formula for $\xi^0(z)$ that $I(\xi^0(t))$ increases along $\arg t = \text{const}$ and decreases along $I(\sqrt{a^2+t^2}) = \text{const}$. Suppose $I(\xi^0(z)) = a\sigma$. A suitable path is then R_I: $\arg t = \sigma$ and the line R_z:

$$I(\sqrt{t^2+a^2}) = I(\sqrt{z^2+a^2}) .$$

With this position

$$|\exp[2i(\xi^0(z) - \xi^0(t))]| < 1 .$$

Consequently from (A.8) we have when a is large:

$$\left|\frac{1-\exp[2i(\xi(z)-\xi(t))]}{2i}\right| < K = \text{const} .$$

R has a corner T where R_1 and R_2 meet. We have $T = O(a)$. We need now some simple bound on $J(t)$. If $0 < t < T$ one can use (A.7), if $t > T$ then $s(t) > C't$ is also needed. Omitting here the details we arrive at the conclusion that:

$$\int_R |J(t)\,s(t)|\frac{\mathrm{d}t}{t} = O(a^{-1}) .$$

The above estimate holds uniformly for all z such that $I(\xi(z)) = a$. It follows from the lemma that on this line Q one has $X_1(z) = O(1)$ uniformly. Also on Q we have:

$$|\varphi(ia, z)| < C\sqrt{z_>}\exp[-a\sigma] ,$$

where C is some constant, $z_>$ the largest between z, a.

Finally from (A. 6) we derive:

$$C(\lambda)\,S^0(\lambda) - C^0(\lambda)\,S(\lambda) = -\frac{1}{4\lambda^2}\int_0^\infty U(z)\,\varphi^0(\lambda, z)\,\varphi(\lambda, z)\,\mathrm{d}z . \tag{A.12}$$

In (A.12) the integration can be carried out along Q and the above found inequality introduced. Dividing both sides by $C(\lambda)\,C^0(\lambda)$ the result follows. More details on this technique can be found in ([6]).

Appendix IV

We suppose now that λ is large positive. Our starting point is eq. (IV) (Appendix III). We take firstly $x \leqslant \lambda$. From the general theory of Bessel functions it is easy to prove that $\varphi^0(\lambda, x) > 0$ and that for sufficiently large

λ, $\varphi^0(\lambda, x)x^{-M}$, $M > \frac{1}{2}$ is an increasing function. From the identity:

$$G(x, y, \lambda) = \varphi^0(\lambda, x)\,\varphi^0(-\lambda, y) - \varphi^0(\lambda, y)\,\varphi^0(-\lambda, x) =$$

$$= 2\lambda\,\varphi^0(\gamma, x)\,\varphi^0(\lambda, y)\int_y^x \frac{\mathrm{d}z}{\varphi^0(\lambda, z)^2}\,, \qquad \infty \geqslant y,$$

replacing $\varphi^0(\lambda, z)$ with the smaller quantity $\varphi^0(\lambda, y)(z/y)^M$ we obtain the bound:

$$|G(\lambda, x, y)| < \frac{y}{M - \frac{1}{2}}\,\frac{\varphi^0(\lambda, x)}{\varphi^0(\lambda, y)}\,.$$

Putting then $\varphi(\lambda, x) = \zeta(\gamma, x)\,\varphi^0(\lambda, x)$ we find the inequality:

$$|\zeta(\lambda, x)| < 1 + \int_0^x |y\,U(y)|\,\zeta(\lambda, y)\,\frac{1}{M - \frac{1}{2}}\,\mathrm{d}y\,.$$

From the lemma it follows then

$$\varphi(\lambda, x) \sim \varphi^0(\lambda, x)\;O(1)$$

uniformly in $0 < x < \lambda$. Take then the identity (A.6):

$$C(\lambda) = C^0(\lambda) + \frac{1}{2\lambda}\Bigg[\int_0^\lambda U(x)\,C^0(\lambda, x)\,\varphi^0(\lambda, x)\,O(1)\,\mathrm{d}x +$$

$$+ \int_\lambda^\infty U(x)\,C^0(\lambda, x)[\,C(\lambda)\,S(\lambda, x) - S(\lambda)\,C(\lambda, x)]\,\mathrm{d}x\Bigg]\,.$$

In this formula $C^0(\lambda, x)\varphi^0(\lambda, x) \sim C^0(\lambda)/(\sqrt{1 - (z/\lambda)^2})$ as it can be deduced from the power series or from the asymptotic expansions of Bessel functions. It can be moreover shown from the Schlaefli representation that:

$$\text{(A.14)} \qquad \left|\frac{C^0(\lambda, x)}{\sqrt{x}}\right|,\; \left|\frac{S^0(\lambda, x)}{\sqrt{x}}\right| < C = \text{constant}, \quad x \geqslant \lambda.$$

