The Many $\pi$-Meson Problem

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It is shown that the properties of a system of $N$ $\pi$-mesons are to a considerable extent determined by three quantum numbers $(N_1N_2N_3)$, the "correlation numbers" whose sum equals $N$. In a state with definite correlation (specific values of $N_1N_2N_3$) these numbers relate to the number of triples: $3\pi$-subsystems with $I = 0$; the number of pairs: $2\pi$ systems with $I = 1$ and the remaining singles out of which such a state can be composed by a well-defined prescription. For states with definite correlation, the correlation numbers dictate which $I$-spin values such states can possibly have. For systems with $I < 2$ ("$N\pi$-clouds") the correlation numbers determine the $I$-spin uniquely.

The branching ratios for an $N\pi$-cloud into the various charge distributions compatible with given $N$ and total charge are not uniquely determined if one only gives the $I$-spin of a specific state. However, it is shown that one gets unique results if one is only given the correlation numbers. General methods are developed for determining these ratios as functions of $N_1N_2N_3$. Tables are given for these ratios for $N = 2$–8.

In Section II all results are compiled and rules are given for practical use. All mathematical proofs are found in Section IV. It is shown there that the present results can all be obtained without recourse to the explicit use of vector addition coefficients. The basic tools are Young's (nonorthogonal) representations of the symmetric group and Thrall's theorem on its orthonormal idempotents.

Applications to annihilation processes are discussed (Section III) with special reference to $\bar{p}-p$ and $\bar{p}-d$ annihilation. Zero- and two-prong star theorems are given. A statistical model for $\bar{p}-p$ and $\bar{p}n$-annihilation predicts that, with increasing $N$, the ratio of the average number of charged $\pi$'s to the average number of neutral $\pi$'s produced in $N\pi$-annihilation becomes nearly equal to 2. However, the predominance of specific correlations could lead to substantially different values for this ratio.

The connection between charge correlations and spatial correlations as well as some dynamical aspects of the problem are briefly discussed in Section V.

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Experimental information obtained in recent years on the traversal of anti-protons through matter (1-3) has shown that, at moderate \( \bar{p} \)-energies, the average number of \( \pi \)-mesons produced in \( \bar{p} \)-annihilation is about 5 or 6. Thus the large majority of these events leads to quite complex many-body states. It would be desirable to have methods of description of such states and to have ways to ask questions about them that take this complexity into account in some well-organized way. This need is also evident when one thinks of the work to be done with the new machines of very high energies which will be in operation in the near future. Whatever novelties may be revealed at these energies, it is clear that the many-body aspect of fundamental particle reactions is bound to come into play in many instances.

General theoretical discussions of the annihilation process have so far mainly been concerned with selection principles that follow from conservation of isotopic spin, of angular momentum, and of parity. These selection rules are only of practical interest if one is given additional information about the initial state from which the annihilation proceeds and, even then, the results are in the main only useful for final states of low multiplicity. As the multiplicities actually are rather large and, as, especially for higher energies, angular momentum and parity conservation do not lead to sufficiently simplifying restrictions, it is of some interest to seek for other approaches to extract information from the phenomena. A possible direction is to look for correlations of various kinds between the final products, among themselves as well as with the initial particles. Some simple consequences of invariance arguments for spatial correlations have been mentioned recently (4). It is the purpose of the present paper to discuss charge correlations.

The question is the following. Consider the reactions

\begin{align*}
\bar{p} + p &\rightarrow N\pi, \\
\bar{p} + n &\rightarrow N\pi.
\end{align*}
For given $N$ several combinations of charged and neutral $\pi$'s may be produced in accordance with all conservation laws. We ask what determines the charge correlations, that is, the relative probabilities of the channels corresponding to the various alternatives of charged and neutral $\pi$'s.

In some general way the charge correlations must depend on the isotopic spin, of course. However, it will turn out first, that the correlations are not uniquely determined by the isotopic spin which plays here only a secondary role and secondly, that there exists a triple of quantum numbers $(N_1N_2N_3)$ in terms of which the charge correlations can be fully specified. It is the main purpose of this paper to describe in full detail how to construct $N\pi$-states in such a way that these three "correlation numbers" come into evidence. The circumstance that the total $I$-spin in the final states of reactions (1) and (2) is restricted to $I = 0$ or 1 will turn out to lead to simplifying features. It will thus be helpful to introduce the following definition.

**Definition 1.** An $N\pi$-cloud is an assembly of $N$ $\pi$-mesons for which $I < 2$. Where this restriction is inessential we continue to talk of $N\pi$-states.

It may be illuminating to indicate right away, though rather loosely, what the correlation numbers refer to. To this end we first note that one can, of course, construct all isotopic states of an $N\pi$-cloud by compounding in all possible distinct ways the $N$ individual $I = 1$ spins to get the desired resultant $I = 0$ or 1. The number $p(N)$ of accessible states increases rapidly with $N$ as is seen from the second column of Table I. (The computation of $p(N)$ is a simple counting problem.) For some given large $N$ there are many possibilities of organizing such a construction of states by means of vector addition. One can start by taking subsets of particles for which one writes down all its possible $I$-spin states, then combines one subset with another and so on; and there are in general several ways of choosing the subsets. Now there exists one unique prescription to do this which is simpler than all others because, as we shall see later, it enables one to write down all states in a way which is faster and perhaps more instructive than the standard way via the Clebsch-Gordan-Wigner coefficients. Moreover, this prescription will turn out to lead to quite substantial short cuts in the discussion of the charge correlations. The procedure is as follows:

First take $N_3\pi$-triples out of the $N\pi$-collection and compose each triple by itself to $I = 0$. Of the remaining $N - 3N_3$ particles pick $N_2 - N_3$ pairs and compose each pair by itself to $I = 1$. We are then left with a number of "single" $\pi$-mesons and denote this number by $N_1 - N_2$. For given $N$ we can choose the numbers $(N_1N_2N_3)$ in a variety of ways provided only that we satisfy the relations

$$N_1 + N_2 + N_3 = N, \quad N_1 \geq N_2 \geq N_3 \geq 0. \quad (3)$$

For a given partition of $N$ which satisfies Eq. (3) one can form products of $\pi$-mesons.
[N\textsubscript{3}]\pi-wave functions of the $I = 0$ kind, $(N\textsubscript{2} - N\textsubscript{3})\pi$-wave functions of the $I = 1$ kind, $(N\textsubscript{1} - N\textsubscript{2})$ single $\pi$-wave functions]. In general there are many such products because one still has options which of the $\pi$'s participate in forming triples, which ones go into the pairs, which ones are to remain single.\footnote{It should be borne in mind that we are constantly referring to $N\pi$-states of some prescribed configuration. For example in momentum space we will have: one meson with momentum $p_1$, $\cdots$, one meson with $p_N$. These configuration labels make for the distinguishability we are referring to here. See further Section II.}

The usefulness of working with the correlation numbers now lies in the following set of properties, to be shown in what follows.

(a) For given $(N\textsubscript{1}N\textsubscript{2}N\textsubscript{3})$ there exist simple rules for forming the products and for taking linear combinations of them which lead to a certain number of mutually orthogonal cloud states; this number is an explicit function of $(N\textsubscript{1}N\textsubscript{2}N\textsubscript{3})$. When we are dealing with one such state we shall say: it belongs to the class $(N\textsubscript{1}N\textsubscript{2}N\textsubscript{3})$.

(b) Cloud states that belong to different classes are orthogonal.

(c) A complete and orthogonal set of $N\pi$-cloud states is obtained by taking the aggregate over all partitions which satisfy Eq. (3).

(d) All cloud states belonging to a given partition have the same $I$-spin; either they all have $I = 0$ or all $I = 1$. In other words: if one is given the correlation numbers, the $I$-spin follows gratuitously. The rule is: if $(N\textsubscript{1} - N\textsubscript{3})$ and $(N\textsubscript{2} - N\textsubscript{3})$ are both even we have $I = 0$, otherwise $I = 1$.

(e) All states belonging to a given correlation have the same charge correlations.

(f) If two states belong to different triples $(N\textsubscript{1}N\textsubscript{2}N\textsubscript{3})$ they have almost always different charge correlations. We shall specify below what “almost” means.

It will be clear that for $N = 2$ the correlation concepts are trivial. Nevertheless it is helpful for the understanding of the case of general $N$ to treat even the 2-body problem from the present view. We will in fact start Section II with a discussion of $N = 2$ in a way which would be excessively laborious if it were not for the fact that it helps pave the way for the general discussion.

Observe that once the statement (e) will have been proved, we can save a lot of labor in computing charge correlations, as now we need only do this for one state of each class; and for large $N$ the number of classes is much smaller than the number of cloud states, as is evident from the comparison of the second and third columns of Table I. For example, by the time we reach $N = 10$, the number of calculations is cut down by two orders of magnitude. (The consideration of $N$-values of this order may perhaps not seem extravagant in the near future.) However, this is not the end of the labor-saving devices.

It will namely be shown that if we know only the charge correlation in certain special classes which we shall denote by “principal classes,” we will know the charge correlation in all classes without further calculation. The definition of
what constitutes a principal class will be given in Section II (b). Thus we can confine our attention to principal classes.

In turn, for principal classes we shall give a number of formulas which give certain charge correlations for many principal classes at the same time, see Section II, Eqs. (28)-(38).

The statements (a)-(e) all refer to cloud states. It may be asked what role the correlation numbers play if we drop the restriction \( I < 2 \). This question is of practical relevance, for example for the consideration of multiple meson production in \( \pi \)-nucleon and nucleon-nucleon collisions. We shall come back to this question in some detail in later work, but we give in Section II the basic theorem for the application of correlation concepts to all \( N\pi \)-states. The theorem establishes a relation between the correlation numbers of a general state and the \( I \)-spin values that are compatible with them.

In giving this outline of the program, we have so far only stressed the methodological aspects of working with correlation numbers and the classes which correspond to them. We will next ask: is there any physics in these correlation numbers?

The question is perhaps best approached by making an analogy. If one deals with the classification of atomic spectra, a first guide is the conservation of total angular momentum, but this leaves much degeneracy in the problem. The next phase is the introduction of particular coupling schemes like Russell–Saunders or \( jj \)-coupling. One can then compare the predictions of a specific scheme with experiment and if this comparison reveals that some particular scheme is “good,” this reflects on specific dynamical features of the system in hand (e.g., small spin-orbit coupling goes with Russell-Saunders). The analogy will be obvious. To angular momentum corresponds \( I \)-spin, while the correlation numbers correspond to the \( L \)- and \( S \)-numbers in Russell–Saunders coupling or the \( j \)-s in \( jj \)-coupling. And to the variance of coupling from atom to atom there may correspond a variance of charge correlations from energy to energy and from multiplicity to multiplicity.

