In a sufficiently strong field a saturation takes place. The order of magnitude of this field \( H_s \) is determined by the condition
\[
\alpha^* \approx 1. 
\]
Therefore,
\[
H_s \sim 2.7 \times 10^7 \text{ cm}^{-1}. \tag{6}
\]

Thus, we may expect that the field, in which saturation sets in, is proportional to the temperature. With \( \alpha^* \sim 10^4 \text{ cm}^{-1} \) we get for \( T = 10^4 \text{ K} \) and \( H_s \sim 2000 \text{ gauss} \).

The thermal conductivity of bismuth and hydrogen temperatures was investigated experimentally by de Haas et al. (1). Considering the influence of magnetic field on the thermal conductivity and assuming that the magnetic field acts upon the electronic part of the latter, the authors divide the thermal conductivity into an electronic and a phononic parts [Table IX in (1)]. Their data show that in the temperature interval 68–80 K the electronic thermal conductivity is independent of the temperature, whereas the phononic part increases with decreasing temperature and exceeds that of the electronic one. This is in accord with the results of the present paper.

It was also shown in the paper cited above, that a saturation in the dependence of the thermal conductivity on magnetic field strength sets in at comparatively weak fields, of the order of \( 10^7 \) gauss. Formula (6) of the present paper gives the same order of magnitude.

With decreasing temperature the saturation takes place in weaker fields. The intersection of the curves in Fig. 7 in (1) can be explained by the fact that, after saturation sets in, the thermal conductivity is mainly due to the phonons and in the temperature region considered decreases with increasing temperature.

In the helium temperature region the thermal conductivity in bismuth was investigated by S. Shalit (2). He found a maximum in the thermal conductivity-temperature curve in the region 4–14 K. He has also shown that on the lower temperatures side, in the vicinity of the maximum, the magnetic field does not set upon the thermal conductivity. This is to be understood as an indication of the predominant role of the phononic thermal conductivity even in the vicinity of the maximum; the electronic part becomes prevailing at still lower temperatures.

In conclusion we wish to express our thanks to S. Shalit for informing us of his experimental results before publication, and also to Prof. L. Landau for his interest in our work.

Translated by E. Lifshitz.

REFERENCES

5. S. Shalit, J. Phys. 8, 315 (1944).

ON THE ENERGY LEVELS OF SYSTEMS WITH \( Z > 137 \)

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The properties of the system of finite radius with \( Z > 137 \) are investigated. It is shown that at a charge \( Z_e = Z \), the lower energy level of system becomes equal to \(-mc^2\), which corresponds to the falling of electrons upon the nucleus. The dependence of \( Z_e \) on radius of system is given.

As is well known, the Dirac equation for an electron in a field of a point charge has a discrete energy spectrum only as long as the charge is less than 137 (in electron units). For point charges greater than 137 the equation has no stationary solution for the lower energy level (\( n = 0, \ l = 0 \)).

At \( Z = 137 \) the energy which corresponds to this level becomes zero, as it seen from the usual formula:
\[
W = mc^2 \left( \frac{1}{\sqrt{1 - \gamma^2}} \right). 
\]

It is to be noted, however, that such a critical meaning of "137" takes place only for point charge. It seems to be of interest to investigate the behaviour of energy levels in system with a finite radius.

In this case the appearance of discrete levels takes place not with the value of \( W = 0 \) as in the case of a point charge. On the contrary, with the further increase of \( Z \) the energy of the discrete level increases further and reaches the value \(-mc^2\) for some charge \( Z_{cr} \). Only if \( Z \) becomes greater than \( Z_{cr} \) the first discrete level disappears.

This behaviour of energy levels is not inherent in the electrically charged system. Snyder, Schiff and Weinberg (1) have shown that the value of the lower energy level of electron in a rectangular well also becomes equal to \(-mc^2\), if the depth of the well arrives at some value \( V_e \). Further increase of the well leads to the disappearance of this level and its transition into the continuous spectrum (lower continuum).

For the explanation of this phenomenon* suppose that we increase the depth of the well in the absence of electron. If the depth of the well becomes greater than \( V_e \) in the lower continuum (continuous spectrum with energies \(<-mc^2\) ), which is filled with electrons, there appears one unoccupied level originating from a discrete level. In this case the uncharged vacuum will have one unoccupied level, while in the absence of the well (or if the depth of the well is smaller than \( V_e \), or if the charge in our case is smaller than \( Z_{cr} \), the uncharged vacuum is a vacuum, all levels of which are occupied.