From eq. (III) (Appendix II) it follows that the same kind of inequality holds for $C(\lambda, x)$ and $S(\lambda, x)$. We are now in condition to write:

$$C(\lambda)[1 + O(1)] + S(\lambda)\,O(1) = C^0(\lambda)[1 + O(1)]$$

and similarly

$$S(\lambda)[1 + O(1)] + C(\lambda)\,O(1) = S^0(\lambda)[1 + O(1)]\,.$$

It follows:

$$C(\lambda) = C^0(\lambda)[1 + O(1)] + S^0(\lambda)\,O(1) \quad \text{etc.},$$

and

$$\lim_{\lambda\to\infty} \frac{C(\lambda)^2 + S(\lambda)^2}{C^0(\lambda)^2 + S^0(\lambda)^2} = 1 .$$

Probably our estimate here can be ameliorated at the expense of additional complication. It would be also desirable to have a simpler proof. We turn our attention now to (A.12). Putting $C(\lambda)^2 + S(\lambda)^2 = T(\lambda)^2$ we can write it as follows:

$$\sin\delta(\lambda) = \frac{1}{-4\lambda^2}\int_0^\infty U(x)\,\frac{\varphi^0(\lambda, x)}{T^0(\lambda)}\,\frac{\varphi(\lambda, x)}{T(\lambda)}\,\mathrm{d}x .$$

If $x > \lambda$ then from the corresponding bounds (A.14) we deduce:

$$|\varphi^0(\lambda, x)| < 2C\lambda T^0(\lambda)\cdot\sqrt{x}\,; \qquad |\varphi(\lambda, x)| < 2C'\,\lambda T(\lambda)\cdot\sqrt{x}\,,$$

C, C' being, as usual, two suitable constants. It follows that the contribution to $\sin\delta$ coming from the interval $\lambda \ldots \infty$ is of the order of

$$CC'\int_\lambda^\infty x\,|U(x)|\,\mathrm{d}x = O(1) .$$

Actually a more elaborate analysis shows that this bound is pessimistic and one has actually $O(\lambda^{-1})$. Between $0 \ldots \lambda$ one has

$$|\varphi(\lambda, x)\,T^{-1}(x)| < H\,|\varphi^0(\lambda, x)\,T^{0^{-1}}(\lambda)|.$$

This part of the integral is certainly smaller than:

$$\frac{1}{4\lambda^2}\,H\int_0^\infty |U(x)|\,\frac{\varphi^0(\lambda, x)}{T^0(\lambda)^2}\,\mathrm{d}x .$$

Since $|U(x)| < Mx^{-2}$ from (2.4) we get the upper bound:

$$\frac{H\pi M}{4\lambda} .$$

If $U(x)$ is of the (3.2) type then the above technique yields easily the stated result. It is not clear to us if the above analysis is included in Carter's result in the sense that his bound may hold regardless of λ being not an half-integer.

A few words on the use of the Phragmen-Lindeloef theorem in extending these results to complex λ are also needed.

Take

$$F^+(\lambda) = \frac{C(\lambda) + iS(\lambda)}{C^0(\lambda) + iS^0(\lambda)} = \frac{T(\lambda)}{T^0(\lambda)}\exp[-i\delta(\lambda)] .$$

This function is easily seen to be regular in $R(\lambda) > 0$. We have also $F^+(\lambda) \to 1$ along $\arg\lambda = 0$ and $\arg\lambda = -(\pi/2)$. $F^+(\lambda)$ is also bounded by an exponential function in the intermediate angles and is therefore bounded by a constant. From Montel's theorem the limit holds in all

$$0 > \arg\lambda > -\frac{\pi}{2},$$

$F^+(\lambda)$ has also no zeros in this sector since if $F^-(\lambda)$ is the adjoint function, then:

$$\exp[-2i\delta] = \frac{F^+(\lambda)}{F^-(\lambda)}$$

and also this function would vanish since (F^+, F^-) cannot vanish simultaneously, for also $C(\lambda)$ and $S(\lambda)$ would vanish and also $\varphi(\lambda, x)$ because of (1.5).

But

$$|\exp[-2i\,\delta(\lambda)]| = \exp[2I[\delta(\lambda)]] > \exp[\pi I(\lambda)] > 0$$

and the above claim is clearly impossible, Along $\arg\lambda = \pi/2$ one has $|F^+(\lambda)| = O(\exp[\pi(\lambda) - 2\sigma(\lambda)])$. If one applies Carleman's theorem then unless $\sigma = \pi/2$, one has always zeros of F^+ in this sector. The properties of $F^-(\lambda)$ are of course the same of $F^+(\lambda)$ provided one changes $\arg\lambda$ into $-\arg\lambda$. All extensions to complex λ can be carried out with the same technique.