Thus we answer the question just raised as follows. The detailed way in which the charges are correlated in the process of nucleon-antineutron annihilation (as a function of \( N \) and of \( \eta \)-energy) depends on the dynamics of the interaction process. More specifically the states of many \( \pi \)'s that are involved here belong to the high-frequency part of the \( \pi \)-clouds surrounding nucleons and antinucleons. Little if anything is known about the dynamics of the inner region of the \( \pi \)-cloud.\(^3\) It can therefore not be the purpose of this paper to anticipate the actual answers as to what the charge correlations will turn out to be. In other words, it is beyond

\(^3\) A promising approach to the treatment of the outer region of nucleon-antineutron interaction has been made by Ball and Chew (\( \theta \)). Here the inner region is phenomenologically represented by a black sphere.
our present knowledge whether or not it will turn out that a π-cloud (for some multiplicity and at some energy) is in a state of "pure" or nearly pure correlation. Wherever this might indeed be the case the present choice of an orthogonal set of states is a "good" one. On the other hand, this choice may easily be as bad as using Russel-Saunders base states for jj-coupling. In the absence of further information it seems indicated, nevertheless, to proceed in the present way which commends itself by its simplicity. Moreover, if we are interested in the verification of the predictions of the statistical model—to be discussed in Section III—one orthogonal set of states will be as good as another and in this case there can be no doubt that the present technique provides the fastest way to the answers.

It may further be noted that the full classification of cloud states to be given in this paper is not purely an enumerative process. It will become evident that this classification also fully answers the question which is the most general form of the effective Hamiltonian for a nucleon-antinucleon-Nπ-system. This will be briefly discussed in Section V. There it will also be seen that there may exist connections between charge correlations and spatial correlations (like angular distributions) in the annihilation process. We repeat, however, that the device of introducing the correlation numbers has in itself no predictive dynamical qualities. Rather, in our view, should the program discussed in this paper be seen from the opposite end: it is the aim to show that the charge correlations may contain useful information and to describe how this information may be analyzed by rules for "π-meson spectroscopy," in the hope that the spectroscopical information so obtained may perhaps contribute to a further understanding of the force laws. To give an instance how this might come about: suppose that there exist strong π-π correlations in the I = 1 state. One might then look for evidence for this in the charge correlation pattern by comparing the results of annihilation experiments with correlations in classes where the number of pairs (in the sense defined above) is optimal.

Obviously, the methods to be employed here are akin to those used in the analysis of complex atomic and nuclear spectra. In fact the present problem may be looked upon as the filling of an "infinite p-shell." We have a p-shell because of the I = 1 character of π-mesons; the shell is infinite because we deal with Bose statistics. Thus we can relate some of the results obtained here with those obtained for the p-orbital part of atomic or nuclear states with limited occupation, where the limitations are due, of course, to the exclusion principle.

Mathematically, the methods employed in all such problems are always based on the recognition, in first instance due to Schur (8), of the deep connections

4 As has been suggested by Frazer and Fulco (7). Actually this π-π interaction is assumed to be strong only in the angular momentum = 1 states. The question of the incorporation of angular momentum considerations in the present framework will be discussed in Section V.
between the representations of the symmetric groups (groups of permutations) and those of the continuous groups of unitary transformations. The possibility to move back and forth at convenience from the discrete permutation groups to the continuous unitary transformation groups is one of the main themes in Weyl's monumental books on group theory (9) and quantum mechanics (10). The methods of Racah (11) rest largely on this connection and so does the present related work.

The main tools to be used in this paper are Young's nonorthogonal representations of the symmetric group, rather than the orthogonal "Yamanouchi representations" (12). The latter are beautifully summarized in an instructive paper by Jahn and van Wieringen (13). (Part of Section IV will be devoted to an expose of these representations.) In the paper just mentioned, the p-shell problem for limited occupation is discussed by means of building up by fractional parentage. As is well known, this method proceeds by constructing states for \( N + 1 \) particles and of given symmetry (or, which is the same, of given class in the terminology adopted above) by starting from \( N \)-particle states, supposed to be in a form where the symmetry is manifest, and vector coupling the \( (N + 1) \)th-particle to these \( N \)-particle states. This method provides some short cuts as compared to a less well-organized vector composition. For the present purposes of the infinite shell even the fractional parentage method is too cumbersome for large \( N \) and the methods to be used hereafter avoid any such \( N \to N + 1 \) procedure altogether.

This is achieved by observing that, especially for large \( N \), it is much simpler to focus the attention on the role of the symmetry classes, that is on the permutation group than on the \( I \)-spin, that is on the group \( R_+ (3) \) of proper rotations in a three-space. Of course, the present work fits entirely within the conventional scheme of charge independence. Thus at some point or other it must be necessary to use rotational invariance. However, as we shall see in Section IV, all we have to do is something like this: given \( N \) three-dimensional vectors, write down a scalar, any scalar, or a vector, any vector that is composed out of these \( N \) vectors. (It will help to pick the scalar or vector with some foresight!) After that is done "by hand," the symmetric group takes over. If some simplification may have been achieved here, it is because we confine the rotation group and the symmetric group to their respective "natural realms." The present author holds it not excluded that an examination of the entirely practical question: when to use rotations, when to use permutations: may lead to other simplifications in the methods of analysis of complex systems.

It is the plan of this paper to present in Section II the qualitative features of
the methods and the results and to leave all mathematical details for Section IV. It is hoped that, at the price of some duplication between these sections, the overall picture may emerge more clearly.

II. RESULTS

(a) THE CASE N = 2. STATEMENT OF THE PROBLEM

We start by defining what shall be meant by a configuration of $N\pi$-mesons.

**Definition 2.** A configuration of an $N\pi$ system is a labeling of each of the $N$ particles by all quantum numbers other than charge which specify its state completely and where these quantum numbers have distinct values from particle to particle. For example, we can label by a set of momenta: $(p_1, \cdots, p_N)$ with $p_1 \neq p_2 \cdots \neq p_N$ is then a configuration. Or one can label by space coordinates, etc. When some of the labels become equal we have a degenerate configuration which is a limiting case of a nondegenerate one. Subtleties like the normalization of wave functions in the limiting cases will not be relevant for any of the applications made in this paper.

Consider the case $N = 2$. How many $2\pi$-states (in the usual quantum mechanical sense) are there that correspond to a given configuration? The answer is $3^2 = 9$, as we have three choices for the charge that goes with $p_1$ and likewise for $p_2$. Charge independence leads us to take the following linear combinations of these nine $2\pi$-states of given configuration. One corresponds to $I = 0$ and is given by

$$
\psi_0 = 3^{-1/2}(\pi_1^+ \pi_2^- - \pi_1^0 \pi_2^0 + \pi_1^- \pi_2^+). \tag{4}
$$

Three correspond to $I = 1$ and its various $I_\tau$-values, for example,

$$
\psi_1^0 = 2^{-1/2}(\pi_1^+ \pi_2^- - \pi_1^- \pi_2^+), \tag{5}
\psi_1^{-1} = 2^{-1/2}(\pi_1^0 \pi_2^- - \pi_1^- \pi_2^0).
$$

The remaining five correspond to $I = 2$, for example,

$$
\psi_2^0 = 6^{-1/2}(\pi_1^+ \pi_2^- + 2\pi_1^0 \pi_2^0 + \pi_1^- \pi_2^+). \tag{6}
$$

The labels 1 and 2 are the “configuration labels.” A prescription to find the coefficients in these linear combinations is the Clebsch–Gordan–Wigner composition of two $I = 1$ vectors to give the desired $I, I_\tau$.

If we simultaneously interexchange the configuration labels and the charge labels all $\psi$'s remain unaltered. If we either exchange the charge labels only or the configuration labels only, the $I = 0, 2$ states are symmetric, the $I = 1$ states antisymmetric under each of these separate operations.

For the $I = 0$ state we can look upon the symmetry as the invariance under a
special rotation in $I$-space: let us denote the isotopic spin vector of a $\pi$-meson by $\mathbf{x}$ with real components $x, y, z$, so that, in the usual way

$$\pi^+ = \frac{x + iy}{\sqrt{2}},$$
$$\pi^- = \frac{x - iy}{\sqrt{2}},$$
$$\pi^0 = z. \quad (7)$$

The desired interchange of $\pi^+$ and $\pi^-$ is brought about by the rotation

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (8)$$

The same rotation for $\psi_1^0$ also gives us the symmetry we are interested in. But for $\psi_1^{-1}$ there is no real rotation in $I$-space which brings about the interchange, because in that case we want to interchange $z$ and $2^{-1/2}(x - iy)$. There is a complex transformation which does the work, however, namely:

$$B = \begin{pmatrix} 1/2 & i/2 & 1/2^{1/2} \\ -i/2 & 1/2 & i/2^{1/2} \\ 1/2^{1/2} & -i/2^{1/2} & 0 \end{pmatrix}. \quad (9)$$

This transformation is unitary: $B^+ = B^{-1}$. (Note that the determinant of $B$ is not equal to 1.) Also $A$ can be considered as a special case of a unitary matrix.

We see, therefore that we can look upon the symmetry properties of the states in question in the following ways:

(a) We interchange the configuration labels 1, 2; that is we make a permutation;

(b) We ask for the properties of the isotopic wave functions under three-dimensional unitary transformations. We note a one-to-one correspondence between (a) and (b). This correspondence is general, in that it also holds for the $N$-body problem;

(c) For particular states like $\psi_0$ we can also look upon the symmetry as the result of an ordinary $I$-spin rotation. This is special. We must not try to mix up rotational properties with permutational symmetries.

It is therefore absolutely essential for the discussion of general $N$ to agree that when we talk about the symmetry properties of states we shall always mean the result of permuting the configuration labels (we shall call these labels: positions) not of the charge labels. Correspondingly we shall say: the states $I = 0, 2$ go with the symmetric configurations, $I = 1$ with the antisymmetric one.

The purpose of the convention is to make a complete separation between what
the isotopic spin rotations act on (charge) and what the permutations act on (position). This kind of separation is not particular to the present problem. It is also relevant when we talk about atomic or nuclear spectra, where the rotations go with the angular and the permutations can go with the radial variables.

We have now endowed the configuration with a property of symmetry or anti-symmetry. If we had only $\pi^0$s, say, this distinction would be useless, the antisymmetric configuration is identically zero. Thus the “configuration with symmetry” is a useful notion only when the objects “filling the configuration” are taken from a collection that contains particles of various charge. But it does not matter from the point of view of configurational symmetry, that the objects are more specifically $\pi$-mesons with their charge independence properties. Let us now forget, as it were, that we are dealing with $\pi$-mesons and distinguish between the order (1, 2) and the order (2, 1) of configurations. It will depend on the filling whether these ordered configurations are identical or not. Thus we can summarize all these very familiar properties of 2$\pi$-systems as follows.

(a) A 2-body system has two kinds of configurations the symmetric and the antisymmetric one (for at least two kinds of charges);
(b) We can “fill” the configuration with 2 $\pi$-mesons and have
   (a) The symmetric configuration can contain $I = 0$ or $I = 2$,
   (b) The anti-symmetric configuration can contain $I = 1$,
   (c) Different $I_\pi$-values corresponding to the same $I$ belong to the same configuration.

While most often one first asks for a state with given $I, I_z$ and then asks for its symmetry properties we have turned the order of questions around. It is this reversal of order which will make the understanding of the N-body problem much simpler. We shall now in fact state the $N\pi$-meson problem in this way. There are $3^N$ states for given configuration. We ask

(1) What are distinguishing symmetry properties of the configurations?
(2) Which $I$-spin can be contained in a configuration with prescribed symmetry?
(3) Can a given $I$-spin value occur more than once in a configuration of given symmetry? (This question is trivially answered for $N = 2$ but for large $N$ things are more complicated in this respect.)