If an electron happens to be in such a well (or in the field of such a charge), it will jump on this unoccupied level, vacuum acquires a charge \(-e\), and the electron, since this is the level of an unobservable vacuum, will pass into the unobservable state. Thus, the result is the disappearance of the electron and the decrease of the charge of the vacuum (or, in other words, of our system) by one electron.

* This interpretation is given in the paper of Schiff, Snyder and Weinberg.
charge. It may be said that the electron falls onto a nucleus \(^*\).

In this paper we shall give the calculation of the value of \(Z_{\text{cr}}\) for a given finite radius \(r_0\). Keeping this problem in mind, we shall, firstly, give the solution of the Dirac equation in this case.

We shall investigate in the following the simplest model with the potential

\[
U = -\frac{Z_0 z^2}{r} \quad \text{for} \quad r < r_0
\]

and

\[
U = -\frac{Z_0 z^2}{r} \quad \text{for} \quad r > r_0.
\]

Dirac equations for radial function (multiplied by \(r\)) of electron with angular momentum \(l = 0\) have the form:

\[
\begin{align*}
\frac{df_1}{dr} - \frac{1}{r} f_1 &= (-x - \omega + u) f_2, \\
\frac{df_2}{dr} + \frac{1}{r} f_2 &= (-x + \omega - u) f_1,
\end{align*}
\]

where \(x, \omega\) and \(u\) are the rest, total and potential energies respectively, measured in \(hc\) units.

For small \(r < r_0\), it is easy to see that the solutions of equations (1) are:

\[
\begin{align*}
f_1 &= a \sin kr, \\
f_2 &= a \cos \left( kr - \frac{\sin kr}{kr} \right),
\end{align*}
\]

where \(k = (\omega - u)^{\frac{1}{2}} - x^2\) and \(a\) is a constant. If we choose \(r_0\) sufficiently small we can put

\[
k^2 = \frac{\gamma^2}{r_0^2}.
\]

In this approximations

\[
\left( \frac{f_2}{f_1} \right)_{r = r_0} \simeq \cot \gamma \frac{1}{kr_0} \approx \cot \gamma - \frac{1}{\gamma}.
\]

We denote

\[
\gamma = \frac{Z_0 z^2}{hc}
\]

and

\[
\gamma \cot \gamma - 1 = \zeta(\gamma)
\]

then

\[
\left( \frac{f_2}{f_1} \right)_{r = r_0} = \gamma^2(\gamma).
\]

For \(r > r_0\), but, in general, small, the solution of (1) has the form:

\[
\begin{align*}
f_1 &= b \left( r^{i\gamma} + \beta r^{-i\gamma} \right), \\
f_2 &= \frac{b}{i} \left[ (1 - i\gamma) r^{i\gamma} + \beta (1 + i\gamma) r^{-i\gamma} \right],
\end{align*}
\]

where \(b\) and \(\beta\) are constants, the later can be determined from the conditions of continuity for \(r = r_0\) and \(g^2 = \gamma^2 - 1\).

We have from (7):

\[
\left( \frac{f_2}{f_1} \right)_{r = r_0} = \gamma \frac{1 + \beta r_0^{2i\gamma}}{(1 - i\gamma) + (1 + i\gamma) \beta r^{-2i\gamma}},
\]

comparing (6) and (8) we obtain:

\[
\beta = \frac{\zeta(\gamma) (1 - i\gamma) - 1}{\zeta(\gamma) (1 + i\gamma) - 1} r_0^{2i\gamma}.
\]

Thus, for \(r > r_0\) (near the origin) the wave functions are:

\[
\begin{align*}
f_1 &= A \left\{ (1 - \zeta(\gamma) (1 + i\gamma)) \left( \frac{r}{r_0} \right)^{i\gamma} - \text{compl. conj.} \right\}, \\
f_2 &= A \left\{ \frac{1 - i\gamma}{\gamma} \left[ (1 + \zeta(\gamma) (1 + i\gamma)) \left( \frac{r}{r_0} \right)^{i\gamma} - \text{compl. conj.} \right] \right\}.
\end{align*}
\]

For obtaining the equation for the energy levels we must investigate the behaviour of the solution at infinity.