APPENDIX V

We shall sketch here briefly the low energy limit. The potential will be supposed to decrease exponentially. In this case a simple generalization of known arguments yields the limit:

(A.15) $$\sin_{k\to 0}\delta(\lambda) \sim k^{2\lambda}\eta(\lambda),$$

where $\eta(\lambda)$ is finite.

This limit follows from (A.12) and from the similar indentity:

(A.16) $$C(\lambda)\,S^0(-\lambda) - C^0(-\lambda)\,S(\lambda) = \frac{1}{2\lambda} + \frac{1}{4\lambda^2}\int_0^\infty \underset{0}{U}(r)\,r^{-\lambda+\frac{1}{2}}\underset{0}{\varphi}(\lambda, r)\,\mathrm{d}r = \frac{\alpha(\lambda)}{2\lambda}.$$

$\underset{0}{U}(r)$ is here the usual potential and r the distance $(x = kr)$. Also $k^{-2}\underset{0}{U}(r) = U(x)$. $\underset{0}{\varphi}(\lambda, r)$ is that solution of:

$$\underset{0}{\varphi''} - \frac{\lambda^2 - \frac{1}{4}}{r^2}\underset{0}{\varphi} - \underset{0}{U}(r)\underset{0}{\varphi} = 0,$$

which behaves like $r^{\lambda+\frac{1}{2}}$ for small r.

(A.12) can be correspondingly written as (k small):

$$C(\lambda)\,S^0(\lambda) - S(\lambda)\,C^0(\lambda) = -\frac{1}{4\lambda^2}\int\limits_0^\infty \underset{0}{U}(r)\,r^{\lambda+\frac{1}{2}}\,\varphi(\lambda, r)\,\mathrm{d}r\cdot k^{2\lambda} = \beta(\lambda)\,k^{2\lambda}\,.$$

It follows that if $R(\lambda) > 0$:

$$\tau^{0^2}(\lambda)\eta(\lambda) = \frac{\beta(\lambda)}{\alpha(\lambda)}\,,\qquad \frac{\tau(\lambda)}{\tau^0(\lambda)} = \alpha(\lambda)\,.$$

Therefore $\eta(\lambda)$ is a meromorphyc function of λ in $R(\lambda) > 0$. If λ is large $\alpha(x) \to 1$ and $\beta(\lambda)$ becomes in general large. A good approximation to $\varphi(\lambda, r)$ is then simply $r^{\lambda+\frac{1}{2}}$. Also one can show that if $\underset{0}{U}(r)$ is a σ potential then $T^0(\lambda)^2\eta(\lambda) = 0(\exp[-2\sigma|\lambda|])$ for λ large imaginary. Take now Sommerfeld's integral (6.4). In the low energy limit

$$-z = \frac{\tau^2}{2k^2} - 1 \sim \frac{\tau^2}{2k^2}\,,$$

if τ is kept constant. But if $-z$ is large positive then:

$$P_{\lambda-\frac{1}{2}}(-z) \sim \frac{\Gamma(\lambda)\,T^{2\lambda-1}k^{1-2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda+\frac{1}{2})}\,.$$

A factor k^{-1} was omitted for simplicity in (6.4). Having care of this factor and others, Sommerfeld's representation takes the limiting form:

$$f(0, \tau) = \frac{1}{\tau}\int\limits_{-i\infty}^{i\infty}\lambda\,\mathrm{d}\lambda\,\frac{\beta(\lambda)}{\alpha(\lambda)}\left(\frac{\tau}{2}\right)^{2\lambda}\frac{\sqrt{\pi}}{\Gamma(\lambda)\,\Gamma(\lambda+\frac{1}{2})\cos\pi\lambda}\,. \tag{A.17}$$

The discussion of this integral in no point is essentially different from the general case already treated in Section **6**, and it is actually simpler because it involves the theory of elementary Mellin transforms.

In (A.17) the path avoids the zeros of α. The resulting analiticity domain is the sector $|\arg\tau| < \sigma$ plus the interms of the circle $|\tau| < \alpha$.

RIASSUNTO

Nel presente lavoro viene definita l'interpolazione degli sfasamenti nello scattering da potenziale per valori generalmente complessi del momento orbitale. Tale definizione si presta particolarmente a discutere le proprietà analitiche (tra cui la rappresentazione di Mandelstam) dell'ampiezza di scattering giovandosi all'uopo di un metodo dovuto a Watson sueccessivamente perfezionato da Sommerfeld.