Before we turn to these general problems, we make one last remark about $N = 2$. The Eqs. (4), (5), but not (6) refer to cloud states. We see that for a 2$\pi$-cloud the symmetry determines the $I$-spin and vice versa. For general $N$ the vice versa has no application, but it will turn out that also for an $N\pi$-cloud its symmetry (suitably defined) determines the $I$-spin uniquely.

If we only ask for the properties of an $N\pi$-system with respect to all quantum numbers (configuration and charge), all we can say is that every state is overall totally symmetric. If we distinguish between configuration and charge, a general overall totally symmetric
state breaks up into a superposition of orthogonal states, each of which has a configurational symmetry and an \( I, I_z \)-assignment. We can consider finer distinctions. For example, instead of labeling by \( p \), one can also label by \( (l, m, p) \) where \( l \) is the angular momentum of a particle (relative to an extraneous origin), \( m \) one component of \( l \) and \( p = |p| \). We can then ask for states labeled by \( I, I_z \) and angular momentum on the one hand, by the remaining configurational properties referring now to \( p \) only on the other. In turn, a state previously specified by \( I, I_z \) and configuration (in our sense) with symmetry now breaks up in more component parts. This more complex problem has recently been discussed by Halpern (14) who considered \( N = 2, 3 \) and \( l \) values 0 or 1. We shall come back in Section V to Halpern's problem. Until then the notion of configuration as given by Definition 2 will be strictly enforced.

(b) About Classes and Their Isotopic Content

For arbitrary \( N \) we can generalize the notion of symmetric configuration, just encountered for \( N = 2 \) by considering configurations which are symmetric between all \( N \) positions, the totally symmetric configurations. We will want to know for what \( I \)-values totally symmetric configurations exist. The answer to this question will come a little later. The notion of antisymmetric configuration like Eqs. (5) for \( N = 2 \) cannot be generalized for arbitrary \( N \) to configurations antisymmetric between all positions. There is in fact no such configuration for \( N \geq 4 \).

This is so because if \( N \geq 4 \) we must at least at two distinct positions find particles of like charge in which case there can be no question of antisymmetrization between these positions, due to Bose statistics.

As is well known, there are more general procedures for symmetrization and/or antisymmetrization for \( N \geq 3 \), namely, one can consider configurations that are symmetric only between groups of positions where each group is smaller than the total number of positions, and where different groups do not have positions in common; and at the same time one can antisymmetrize only between other groupings of positions where again the groups do not overlap and are each smaller than the total number of positions. The detailed recipe to obtain all such configurations is as follows.

Consider a partition \( N, N = N_1 + N_2 + N_3 + \cdots \) where we order the partition by always taking \( N_1 \geq N_2 \geq N_3 \cdots \). Corresponding to this partition we draw a picture, the so-called Young tableau: first draw \( N_1 \) squares in a row, then underneath that \( N_2 \) squares starting from the same initial column, then \( N_3 \) squares underneath, etc. Examples: see Fig. 1 for \( 5 = 3 + 2 \) and \( 7 = 3 + 3 + 1 \). Thus for given \( N \), different Young tableaux (or tableaux for short) consist of \( N \) squares arranged in different shapes.

For each tableau we next construct a set of labeled tableaux in the following way. Take the numbers 1, \( \cdots \), \( N \) (these are our configuration labels or positions) and put one number in each square. Do this in all possible ways subject to the conditions: reading each row from left to right and each column from top to bottom the numbers must appear in an increasing sequence. Example: for \( 5 = 3 + 2 \) there are 5 labeled tableaux, drawn in Fig. 2, (a)–(e).
The labeled tableaux are the tools for organizing symmetrizations and antisymmetrizations. Thus after having chosen a partition and drawn the tableau we consider now a specific labeled tableau. The recipe is: positions occurring in the same row are subject to symmetrization, positions in the same column to antisymmetrization. Each labeled tableau is therefore the pictorial representation of an operator which commands us to symmetrize in certain positions, antisymmetrize in others.

The example of Fig. 2 illustrates the distinction (true in general) between different labeled tableaux belonging to the same tableau. Thus in Figs. 2(a), (b), (c) the positions 1, 2 are candidates for symmetrization in (d), (e) the same positions are candidates for antisymmetrization. In turn Figs. 2(a), (b) are distinguished by the fact that there is symmetry, antisymmetry, respectively, between positions 1 and 3, etc. The distinction between labeled tableaux belonging to different partitions (of the same \( N \)) is even more obvious. Thus if we compare Fig. 2(f) with any other of the group (a)—(e) it is clear that we deal with distinct prescriptions, as (f) asks us to antisymmetrize in 3 positions, while the others antisymmetrize in only 2.

It should be emphasized, however, that the operation which a labeled tableau tells us to perform is not yet fully specified. Indeed it is still necessary to state
in which order the various operations are to be performed. There are various conventions current in the literature. All ordering conventions have this in common that when one convention is consistently applied we eventually obtain a set of states with these properties: states with configurational symmetry dictated by the labeled tableaux and the particular ordering convention chosen are linearly independent (irrespective of whether we compare labeled tableaux belonging to the same or to different partitions). And with the labeled tableaux plus fixed ordering convention we get a set of "configurations with symmetry" which is complete. But while we have linear independence, we do not have in general that the states are orthogonal. The orthogonality can be achieved, however, by choosing the ordering convention judiciously. It is the Yamanouchi prescription of ordering \((12)\) which does just that. For the survey of the qualitative aspects it is not necessary to go into details right here. (See further Sections IV(a) and (e).)

We observed above that for \(\pi\)-mesons we can never antisymmetrize in more than three positions. Accordingly we must confine ourselves to tableaux of at most three rows, or, which is the same, to partitions of \(N\) in at most three parts. We are now ready to give the following definition.

**Definition 3.** A class of an \(N\pi\)-configuration is defined by a partition, specified by Eq. \((3)\), of \(N\) in at most three parts. The class is denoted by \((N_1N_2N_3)\). \(N_3\) may be zero in which case we write \((N_1N_2)\). \(N_2\) may also be zero in which case we write \((N)\). Configurations belonging to a given class have mixed symmetry antisymmetry properties as indicated by the recipe given above.

Special examples: the class \((N)\) has a one-row tableau and refers to the totally symmetric configuration. A totally antisymmetric configuration consists of one column only. The largest \(N\) for which there exists a totally antisymmetric class is \(N = 3\), the class is \((111)\). These one row and one column tableaux have the special property that to each tableau there is clearly only one labeled tableau. The configurations we met in the example \(N = 2\) are of this kind. The symmetric configuration for \(N = 2\) is "the whole class" \((2)\), the antisymmetric one is the whole class \((11)\). For all other tableaux there is more than one configuration with symmetry going with the same tableau, as there are more than one labeled tableaux. We ask for the number \(\rho(N_1N_2N_3)\) of such configurations. The result is this (for details see Section IV).

**Theorem A.** The number of configurations with symmetry belonging to a given class is given by

\[
\rho(N_1N_2N_3) = \frac{N!(N_1 - N_2 + 1)(N_1 - N_2 + 2)(N_2 - N_3 + 1)}{(N_1 + 2)!(N_2 + 1)!N_3!}. \tag{10}
\]

Remarks: (1) Special instances of Eq. \((10)\) are: if \(N_3 = 0\),

\[
\rho(N_1, N_2) = \frac{N!(N_1 - N_2 + 1)}{(N_1 + 1)!N_2!}. \tag{11}
\]
and, if also $N_2 = 0$:  

$$\rho(N) = 1.$$  \hspace{1cm} (12)  

(2) We have the following recursion formula  

$$\rho(N_1N_2N_3) = \rho(N_1 - 1, N_2N_3) + \rho(N_1, N_2 - 1, N_3) + \rho(N_1N_2, N_3 - 1),$$  \hspace{1cm} (13)  

where the equality in quotes refers to the additional stipulation: if in any of the three terms on the right-hand side of Eq. (13) one of the triple of numbers in the argument is smaller than its successor the corresponding $\rho$-value = 0.

Examples: $\rho(421) = 35 = \rho(321) + \rho(411) + \rho(42)$,  

$$\rho(441) = 13 = \rho(431) + \rho(44).$$

So far we have only considered configurations and the way these are grouped in classes, and have not yet mentioned isotopic spin. We have recognized the existence of the "correlation numbers" $(N_1N_2N_3)$. For $N = 2$ it is now possible to state in the present language: to the class (2) corresponds $I = 0, 2$, to the class (11) $I = 1$. Thus for $N = 2$ we can simultaneously assign correlation numbers and $I$-spin and there is in fact some redundancy in doing both. We must ask if it is true in general that correlation and $I$-spin commute. The answer is affirmative; this provides the very basis for the classification of spectra of complex systems involving identical particles.

Thus we now consider the quantum numbers $I, I$, together with $(N_1N_2N_3)$. The content of the positions becomes relevant at this stage. We wish to know which $I$ spins fit together with a given set of correlation numbers.

**Definition 4.** The isotopic content of a configuration belonging to the class $(N_1N_2N_3)$ is a function $\psi(I; N_1N_2N_3)$ whose value is equal to the number of times that $I$ is contained in the configuration. Examples: if the configuration is of class $(N_1N_2N_3)$ and does not contain $I = 0$, then $\psi(0; N_1N_2N_3) = 0$. If it contains $I = 4$ twice then $\psi(4; N_1N_2N_3) = 2$. In our initial example, $\psi(0; 2) = \psi(2; 2) = \psi(1; 11) = 1; \psi(1; 2) = 0$, etc.

The question immediately arises: what about the relative isotopic content of different configurations belonging to the same class? This question is answered by the following theorem.

**Theorem B.** The isotopic content is a class property. That is, all configurations belonging to the same class have the same content. (See further Section IV). Thus we may speak of the isotopic content of a class, as was indeed already implied when the arguments of $\psi$ were written down.

**Remark:** The isotopic content is independent of $I$.  

We next state the properties of $\psi$ in the form of two theorems. (For details see Section IV.)

**Theorem C.** $\psi$ can be brought into the form

$$\psi(I, N_1 N_2 N_3) = \phi(I, N_1 - N_3 + 2) - \phi(I, N_2 - N_3 + 1) - \phi(I, N_1 - N_2 + 1). \quad (14)$$

We shall presently give the explicit form of $\phi$ but first note the following fundamental properties of $\psi$ which follow directly from Eq. (14):

$$\psi(I; N_1 N_2 N_3) = \psi(I; N_1 - N_3, N_2 - N_3, 0) = \psi(I; N_1 - N_3, N_1 - N_2, 0). \quad (15)$$

The "0" is superfluous in a way. We have here relations between classes that belong to different numbers of particles, namely the classes $(N_1 N_2 N_3), (N_1 - N_3, N_2 - N_3), (N_1 - N_3, N_1 - N_2)$ belonging to $N; N - 3N_3; 3N_1 - N$ particles, respectively. Example: the isotopic content of the class $(431)$ of eight particles is the same as that of the class $(32)$ of five particles and both are equal to the isotopic content of the class $(31)$ of four particles. (There is a simple interpretation for Eq. (15), see Section IV.)