---

\(^*\) In the case of an integer spin a "fall" of this sort is connected with the complex eigenvalues. The Dirac equation cannot have complex eigenvalues.

\(^*\) For this purpose we shall bring the equations in a canonical form of the equations for Whittaker functions (\(^\dagger\)).

and in this case the "fall" will be of such peculiar character.

\(^\dagger\) Normalization constants \(a\) and \(b\) have no importance for us, and we shall not determine them later.
Instead of functions $f_1$ and $f_2$, we introduce new functions $G_1$ and $G_2$ defined by the equations (1):

\[
G_1 = f_1 \sin \frac{\theta}{2} + f_2 \cos \frac{\theta}{2},
\]
\[
G_2 = -f_1 \sin \frac{\theta}{2} + f_2 \cos \frac{\theta}{2},
\]

where $\theta = \arccos W/mc^2$ ($W$ is the energy).

We introduce now the new variable

\[
x = 2 \left(1 - \frac{W}{mc^2} \right)^{1/2} \frac{mc}{\hbar} r_0.
\]

For this function we obtain the following equations:

\[
\frac{dG_1}{dx} + \frac{1}{x} G_1 = -\frac{1}{2} G_1 + \frac{y}{x \sin \gamma} (G_1 \cos \gamma - G_2),
\]
\[
\frac{dG_2}{dx} + \frac{1}{x} G_2 = -\frac{1}{2} G_2 + \frac{y}{x \sin \gamma} (G_1 \cos \gamma - G_2).
\]

Eliminating from these equations $G_1$ and $G_2$ correspondingly, we arrive at the equations of the second order:

\[
\frac{d^2G}{dx^2} + \frac{1}{x} \frac{dG}{dx} + \left[ -\frac{1}{4} + \frac{\gamma \cot \gamma \pm \frac{1}{2}}{x} \right] G = 0.
\]

where the double sign $(\pm)$ in the second term in the square bracket corresponds to the two equations for $G_1$ and $G_2$ respectively.

If we put now

\[
G = x^{-1/2} G'
\]

we arrive at the Whittaker equations for $G'$:

\[
\frac{d^2G'}{dx^2} + \left[ -\frac{1}{4} + \frac{\gamma \cot \gamma \pm \frac{1}{2}}{x} \right] G' = 0.
\]

As it is shown in (1), the solutions of (16) are Whittaker functions:

\[
G' = W_{\nu, \gamma} \text{e}^{\pm \gamma x/2} (x).
\]

We have chosen the solution which disappears at infinity as $e^{-\gamma x/2}$. The second solution of (16) increases at infinity as $e^{\gamma x/2}$ and does not satisfy the conditions of our problem.

Using the expansion of Whittaker function we obtain from (16) and (17), finally (for small $x$):

\[
G_1 = c \left\{ \frac{\Gamma(-2\nu)}{\Gamma(-\nu - \gamma \cot \gamma)} x^\nu + \text{compl. conj.} \right\},
\]
\[
G_2 = c \left\{ -1 + \frac{\gamma}{\sin \gamma} \right\} \left\{ \frac{\Gamma(-2\nu)}{\Gamma(-\nu - \gamma \cot \gamma)} x^\nu + \text{compl. conj.} \right\},
\]

$c$ is a new normalization constant.

It is possible also to construct the function $G_1$ and $G_2$ from the solutions (10) by means of equations (11). We obtain immediately:

\[
G_1 = A \sin \frac{\theta}{2} \left\{ \left( 1 + \frac{1 - i\gamma}{\gamma} \cot \frac{\theta}{2} \right) \left[ 1 - \gamma (\gamma + 1) \right] \left( \frac{r_0}{\gamma} \right)^{\gamma} - \text{compl. conj.} \right\}.
\]

and a similar expression for $G_2$.

It is obvious that apart from normalization constants the functions (18) and (19) must be identical. This is just the condition we have looked for, which determines the energy levels.

As $G_2$ can be obtained from $G_1$ by means of (13), it is sufficient to compare the expressions for $G_1$ only.