Thus there exists a collection of classes which is a subset of the collection of all classes of all particle numbers $N$ such that if we know the isotopic content of each class in the subset we know the content of every class. This subset is the collection of principal classes.

**Definition 5.** A principal class corresponds to a partition of $N$ in at most two parts and is of the form $(2p + q, p)$ where $3p + q = N$ and where $p$ and $q$ may run independently through all values $0, 1, 2, \cdots$.

Examples: All classes $(N)$, the totally symmetric ones, are principal classes. Then follow the classes $(21)$, $(31)$, $(41) \cdots$; then come $(42)$, $(52)$, $\cdots$; etc.

It is instructive to ask: how many new principal classes do we meet when we let $N$ increase by 1? We start from $N = 2$ where the new principal class is $(2)$. The other class is $(11)$ which according to Eq. (15) has the same content as $(1)$; that is it has $I = 1$. For $N = 3$ we now meet the new classes $(3)$ and $(21)$. In addition there is here the class $(111)$. But the latter has by Eq. (15), the same content as the class "$(0)$" which is the one class empty of particles, that is, the vacuum which by definition is nondegenerate and has $I = 0$. Thus $(111)$ has $I = 0$. In fact all classes $(N_1 N_1 N_1)$ contain $I = 0$, once and only once, and nothing else. Going next to $N = 4$, the new principal classes are $(4)$ and $(31)$. The remaining classes are not of the principal kind. The class $(22)$ can be related to

* After this work was completed, I found that the Theorems C and D are implicitly contained in a paper by Racah (15). The methods employed in Section IV to prove these theorems are different from those used in the cited paper.
(2) whence (22) contains $I = 0$ and 2, each of these only once as we already know from our example $N = 2$; while, finally (211) can be related to (1) that is, it contains $I = 1$ once and nothing else. And so on.

The last column of Table I contains the number of new principal classes. This number will turn out to be indicative of the number of new charge correlation calculations that we shall have to make by going from $N$ to $N + 1$. This we shall see below, but we emphasize from the start: it is not the number of classes, but the number of new principal classes that counts.

Thus we now confine our attention to the principal classes and hence ask for $\psi(I; 2p + q, p)$. It turns out that one has to distinguish the even and odd values of both $p$ and $q$.

**Theorem D.**

\begin{align*}
(p \text{ even, } q \text{ even}): \\
\psi &= 1 + \min\left(\frac{p}{2} ; \left[\frac{I}{2}\right]\right) + \min\left(\frac{p + q}{2} ; \left[\frac{I}{2}\right]\right) \\
&\quad - \min\left(p + q + \frac{1}{2} ; \left[\frac{I + 1}{2}\right]\right). \\
(p \text{ even, } q \text{ odd}): \\
\psi &= \min\left(\frac{p}{2} ; \left[\frac{I}{2}\right]\right) + \min\left(\frac{p + q + 1}{2} ; \left[\frac{I + 1}{2}\right]\right) \\
&\quad - \min\left(p + q + \frac{1}{2} ; \left[\frac{I}{2}\right]\right). \\
(p \text{ odd, } q \text{ even}): \\
\psi &= \min\left(\frac{p + 1}{2} ; \left[\frac{I + 1}{2}\right]\right) + \min\left(\frac{p + q + 1}{2} ; \left[\frac{I + 1}{2}\right]\right) \\
&\quad - \min\left(p + q + \frac{1}{2} ; \left[\frac{I + 1}{2}\right]\right). \\
(p \text{ odd, } q \text{ odd}): \\
\psi &= \min\left(\frac{p + 1}{2} ; \left[\frac{I + 1}{2}\right]\right) + \min\left(\frac{p + q}{2} ; \left[\frac{I}{2}\right]\right) \\
&\quad - \min\left(p + q + \frac{1}{2} ; \left[\frac{I}{2}\right]\right).
\end{align*}

Here $\min (x; y) = x$ for $x \leq y$ and $= y$ for $x \geq y$. And $[x]$ is the largest integer contained in $x$.

We now collect the information. Equation (10) gives the number of configura-
tions per class. In each configuration of given principal class such I-values can be contained as are dictated by Eq. (16)-(19). The contention is that in this way we get a complete enumeration of all \(N\pi\)-states for given \(N\). Example: In Table II we have collected all relevant numbers for \(N = 8\). We have chosen this rather large \(N\)-value to show what general features emerge. To each class (those classes marked with an asterisk are the new principal classes) we give the value of \(\rho\). "Which I" tells the I-values that fit in the class. The notation \(2^2\) means: \(I = 2\) occurs twice. \(\sigma\) is the number of states which can fit into the class and is obtained as follows. Take a fitting I-value, multiply by \((2I + 1)\) and by the number \(\rho\) (as each of the \(\rho\) members of the configuration gives a different state of the same \(I, I_z\)). Thus

\[
\sigma(N_1N_2N_3) = \rho(N_1N_2N_3) \sum I (2I + 1)\psi(I, N_1N_2N_3).
\]  

(20)

Summing over all classes we get a total number 6561 = \(3^8\) of states, as it should be. In general

\[
\sum \sigma(N_1N_2N_3) = 3^N
\]

(21)

where the sum goes over all partitions satisfying Eq. (3). We have denoted by \(n(I)\) how often the isotopic spin I occurs, as follows from the \(\rho\)- and "which I" columns.

We shall now state without proof a few subsidiary results that can be obtained from Theorem D.

**Theorem D1.** The maximal I-spin contained in a principal class is \(I = 2\rho + q\). It occurs only once.

_Example._ The maximal I-spin in (31) is \(I = 3\). The same is true for (32), (431), etc.

**Theorem D2.** Only for \(I \geq 2\) is it possible that I-spin values occur more than once in a given class.

_Examples._ The first time that an I-spin occurs twice happens for \(N = 6\): The class (42) contains \(I = 0, 2^2, 3, 4\). More than one I-spin may occur more than once, as in class (73) which contains \(I = 1, 2, 3^3, 4^3, 5^3, 6, 7\). It is not until \(N = 12\) that an I-spin occurs three times. Thus the class (84) contains \(I = 0, 2^2, 3, 4^3, 5^3, 6^2, 7, 8\). And so on. The double occurrence of angular momentum 2 for \(N = 6\) has been known in nuclear physics for many years (16).

**Theorem D3.** The smallest number of particles for which an I-spin value occurs \(n > 1\) times is equal to \(6(n - 1)\). The (principal) class in which it happens is \((4n - 4, 2n - 2)\). The I-spin to which it happens is \(2n - 2\).

**Theorem D4.** The only principal classes which do not contain any I-spin more than once are the classes \((N)\) and \((N_1, 1)\). The class \((N)\) contains \(I = N, N - 2, N - 4, \ldots, 1(0)\) for odd (even) \(N\). The class \((N_1, 1)\) contains \(I = 1, 2, 3, \ldots, N_1\). Remark: If an I-spin occurs more than once in a given class, the
specification by \(I, I_s\) and configuration of a state is no longer enough. One needs a further label to keep the multiply-occurring \(I\)-values apart. A systematic approach to this question has been started by Racah. However, we shall not pursue this point further in the present paper, as from here on our concern will be exclusively with cloud states, for which multiple occurrence never takes place.

(c) The \(N\pi\)-Cloud

Our example \(N = 8\) shows that in a given class either \(I = 0\) or \(I = 1\) occurs but never both these \(I\)-values. This is a general feature which is easily shown from Eqs. (16)-(19).

Theorem D5. If \(p\) and \(q\) are both even, the principal class contains \(I = 0\) once and only once and does not contain \(I = 1\). If either \(p\) or \(q\) or both \(p, q\) are odd the principal class contains \(I = 1\) once and only once and does not contain \(I = 0\). In other words: the class of the configuration determines the \(I\)-spin of the cloud. It follows that the total number of states of a cloud is equal to the total number of configurations. Because there is a unique correspondence: correlation \(\rightarrow I\)-spin we are now entitled to talk of the class of a cloud state, where earlier we only talked of the class of a configuration. We are now completely organized for the discussion of the properties of the \(N\pi\)-cloud and summarize the state of affairs as follows.

The states of the \(N\pi\)-cloud can be grouped in classes. The classes are distinguished by the correlation numbers \((N_1N_2N_3)\). The correlation numbers determine the \(I\)-spin: if and only if \(N_1 - N_3\) and \(N_2 - N_3\) are even, \(I = 0\), otherwise \(I = 1\).

The number of states in a given class is equal to \(\rho(N_1N_2N_3)\) given by Eq. (10). And if we follow the detailed ordering of symmetrizations and antisymmetrizations to be dictated by the Yamanouchi procedure, the states are orthogonal. In the following subsection (d) it is tacitly understood that we are in this representation.

From a theoretical point of view, the analysis of the bursts of \(\pi\)-mesons produced in antinucleon-nucleon annihilation is therefore somewhat simpler than the discussion of multiple \(\pi\)-production in nucleon-nucleon and \(\pi\)-nucleon collisions, as in the latter case one can have either \(I = 0, 2\) or \(I = 1, 2\) in the same configuration, and as, moreover, \(I = 2\) values may occur more than once.

We are now ready to discuss the charge correlations in the \(N\pi\)-cloud.

(d) About the Correlation Coefficients

In order to get a clear statement of the problem, we revert once more to the example \(N = 2\) and consider the particular state \(\psi_0\) given by Eq. (4). From this equation we read off that the probability to find a \(\pi^+\) with momentum \(p_1\) and a

\[\text{See Ref. 11. The method followed is the one of group interpolation.}\]
\( \pi^- \) with momentum \( p_2 \) is equal to the probability of finding a \( \pi^+ \) with momentum \( p_2 \) and a \( \pi^- \) with momentum \( p_1 \), and both are equal to the probability of finding two \( \pi^0 \)'s with momenta \( p_1 \), \( p_2 \) respectively.

We can put this statement a little weaker: given a configuration, that is, two momenta \( (p_1, p_2) \). The probability of finding this configuration occupied by a \( \pi^+ \) and a \( \pi^- \) is twice as large as the probability of finding \( 2\pi^0 \) (which is the 2/1 ratio in \( K_+^\circ \)-decay into \( 2\pi \) if \( \Delta I = \frac{1}{2} \) is valid). This 2/1 ratio is then the branching ratio in the configuration. Similarly, when we talk of branching ratios in what follows, the meaning is always that we compare two charge channels with the same \( N \), the same total charge \( Q \) and in the same configuration. For example we compare the rate of \( (2\pi^+, 3\pi^-) \) production with that of \( (\pi^+, 2\pi^-, 2\pi^0) \) in a configuration given by five momenta \( p_1, \cdots, p_5 \) but we do not specify which charge goes with which momentum.

Of course, the ratios so obtained also refer to what one finds if one integrates over five momentum intervals and \textit{a fortiori} they refer to total rates per channel, that is, to the result of integrating over all configurations. Total rates are, of course, the easiest to determine experimentally, but it seemed worthwhile to state that the numbers we are going to obtain for branching ratios can be used for much more detailed analyses than total rates only. It need hardly be emphasized that the integration over configurations implies some loss of information, interesting correlation effects could conceivably escape notice this way. (See also Section III.)

Thus we now consider a second way of partitioning the number \( N \) of particles in an \( N\pi \)-cloud, namely with respect to the electric charge. From here on we consider the charge \( Q \) of the cloud as given, \( Q = \pm 1 \) or 0. A charge partition of the cloud is the triple of numbers \( \{m^+, m^-, m_0\} \) which satisfy

\[
\begin{align*}
m^+ + m^- + m_0 &= N, \\
m^+ - m^- &= Q.
\end{align*}
\]  

(22)

Consider some specific state of an \( N\pi \)-cloud for given \( N \) and \( Q \). We wish to know: if the cloud is in this state what are the relative probabilities for the various charge partitions \( \{m^+, m^-, m_0\} \) compatible with Eq. (22)? In Section IV we prove the following theorem.

**Theorem E.** The relative probabilities for the various partitions \( \{m^+, m^-, m_0\} \) which satisfy Eq. (22) and which refer to some state belonging to the class \( (N_1N_2N_3) \) are the same for all states of this class. In other words: the charge correlations have the class property.

This leads us to introduce the following definition.

**Definition 6.** Correlation coefficients, denoted by \( [N_1N_2N_3 \mid m^+, m^-, m_0] \) are the branching ratios of cloud states which belong to the class \( (N_1N_2N_3) \) into the
charge partitions which satisfy Eq. (22). These coefficients are appropriately
normalized:

\[ S[N_1N_2N_3 | m_+, m_-, m_0] = 1. \] (23)

Here the symbol \( S \) denotes summation over all charge partitions compatible
with Eq. (22).

**Examples.** From Eq. (4) we read off the correlation coefficients for the \( 2\pi \-
cloud state of class (2) and the charge partitions* (110) and (002):

\[ [2 | 110] = \frac{2}{\sqrt{6}}, \quad [2 | 002] = \frac{1}{\sqrt{6}}. \] (24)

The \( I = 1 \) state of Eq. (5), has symmetry (11) in the present language. Ac-


a complicated notation for a simple fact, surely, but a simple notation when ap-
plied to complicated many \( \pi \)-systems.

**Theorem F.** All correlation coefficients can be found from those referring to
principal classes by means of the following two identities

\[ [N_1N_2N_3 | m_+, m_-, m_0] = [N_1 - N_3, N_2 - N_3 | m_+ - N_3, m_- - N_3, m_0 - N_3], \] (26)

\[ [N_1N_2 | m_+, m_-, m_0] = [N_1, N_1 - N_2 | N_1 - m_-, N_1 - m_+, N_1 - m_0]. \] (27)

(Note the switch of \( m_- \) and \( m_+ \) on the right-hand side of Eq. (27.) By referring
to definition 5, one sees immediately that indeed we are back once again at the
principal classes.

**Examples of Eq. (26).** (A) The branching ratio

\[ \bar{p}p \to (\pi^+, \pi^-, 3\pi^0)/(2\pi^+, 2\pi^-, \pi^0) \]

with final state in class (311) is the same as \( \bar{p}p \to (2\pi^0)/(\pi^+ + \pi^-) \) with final
state of class (2). Hence the first-mentioned ratio is equal to \( \frac{1}{2} \zeta \), see Eq. (24).

(B) If the final state is of class (221), \( \bar{p}p \)-annihilation can only yield \( (2\pi^+, 2\pi^-, \pi^0) \) because the state (11) can only give \( (\pi^+\pi^-) \), see Eq. (25). (C) Note the special
case of the classes \( (N_1N_2N_3) \). The only charge channel is \( (N_1\pi^+, N_1\pi^-, N_1\pi^0) \).

(D) The ratio \( \bar{p}n \to (\pi^+, 2\pi^-, 3\pi^0)/(2\pi^+, 3\pi^-, \pi^0) \) in a state of class (411) is the
same as \( \bar{p}n \to (\pi^-, 2\pi^0)/(\pi^+, 2\pi^-) \) in the symmetric state (3).

**Examples of Eq. (27).** (A) the ratio \( \bar{p}p \to (\pi^+, \pi^-, 3\pi^0)/(2\pi^+, 2\pi^-, \pi^0) \), class
(32) is the same as \( \bar{p}p \to (2\pi^+, 2\pi^-)/(\pi^+, \pi^-, 2\pi^0) \), class (31). (B) The ratio

* In order to avoid confusion, null-values of one or two of the \( m \)'s must always be written
down as such. This is unlike the notation for the \( N_i \); see Definition 3.
Remark. Whenever any of the three numbers $m - N_3$ on the right-hand side of Eq. (26) or any of the three numbers $N_1 - m$ in Eq. (27) becomes negative the corresponding correlation coefficient is meant to be zero. For example the class (221) does not contribute to the channel $(\pi^+, \gamma, 3\pi^o)$.

The final step is now to give expressions for the correlation coefficients of the principal classes. According to Definition 4, a principal class is characterized by two numbers, $p$ and $q$. We give here the correlation coefficients for arbitrary $q$ and $p = 0, 1, 2$. This is enough to construct complete tables of correlation coefficients for all classes up to and including $N = 8$. One has to distinguish between $q$ even or odd. Furthermore the coefficients for charge $Q = 0$ and $Q = -1$ are best dealt with separately. We call the corresponding coefficients $\hat{p}_p$- and $\hat{p}_m$-coefficients, respectively. It is shown in Section IV how to obtain the following expressions in terms of $\alpha(m,n)$ defined by

$$\alpha(m,n) = \frac{\binom{2m}{m}}{\binom{2n}{n}} \quad \text{for } m \leq n$$

$$= 0 \quad \text{otherwise.}$$

$\hat{p}_p$-coefficients.

$$[2n \mid n - m, n - m, 2m] = \frac{2^{2n-2m}}{2n + 1} \alpha(m,n),$$

$$[2n + 1 \mid n - m, n - m, 2m + 1] = 3 \cdot 2^{2n-2m+1} \cdot \frac{2m + 1}{(n + 1)(2n + 3)} \alpha(m,n + 1),$$

$$[2n + 1, 1 \mid n - m + 1, n - m + 1, 2m] = 3 \cdot 2^{2n-2m+1} \cdot \frac{n - m + 1}{(n + 1)(2n + 3)} \alpha(m,n + 1),$$

$$[2n + 2, 1 \mid n - m + 1, n - m + 1, 2m + 1] = 3 \cdot 2^{2n-2m+1} \cdot \frac{n - m + 1}{(n + 1)(2n + 3)} \alpha(m,n + 1),$$

Equations (28)-(38) are written in such a way that $n$ runs through $0, 1, 2, \ldots$. 
\[ [2n + 2, 2 | n - m + 2, n - m + 2, 2m] \]
\[ = 2^{2n-2m+1} \frac{4mn + 5m - n - 1}{(n + 1)(2n + 3)(2m - 1)} \alpha(m, n + 1), \tag{33} \]
\[ [2n + 3, 2 | n - m + 2, n - m + 2, 2m + 1] \]
\[ = 3.2^{2n-2m+1} \frac{4mn + 7m + n + 1}{(n + 1)(2n + 3)(2n + 5)} \alpha(m, n + 1). \tag{34} \]

\textit{pn-coefficients.}
\[ [2n + 1 | n - m, n - m + 1, 2m] \]
\[ = 3.2^{2n-2m+1} \frac{n - m + 1}{(n + 1)(2n + 3)} \alpha(m, n + 1), \tag{35} \]
\[ [2n + 1, 1 | n - m + 1, n - m + 2, 2m - 1] \]
\[ = 3.2^{2n-2m+2} \frac{n + m + 1}{(n + 1)(2n + 3)} \alpha(m - 1, n + 1), \tag{36} \]
\[ [2n + 2, 1 | n - m + 1, n - m + 2, 2m] \]
\[ = 3.2^{2n-2m} \frac{(n + m + 1)}{(n + 1)(2n + 3)} \alpha(m, n + 1), \tag{37} \]
\[ [2n + 3, 2 | n - m + 2, n - m + 3, 2m] \]
\[ = 3.2^{2n-2m} \cdot \frac{-m^2(4n + 7) + m(2n + 3)(2n + 5) - (n + 1)(n + 2)}{(2m - 1)(2n + 1)(2n + 3)(2n + 5)} \cdot \alpha(m, n). \tag{38} \]

Remark. The well-known results about \(K_{\pi 3}\) decay are also contained in these coefficients. Thus, if we assume \(\Delta I = \frac{1}{2}\) and that the final \(3\pi\)-state is totally symmetric (17) we read off from Eq. (35), with \(n = 1, m = 0\) or 1 that \(K^- \to (\pi^+, 2\pi^-)/(\pi^-, 2\pi^0) = 4\). The other possible states compatible with \(\Delta I = \frac{1}{2}\) would give a ratio equal to unity. Likewise, under the same two assumptions we get (18) \(K_\pi^0 \to (\pi^+ \pi^- \pi^0)/(3\pi^0) = \frac{8}{3}\). The other possible states compatible with \(\Delta I = \frac{1}{2}\) would forbid the \(3\pi^0\)-decay of the \(K_\pi^0\).

With the help of the rules given in this section, it is easy to see that these expressions are not only sufficient to compute all coefficients through \(N = 8\), but that also a good deal of the work has been done whenever one wishes to go to higher \(N\)-values. For example all coefficients for \(N = 9\) are covered by the above expressions except those for the class (63); for \(N = 10\) we only lack (73), for \(N = 11\) only (83), etc. There exists no obstacle to the calculation of the higher coefficients, but for the present the discussion of \(N \leq 8\) seems ample.
It will be noted that there exist several simple identities between the coefficients written down. These identities will not be discussed further in this paper.

III. DISCUSSION

For $N = 2-8$ the values of the correlation coefficients are given in Tables III. The explanation of the symbols in the tables is as follows. The C-column gives the class $(N_1N_2N_3)$ in question, the PC-column contains the corresponding principal class, $I$ gives the $I$-spin, $\rho$ the number of states belonging to the class, as given by Eq. (10). The triples of numbers under the headings $\rho p$ and $\rho n$ denote the charge partitions $(m_+, m_-, m_0)$. The entries in the tables are the correlation coefficients. The columns marked $\eta$ give the value of

$$\eta = \frac{n_{\text{ch}}}{n_0},$$

where $n_{\text{ch}}(n_0)$ are the averages over the charge partitions for a given class of the number of charged (neutral) $\pi$-mesons produced by annihilation into a state of that class.

The row SM(0) contains the (unnormalized) average charge correlations which obtain if only the states with $I = 0$ are considered and all such states are given equal weight (so that the corresponding classes have weight $\rho$. SM(1) refers likewise to $I = 1$ and SM is the average in the same sense over all states $I = 0$ and 1.

Before we discuss the contents of these tables, it may be well to ask what one can say about the feasibility of observing charge correlations. The crucial point is, of course, the detection of the $\pi^0$'s. We recall how and with what degree of detail the experimental problem of $\pi^0$'s in annihilation has been treated so far.