Noting the different signs in curly brackets of (18) and (19) the following equations can be obtained:

\[
\text{Re} \frac{\left[ 1 - \gamma (\gamma + 1) \right] \left( 1 + \frac{1 + i\gamma}{\gamma} \cot \frac{\theta}{2} \right) \Gamma(-2\nu) x^\nu}{\Gamma(-\nu - \gamma \cot \gamma)} = 0.
\]

* In the substitution of the solutions (18) and (19) in (13) the following relations are useful:

\[
\frac{1 + \gamma + (1 - i\gamma) \cot \frac{\theta}{2}}{1 + \frac{1 + i\gamma}{\gamma} \cot \frac{\theta}{2}} = \frac{1 + \gamma / \sin \gamma}{1 - \gamma / \sin \gamma},
\]
\[
\frac{1 + \gamma - (1 - i\gamma) \cot \frac{\theta}{2}}{-1 + \frac{1 + i\gamma}{\gamma} \cot \frac{\theta}{2}} = \frac{ig - \gamma \cot \gamma}{1 + i(g - \gamma \cot \gamma),}
\]
where according to (12):

\[ x_0 = 2 \left( 1 - \frac{W}{m_0^2} \right)^{1/3} \frac{m_0}{\hbar} r_0. \]  \hspace{1cm} (21)

This equation determines for the given value of \( r_0 \), the values of energy levels of our system.

For our problem there are essential the energy levels close to \(-mc^2\) only. In this case the equation (21) may be simplified. Since \( c t g \varepsilon \) is very great (and negative) we can use the asymptotic expansion of \( \Gamma \)-function (Stirling’s formula), then

\[
\arg \Gamma (-i\gamma - \gamma c t g \varepsilon) = -g \log (-\gamma c t g \varepsilon) = \frac{-g \log \frac{\gamma W/me^2}{(1 - W/me^2)^1/3}}{\gamma c t g \varepsilon}.
\]

Computing the argument of the expression under the \( \text{Re} \) sign in (20) putting \( c t g \varepsilon/2 = 0 \) (since if \( W \approx -mc^2 \varepsilon \approx \pi \)) and using the obvious relation

\[
\arg \Gamma (-2ig) = \frac{\pi}{2} + \arg \Gamma (1 - 2ig)
\]

we obtain:

\[
g \log \gamma \frac{2mc_0}{\hbar} \frac{|W|}{me^2} + \arg \Gamma (1 - 2ig) + \\
+ \arg \tan \frac{g \gamma (\varepsilon)}{1 - \varepsilon (\gamma)} = n\pi
\]

\[ (n = 1, 2, \ldots). \]  \hspace{1cm} (22)

It must be noted that from (20) follows the double sign \((\pm)\) before \( n\pi \), but for a given \( r_0 \) the different signs correspond to \( g \) which differ also only in sign. As the sign of \( g \) is not determined (only \( g^2 \) enters the problem) we retain only the minus sign alone and consider \( g \) as a positive quantity.

The value \( n = 0 \) must be excluded, since in this case we obtain discrete negative energy level for \( n = 0 \) \((Z = 137)\), which tends to \(-\infty\) if \( r \to 0 \). But it can be shown that exact equations have no discrete negative levels even for a point charge.

Putting now in (22) \(|W| = mc^2\) and \( u = 0 \), we obtain the equation for the value of a critical charge for a given “radius” of a nucleus \( r_0 \), i.e. the minimum charge in the field of which the first energy level passes into a lower continuum:

\[
\log \gamma \frac{2mc_0}{\hbar} = \frac{n}{\delta \gamma} \left[ -\arg \tan \frac{\delta \gamma (\varepsilon)}{1 - \varepsilon (\gamma)} - n\pi \right].
\]  \hspace{1cm} (23)

For a very small \( r_0 \) \((\ll \hbar/2 mc)\), equation (23) may be written in approximate form:

\[
\delta \gamma = -\frac{\pi}{1 \log \frac{2mc_0}{\hbar}}
\]  \hspace{1cm} (24)

Thus for a nucleus of finite radius the disappearance of discrete levels (and falling of electrons) happens not for \( Z = 137 \), but for a greater charge, determined by equation (23).

For example, for a value \( r_0 = 1.2 \cdot 10^{-9}, \ Z_\gamma = 200, \) for \( r_\delta = 8.10^{-3}, \ Z_\delta = 175. \)

In conclusion we wish to thank Prof. L. Landau for interesting discussions in connection with this problem.

Translated by authors.

REFERENCES

3. Cf. В. А. Ф о к, Начала квантовой механики \( (V. F o c k, \text{Principles of Quantum Mechanics}) \), Leningrad, 1932, p. 239.