(a) Where Watson's theorem (19) applies, one may use the fact that the average number of charged $\pi$'s produced equals twice the number of the neutrals. This is valid to a good approximation for annihilation on complex nuclei (with corrections for neutron excess). We shall come back presently to the role of this theorem in nucleon-antinucleon annihilation.

(b) Under more general circumstances (which include case (a) as a special category) one often assumes (1-3) that the $\pi^0$-spectrum is the same (apart from normalization) as the $\pi_{ch}$-spectrum, and arrives in this way at an average number of $\pi^0$'s emitted in stars with a given number of visible prongs. For the purpose of obtaining the charge correlations referring to total rates one would like ideally to know the following: given a number of $n$-pronged stars, which percentage of these is accompanied by $0, 1, 2, \ldots, \pi^0$'s? This requires a more refined analysis than only determining the averages mentioned above. Some extent of refinement does not seem out of the question. And, which is perhaps more important, it does not at all seem impossible to modify the experimental arrangements such that considerably more information on the $\pi^0$'s can be obtained.
I shall not discuss this vital question any further in this paper, but would like at this point to express my gratitude to Drs. G. Goldhaber, S. Goldhaber, D. Glaser, and N. Horwitz for orienting discussions on this topic.

We now return to the tables, and first note that the entries of the $\bar{p}n$-type are of practical relevance for the analysis of $\bar{p}$ annihilation in deuterium, where we can distinguish between the two families of channels

\[ \bar{p} + d \rightarrow n + N\pi, \quad (40) \]
\[ p + d \rightarrow p + N\pi. \quad (41) \]

It follows from $I$-spin conservation that the $N\pi$-assembly produced in each of these reactions can only have a total $I = 0, 1$. Thus in deuterium (unlike what happens in the case of a general nucleus) the $N\pi$-system produced constitutes a cloud in the sense of Definition 1. We can therefore apply the tables directly to deuterium, with the understanding that its $\bar{p}p$- and $\bar{p}n$-entries, respectively, refer to the “neutron production” and “proton production” channels of type (40), (41).

It need hardly be stressed that the presence of the final state nucleon does not affect the values of the correlation coefficients. On the other hand, rescattering effects of the $\pi$'s on the residual nucleon may of course modify the correlation patterns of, say, reactions (40) as compared to those of reaction (1). We shall not discuss these modifications in the present paper, they will among other things depend on the ratio of the annihilation volume to the deuteron volume and on the energy distribution in annihilation. However this may be, it is clear that a particular interest attaches to a $\bar{p}$-$d$ experiment.

In this connection it is worth mentioning that there exist relations between the $\bar{p}p$- and the $\bar{p}n$-entries which are a consequence of a somewhat refined version of Watson’s theorem (19). In its application to the present process, the theorem in its simplest form states the following. Let $n_\text{ch}(I)$ and $n_0(I)$ denote the average total number of charged and neutral $\pi$'s, respectively, produced in all channels collectively with isotopic spin value $I$. Then we have

\[ n_\text{ch}^\bar{p}(0) = 2n_0^\bar{p}(0), \quad (42) \]
\[ n_\text{ch}^\bar{p}(1) + 2n_\text{ch}^\bar{n}(1) = 2n_0^\bar{p}(1) + 4n_0^\bar{n}(1), \quad (43) \]

where the superscripts denote whether we are dealing with the reactions (1) or (2), respectively. Equations (42) and (43) also apply to the reactions (40), (41) if we understand “$\bar{p}p$” to refer to Eq. (40) and “$\bar{p}n$” to Eq. (41). Equation (42) follows from the fact that the $I = 0$ state is isotropic. Equation (43) follows from Eq. (42) and from the fact that a 50/50 statistical mixture of $(\bar{p} + \bar{n})$ in interaction with a 50/50 statistical mixture of $(p + n)$ is also isotropic.\footnote{Actually it suffices to consider the interaction between $\bar{p}$ and $(p + n)$. “Isotropic” refers to isotopic spin space, of course.}
If we are able to label the many accessible $I = 0$ and 1 channels with quantum numbers that are good simultaneously with $I$, then Eqs. (42), (43) hold for specific values of these quantum numbers separately. We have immediately available two sets of such commuting quantum numbers: (1) the configuration labels themselves, for example $p_1, \cdots, p_N$; (2) the correlation numbers. Thus a more detailed version of the theorem is

$$n_{\text{ch}}^{\bar{p}p}(0; N_1 N_2 N_3; \text{conf.}) = 2n_{\text{ch}}^{\bar{p}p}(0; N_1 N_2 N_3; \text{conf.}),$$

$$n_{\text{ch}}^{\bar{p}n}(1; N_1 N_2 N_3; \text{conf.}) + 2n_{\text{ch}}^{\bar{p}p}(1; N_1 N_2 N_3; \text{conf.}) = 2n_{\text{ch}}^{\bar{p}p}(1; N_1 N_2 N_3; \text{conf.}) + 4n_{\text{ch}}^{\bar{p}n}(1; N_1 N_2 N_3; \text{conf.}),$$

where "conf." denotes that we are talking about some specific configuration and where the notation otherwise is obvious. (We repeat that a configuration refers to a set of momentum values without regard to charge; see the beginning of Section II (d).) It is a consequence of Eq. (44) that all $\eta$-entries for $\bar{p}p$-reactions in $I = 0$ channels of given class are equal to 2. Equation (45) establishes relations between the $\bar{p}p$- and the $\bar{p}n$-entries.

Next we turn to the rows marked SM which have reference to the statistical model. In the analysis of annihilation stars on the basis of this model, originally proposed by Fermi (20),11 one starts with the following two assumptions:

1. The matrix element is essentially independent of the energies and momenta of the problem12;

2. All individual channels, be it that they have $I = 0$ or $I = 1$, contribute to the total transition probability with the same weight. In this way the total rate of the process (1) becomes proportional to

$$\rho(N) = \rho(1, N) + \rho(0, N),$$

where $\rho(I, N)$ is the number of accessible $N_T$-states with isotopic spin $I$. Likewise for $\bar{p}$ annihilation on neutrons the statistical model implies proportionality to $\rho(1, N)$.

Due to assumption (2), the statistical model makes definite predictions about the charge correlations. In particular it was noted by Fermi (19, 20) that the model implies the following limit-type relations for large multiplicities:

$$n_{\text{ch}}^{\bar{p}p}(1, N) \approx 2n_{\text{ch}}^{\bar{p}p}(1, N); \text{ large } N \text{ (statistically).}$$

Here the single argument $N$ refers to the multiplicity. Likewise we must have

$$n_{\text{ch}}^{\bar{p}n}(1, N) \approx 2n_{\text{ch}}^{\bar{p}n}(1, N); \text{ large } N \text{ (statistically),}$$

where Eq. (48) can be seen as a consequence of Eqs. (45) and (47).

11 For a review of the statistical model, including some refinements, see Milburn (21).

12 Sometimes one includes an essentially kinematical dependence on the over all energy of the system.
These asymptotic ratios are borne out by the $\eta$-entries in the SM-rows. We can now also get a notion of the rapidity with which the asymptotic value $\eta_{BM} \rightarrow 2$ is reached. For $N = 5$ the deviations are within 10%; when we reach $N = 8$ they are $\sim 1\%$. (I do not know of any reason for the curious fact that, for $N = 6$, $\eta_{BM}$ is exactly equal to 2.)

If we now compare the statistical $\eta$-value with its values for specific correlations we note first that the value $\approx 2$ is, of course, not unique for the statistical model and secondly, that $\eta$ may deviate strongly from 2 if certain correlations would predominate. All one could say from an analysis of $\eta$-values only is that if $\eta \approx 2$ the statistical model may not be bad.

Horwitz et al. (3) have subjected their low $p$-energy data to this $\eta$-test and find that $\eta \approx 2$ within the experimental error. This result refers to the total rate of charged versus neutral $\pi$-production, that is, integrated over all configurations. It would be very interesting to see whether the $\eta \approx 2$ value also persists if one treats different configurational intervals separately, for example by choosing a mesh of momentum- and relative angle intervals as fine as is compatible with statistics. It is clear that too much integration over configurational intervals may well tend to mask conceivable correlation effects.

That this indeed may happen is shown by recent results of Goldhaber et al. (22) which reveal spatial correlation effects for which there is no room in the statistical model in its present form. We shall come back elsewhere to refinements of the statistical model which may perhaps take account of these effects. For the present discussion the main point to be emphasized is that experiment indicates that these effects essentially vanish if one "integrates too much."

There are other indications that the statistical model, at least in its form stated here, is not too successful. Thus the size of the annihilation volume has to be taken unphysically large, namely 10 to 15 times the "$\pi$-volume,"

$$(4\pi/3) \cdot (h/m_\pi c)^3$$

in order to account for the observed average multiplicity. Again, the possibility is not excluded that refinements of the statistical model may lead to a more reasonable volume size (23). For the present it is mainly important to note that also the multiplicity is computed on the basis of assumption (2) and that it would be highly desirable to have information as to the degree of validity of the two underlying assumptions separately. Here the experimental study of the charge correlations is a valuable potential source of information.

In the absence of detailed experimental material it does not seem timely to discuss in much more detail the content of these tables which rather speak for themselves. Thus we shall conclude this section by drawing attention to some broad aspects of the problem.

$^{13}$ In this specific instance "integration" means summation over pairs of like and of unlike charge; see Ref. 22.
(1) As was stated in the Introduction, states belonging to different classes almost always give rise to different charge correlations. Referring to the tables we see that for \( N \leq 8 \) it only happens twice that a pair of classes for given \( N \) give the same correlation coefficients. These exceptions are: \( \bar{p}p \)-entries for \( N = 3 \), classes (21) and (111); \( N = 6 \), classes (321) and (222). The case \( N = 6 \) is actually a consequence of the one for \( N = 3 \) as is seen from Eq. (26). The "exceptional identity" which rules this effect is
\[
[N NN | NNN] = [N + 1, N, N - 1 | NNN].
\] (49)
The next time two distinct classes give the same correlations happens for \( N = 9 \). I have verified that for higher \( N \)-numbers than explicitly given here (\( N = 9, 10, 11 \)) no further recurrence of this phenomenon appears. I do not know whether the phenomenon will never appear again (except for \( 3n \) particles where Eq. (49) holds) but I hold it probable that the exceptional identity given here is a lone case.

Actually, if it were our only problem to give a criterion to distinguish between pairs of classes ruled by Eq. (49), life would be very easy. In fact the class \( NNN \) has \( I = 0 \), \( (N + 1, N, N - 1) \) has \( I = 1 \) and therefore the \( \bar{p}n \)-annihilation into states of the former class are forbidden. Thus if we consider simultaneously the \( \bar{p}p \) and \( \bar{p}n \) entries, or, which is the same, if we study the \( d \)-reactions (40), (41) there is not a single exception on record from the rule that charge correlations in principle distinguish between classes.

(2) The number of "triples," in the sense defined in the Introduction, grows as we descend the C-column of each table. Descending likewise in the columns for given charge partition we see that many channels die out and that we end up with only those channels open which give as closely as possible equal numbers of \( \pi^+ \), \( \pi^- \), \( \pi^0 \), respectively. Of course, this trend should not be confused with the property of the statistical model that the average numbers of produced \( \pi \)'s of different charge tend to become equal for large \( N \). The essential distinction is that the statistical model does not have the property of the classes with optimal numbers of triples to forbid the production in ever increasing numbers of charge channels.

(3) The number of "pairs," in the sense defined in the Introduction, is optimal somewhere in the middle of the table, to be more precise, for the class \( (N/2, N/2) \) for even \( N \), \( (\frac{1}{2}(N + 1), \frac{1}{2}(N - 1)) \) for odd \( N \). Note the identities
\[
\begin{align*}
\left[ \frac{N}{2}, \frac{N}{2} | m_+, m_-, m_0 \right] & = \left[ \frac{N}{2} | m_+ - \frac{N}{2}, m_- - \frac{N}{2}, m_0 - \frac{N}{2} \right], \quad (N \text{ even}), \\
\left[ \frac{N + 1}{2}, \frac{N - 1}{2} | m_+, m_-, m_0 \right] & = \left[ \frac{N + 1}{2}, 1 | \frac{N + 1}{2} - m_-, \frac{N + 1}{2} - m_+, \frac{N + 1}{2} - m_0 \right], \quad (N \text{ odd}), 
\end{align*}
\] (50) (51)
which are special instances of Eqs. (26), (27). Thus the Eqs. (29)–(32) contain enough information to write down the correlation coefficients for arbitrary $N$ in classes where the number of pairs is optimal. The tables show clear distinctions between these classes and the ones with optimal number of triples, and also with the statistical model.

(4) The number of singles, in the sense defined as the introduction, is optimal at the top of the tables. These are classes of high symmetry. Their correlation coefficients again differ markedly from the many triple, many pair classes, and from the statistical model.

(5) **Zero-Prong Theorem.** The probability $P_0(N)$ that in an $N\pi$-annihilation all particles appear as neutrals satisfies the inequality

$$P_0(N) \leq \frac{1}{N+1} (N \text{ even}),$$

$$P_0(N) \leq \frac{3}{N+2} (N \text{ odd}).$$

This is a consequence of Eqs. (29) and (30) with $m = n$. The equality is reached if the annihilation has pure correlation of class $(N)$. For the statistical model, we have, for all $N$,

$$P_0^{\text{SM}}(N) = \frac{1}{(N+1)\rho(N)} (N \text{ even}),$$

$$= \frac{3}{(N+2)\rho(N)} (N \text{ odd}),$$

where $\rho(N)$ is given (5) by Eq. (46).

(6) **Two-Prong Theorem.** As exemplified in the tables the number of classes giving rise to two-prong stars is at most equal to 4. Table IV gives the probability $P_2(N)$ for producing a two-prong star in a $\bar{p}p$-event, for each of these four classes and for arbitrary $N$. All results are implied in Eqs. (29)–(34) and (26). From Table IV we deduce:

$$P_2(N) \leq \frac{3}{N-1} (N \text{ even})$$

For $N > 3$

$$P_2(N) \leq \frac{3(N-1)}{N(N-2)} (N \text{ odd})$$

For the statistical model we find with the help of Eq. (10):

$$P_2^{\text{SM}}(N) = \frac{N(4N-3)}{2(N+1)\rho(N)} (N \text{ even}),$$

$$\frac{(N-1)(4N+5)}{2(N+2)\rho(N)} (N \text{ odd}).$$

(53a)
More such theorems can be established, but for the present the indications given here how one finds relations like Eqs. (52), (53) may suffice.

(7) Annihilations involving K-particles. The present formalism is not developed far enough to incorporate the analysis of mixed Kπ-stars. The reason is that in such stars the π-part does not form a cloud in our sense. We will come back to elsewhere to this question.

At low energies (1-3) the fraction of Kπ-stars seems to be less than 10%. Thus, at least at these energies, the present analysis suffices for the vast majority of events.

This concludes the physics contained in this paper. Even though a certain element of systematics has now been brought into the annihilation phenomena, it is evident that these still remain of considerable complexity, as had to be expected. Nevertheless it is hoped, that the present method may serve to find some interesting experimental questions to ask, and, above all, to find out which experimental results may possibly be surprising and which are not.

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IV. PROOFS

(a) SYMMETRIC GROUP AND YOUNG TABLEAUX

All theorems stated in the foregoing are simple consequences, and in a few instances mere transcriptions, of basic properties of the symmetric group $S_N$ of permutations acting on $N$ things, of the three-dimensional unitary group $U(3)$, and their algebras and representations.

The notion of class introduced in Section II is coincident with that of a special set of classes (now in the sense of group theory) of the group $S_N$. A general class of $S_N$ consists of those permutations in $S_N$ which are made up out of a prescribed number of cycles of prescribed length. The specification of a general class of $S_N$ is fully given by a general partition of $N$ as described in Section II, $N_1$ is the number of longest cycles, $N_2$ the number of next longest ones, (which may be equal to $N_1$), etc. The classes $(N_1N_2N_3)$ of our Definition 3 correspond to a subset of the set of classes of $S_N$, restricted by the requirement of partition in at most three parts. From a general theorem about finite groups it follows that to the family of classes $(N_1N_2N_3)$ there corresponds a family of inequivalent irreducible representations of $S_N$ in a one-to-one way. It is furthermore well known (see Ref. 9, p. 213) that the dimension of such a representation is equal to $\rho(N_1N_2N_3)$ as given by Eq. (10). To the $\rho \times \rho$ matrix representations of the
elements of $S_N$ correspond sets of $p$ functions of the things on which the elements act. These $p$ functions are characterized by the representation in hand which transforms them irreducibly into each other.

We now make contact with Theorem A. The things on which the elements of $S_N$ act in our case are the positions. The $p$ functions just referred to are the $p$ "configurations with symmetry." We shall denote these functions by $\phi_a^{[N]}$, where $[N]$ is a shorthand for the partition of $N$ and $a = 1, 2, \cdots, p$. This labeling is complete whenever the content of the configuration is immaterial. Generally, $\phi_a^{[N]} = \phi_a^{[N]}(y_1, \cdots, y_N)$ where the $y_i$ are the "variables of content."

In Section II the number of functions $\phi_a$ was counted by the number of distinct labeled tableaux. It has to be established that this number equals the dimension of the representation of $S_N$. This is not hard to do. (See Ref. 9, Chapter IV and see also Ref. 24.) Also it has to be established that the technique of labeled tableaux will generate a complete set of representations corresponding to partitions in at most three parts. Also this is a known proposition and we shall not discuss the proofs. (See Ref. 24 and Chapter IV of Ref. 9.)

Of course, we are not particularly interested in what the individual elements of $S_N$ do to our configurations, that is to $\phi_a$. Rather do we need only certain linear combinations of these elements which perform the symmetrization operations. In particular we need such combinations which do not transform the $p$ functions $\phi$ into each other, but rather which transform each function into itself. The labeled tableaux are the device to codify which linear combinations do just that. What we need is twofold.

1. To the $\phi_a$ labeled tableau of partition $[N]$ we want to construct an operator $\Omega_a^{[N]}$ which is a linear combination of elements in $S_N$ and which satisfies

$$\Omega_a^{[N]}|^2 = \text{const.} \Omega_a^{[N]}.$$  \hspace{1cm} (54)

2. We need a generating function $f$ of $N$ arguments (positions) such that

$$\Omega_a^{[N]}f = \phi_a^{[N]}(y_1, \cdots, y_N),$$  \hspace{1cm} (55)

where the $\Omega_a^{[N]}$ are considered as operators performing prescribed permutations on the labels $1, \cdots, N$. We have to be careful in the choice of $f$, so as to avoid the trivial result $\Omega_a^{[N]}f \equiv 0$. Then it follows from Eq. (54) that

$$\Omega_a^{[N]}|\phi_a^{[N]}|^2 = \text{const.} \phi_a^{[N]}.$$  \hspace{1cm} (56)

Thus the "idempotent" property (54) just means the reproducibility of $\phi_a^{[N]}$ by $\Omega_a^{[N]}$. [Note. The constant (which is known always to be $>0$) can be trivially worked away by normalizing $\phi_a^{[N]}$ properly. For what follows it is worth emphasizing that for such practical calculations as branching ratios this constant will be immaterial. This will allow us not to have to worry about the normalization of either $\Omega$ or $\phi$.] To construct $\Omega_a^{[N]}$ we first introduce the following definition.
**DEFINITION 7.** The first labeled tableau is the one that corresponds to the lexicographical labeling (see Ref. 9, p. 120). The corresponding $\Omega$ will be denoted by $\Omega_1$. Example: Fig. 2a is the first labeled tableau of the class (32).

A. Young’s prescription for $\Omega_1$ is the following: Let $p_i$ be an element of $S_N$ which does not interchange labels in different rows. Let $q_i$ refer likewise to columns and let $\epsilon_i = +1(-1)$ if $q_i$ is even or odd. Put $P^{[N]} = \sum_i p_i$; $Q^{[N]} = \sum_i \epsilon_i q_i$. Then

$$\Omega_1^{[N]} = P^{[N]} Q^{[N]},$$

and thus

$$\Omega_1^{[N]} f = P^{[N]} Q^{[N]} f = \Phi_1^{[N]}$$

gives us the "first" configuration with symmetry of the class considered, provided we have not chosen $f$ such that $\Omega_1^{[N]} f$ vanishes identically. It is well known (see Ref. 9, p. 121, Eq. (2.6)) that $\Omega_1^{[N]}$ satisfies Eq. (54). In the calculations of correlation coefficients the first labeled tableau plays a major role.

**Remark.** I have found it very useful to use the convention: first antisymmetrize, then symmetrize, as is implied by Eq. (58). One can also invert the order and get a distinct but equivalent set of rules.

Let $A_a$ be the element of $S_N$ which sends the first labeled tableau into the $a$th. ($A_1$ is the unit element of $S_N$.). Then

$$\Phi_a^{[N]} = \Omega_a^{[N]} f = A_a^{-1} \Omega_1^{[N]} A_a f$$

is a set of linearly independent functions which provide a basis for the representation $(N_1N_2N_3)$. Clearly all $\Omega_a^{[N]}$ satisfy Eq. (54) if $\Omega_1^{[N]}$ does.

This follows from Specht’s theorem (25) which says that the functions $\Phi_a^{[N]} = A_a^{-1} \Omega_1^{[N]} A_a f$, characteristic for $(N_1N_2N_3)$, are a set of $n^3$ linearly independent functions provided $f$ is totally unsymmetric. $f(y_1, \ldots, y_N)$ is said to be totally unsymmetric if the $n!$ functions $s_i f$ are linearly independent, where $s_i$ is any element in $S_N$. The condition of total unsymmetry is invoked to generate bases for all representations of $S_N$ with the help of one and the same $f$. We shall use different $f$'s for different representations. The $f$ common to all representations is then a linear combination of those individual $f$'s with arbitrary numerical coefficients. A specific representation projects us into the individual $f$ characteristic for it.

In applying the general theory of representations of $S_N$ to the present case, it is important not to invoke the identities

$$\{x_i \times x_j\} = \{x_j \times x_i\},$$

$$\{x_i \times x_k\} = \{x_k \times x_i\},$$

until all operations pertaining to the group $S_N$ have been performed. Indeed, the Eqs. (60), (61) refer to properties under the group $R_{+}(3)$ and, while they can be expressed by using an element of $S_N$ they have, of course, nothing to do with the latter group. One can circumvent any ambiguity here by performing all calculations involving $S_N$ within the unitary
extension $U(3)$ of $R_+(3)$ where the identities (60), (61) do not hold, if one understands for
example that $(x_1 \cdot x_2)$ means $\sum x_{i\alpha} x_{\alpha\beta}$. Afterwards one may return to the real case as a limit.
I believe that such a technique may be useful. However, for the present it will suffice to
follow the rule not to employ Eqs. (60), (61) until operations referring to $S_N$ are out of the
way.

Example. The simplest more than one-dimensional case occurs for $N = 3$, class
(21). Make up the first labeled tableau and note that $Q = e - \{13\}, P = e + \{12\}$
where $e$ is the unit element of $S_2$ and $\{ij\}$ is the transposition of $i$ and $j$. $f = f(y_1y_2y_3)$ and the permutations will be agreed to act on the indices. Thus

$$\phi_1^{(21)} = f(y_1y_2y_3) - f(y_3y_2y_1) + f(y_3y_1y_2) - f(y_1y_3y_2).$$  \(62\)

As $A_2 = A_2^\dagger = \{23\}$ we have

$$\phi_2^{(21)} = f(y_1y_2y_3) - f(y_3y_1y_2) + f(y_3y_2y_1) - f(y_1y_3y_2).$$  \(63\)

Let us now take Theorem Dd for granted for a moment which says that the class
(21) contains $I = 1$ and $2$ and let us pay attention to content by constructing
the $\phi_a$ for $I = 1$. The functions shall be denoted by $\phi_a(21)$ (1) where the "1" re-
fers to the $I$ spin. The variables of content now are $x_i$ with $x$ defined as in Eq.
(7) and where $i$ denotes a specific value of the configuration label (for example
$p_i$). The choice of $I = 1$ must reflect itself in the choice of $f$ which shall transform
as a vector. We take

$$f = O(x_1 \times x_3) \times x_2,$$  \(64\)

where $O$ means: read the indices 1, 2, 3 in their natural order. We see here a
conflict of notation: we can, of course, write Eq. (64) with the 1, 2, 3 directly
in natural order but then lose the great advantage of vector notation.

We circumvent these problems by using a "two-dimensional" notation. In
what follows we are going to write

$$(i)_{\alpha} \text{ for } x_{i\alpha}, \quad \alpha = 1, 2, 3$$

$$(j)_{\alpha} \text{ for } (x_i \times x_j)_{\alpha} ; \quad (ij) \text{ for } x_i \cdot x_j$$

$$\begin{pmatrix} i \\ j \\ k \end{pmatrix} \text{ for } x_i \cdot (x_j \times x_k).$$  \(65\)

Thus Eq. (62) becomes

$$f = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \times 2.$$  \(66\)

\textsuperscript{14} The notation $\{\}$ is used rather than the conventional ( ) to avoid confusion with the
other symbols used here.
Note: the vector multiplication sign between two columns shall not be treated by the conventions of Eq. (65). In this way we find from Eqs. (62), (63), and (60)

\[ \phi_1 \cong \frac{1}{3} \times 2 + \frac{2}{3} \times 1, \]

\[ \phi_2 \cong \frac{1}{2} \times 3 + \frac{3}{2} \times 1. \]

The notation \( \cong \) means: apart from a constant factor. We shall have no need at all for proper normalization and such constants will be dropped.

(2) The choice (66) of \( f \) in "pre-antisymmetrized form" is a first example of what can be done "by hand." Such devices will make the final computations of this section very simple.

(3) Equations (67) illustrate by counter example the well-known fact that the prescription given above does not in general lead to orthogonal states. The second terms in Eqs. (67) are identical (apart from sign); the first terms are not. It is in fact characteristic for the tableaux that they are quite natural for such nonorthogonal representations. Orthogonalization, though straightforward, leads to more complicated rules (12, 13). We are not much accustomed in physics to work with nonorthogonal states. In the present case they are very useful, however. The reason is the following. The transition from the present to an orthogonal representation is possible by a unitary transformation. If we call \( \psi^{[N]}_a \) the orthogonal states then

\[ \psi^{[N]}_a = \sum_{b=1}^{6} S_{ab} \phi^{[N]}_b, \]

where the matrix \( S \) linearly combines the various nonorthogonal configurations. But \( S \) does not act on the variables of content. Hence for proofs of many general properties which refer to content only, there is no trouble to work with the \( \phi_a \). This we shall see in the next subsection.

(b) Unitary Group and Isotopic Content

Consider a general labeled tableau for the class \( (N_1 N_2 N_3) \) with labels \( i_1, \ldots, i_{N_1} \) in the first column, \( j_1, \ldots, j_{N_2} \) in the second, \( k_1, \ldots, k_{N_3} \) in the third. In the notation of Eq. (65) we adjoin to it a function

\[ F^{(N)}_a(\alpha_1, \alpha_2, \ldots, \alpha_{N_2-N_3}; \beta_1, \beta_2, \ldots, \beta_{N_1-N_2}) \]

defined by

\[ F^{(N)}_a(\alpha_1, \alpha_2, \ldots, \alpha_{N_2-N_3}; \beta_1, \beta_2, \ldots, \beta_{N_1-N_2}) = \left( \begin{array}{c} i_1 \\ i_2 \\ \vdots \\ i_{N_2} \\ j_1 \\ j_2 \\ \vdots \\ j_{N_2} \\ k_1 \\ k_2 \\ \vdots \\ k_{N_3} \end{array} \right) \left( \begin{array}{c} i_{N_2+1} \\ \vdots \\ i_{N_1} \\ j_{N_2+1} \\ \vdots \\ j_{N_1} \\ k_{N_3+1} \end{array} \right), \]

(69)
where the $\alpha$, $\beta$ take on the values 1, 2, 3. The index $a$ marks the particular labeled tableau in hand. Out of these pre-antisymmetrized $F$'s we can, like in the special case of Eqs. (62), (63), construct several $\phi_{a}^{[N]}$ by symmetrizing in the rows; there are in general many $\phi_{a}^{[N]}$ because the $\alpha$, $\beta$ labels are still hanging loose. We can next take linear combinations in the $\alpha$, $\beta$ to obtain, so we have now to prove, functions with certain prescribed $I$-spins. For the moment we denote the functions obtained after symmetrization by $\phi_{a}^{[N]}(c)$ where $c$ enumerates the alternatives for the labels $\alpha$, $\beta$.

At this stage we invoke the capital theorem which builds a bridge between configuration and content: there is a one to one correspondence (apart from equivalence) between the irreducible representations of the group $S_N$ referring to partitions in at most three parts and those of the group $U(3)$ of three dimensional unitary transformations. We apply the theorem by observing that therefore the $\phi_{a}^{[N]}(c)$ must be the base vectors of an irreducible representation of $U(3)$ where the elements $U$ of $U(3)$ act on the variables $\alpha$, $\beta$. An example of such a unitary transformation was given in Eq. (9). Thus we have

$$U\phi_{a}^{[N]}(c) = \sum_{c'} D_{a}^{[N]}_{c'}(U)\phi_{a}^{[N]}(c'),$$

(70)

where the matrix $D_{a}^{[N]}(U)$ is the representative of the operator $U$. Note that $D$ linearly combines the $\phi_{a}^{[N]}(c)$ for different $c$ but the same $a$!

Equation (70) implies that we are entitled to read $I$, $I_z$ (and possible residual quantum numbers) for $c$ if we confine ourselves to the transformations of the $I$-spin group. The latter is the group $R_+ (3)$ of proper orthogonal transformations which is a subgroup of $U(3)$. Hence the $\phi_{a}^{[N]}(c)$ constitute also a representation with respect to $R_+ (3)$. But the restriction $U(3) \rightarrow R_+ (3)$ may (and in general will) make the representation (70) reducible.

The independence of $D$ from $a$ means that a rotation acting on any of the $\phi_{a}^{[N]}(c)$, $a = 1$, $\cdots$, $\rho$ induces the same (reducible) representation for all $a$.

Thus we have now established Theorem B. The argument leading to (70) is, of course, the one that underlies the classification of all complex spectra. We have yet to find which representations of $R_+ (3)$ are contained in Eq. (70) and how often they occur. That is, we must still compute $\psi(I, N_1 N_2 N_3)$. Before we actually do this we prove the identities of Eq. (15).

(c) ISOTOPIC SPIN GROUP $R_+ (3)$ AND SUBSHELLS; DUALITY

Now that Theorem B is established, the identities (15) follow by considering specific labeled tableaux.

(1) Consider the case $[N] = (N_1 N_2 N_3)$ with $N_3 \neq 0$ and take the set of la-

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14 See reference 10, p. 228, where the equivalence is established for the general linear group GL(3) rather than U(3). The restriction to the unitary subgroup of GL(3) does not affect the theorem.
beled tableaux whose first column reads 1, 2, 3, downward. Clearly the various possibilities for labeling the residual squares just correspond one-to-one to all the labeled tableaux for the class \([N'] = N_1 - 1, N_2 - 1, N_3 - 1\). Suppose we drop the first column; this means that we omit a factor with \(I\)-spin zero, see Eq. (65). Such a factor does not change the isotopic content. Thus the particular labeled tableau of class \([N]\) has the same content as all labeled tableaux of class \([N']\) and hence, using also Theorem B:

\[
\psi(I, N_1 N_2 N_3) = \psi(I, N_1 - 1, N_2 - 1, N_3 - 1).
\]

Treat the second column of length 3 (if present) similarly, etc. It follows that the class specified by

\[
\begin{align*}
N_1' &= N_1 - N_2, \\
N_2' &= N_2 - N_3, \\
N_3' &= N_3 - N_3
\end{align*}
\]

has the same isotopic content as does \([N]\). This establishes the first part of Eq. (15) and shows the special significance of the triples which we mentioned in the introduction. The totally antisymmetric state of 3 \(\pi\)-mesons constitutes what amounts nearest to a "closed subshell" in our considerations.

Consider next a class \((N_1 N_2)\), i.e., \(N_3 = 0\) and one of its \(F\)-functions (see Eq. (69)):

\[
F_{a(N_1 N_2)}(\alpha_1 \cdots \alpha_{N_2}, \beta_1 \cdots \beta_{N_1 - N_2}) = \binom{1}{2}_{a_1} \binom{3}{4}_{a_2} \cdots \binom{2N_2 - 1}{2N_2}_{a_{N_2}} \binom{2N_2 + 1}{\beta_1} (N_1 + N_2)_{\beta_{N_1 - N_2}}
\]

which is a product of \(N_2\) "polar vector components" and \(N_1 - N_2\) "axial vector" components. Because we only deal with proper rotations there is really no physical distinction between axial and polar character. Consider the dual transformation by which we interchange axial and polar components, as follows:

\[
F_{a(N_1 N_2)}(\alpha_1 \cdots \alpha_{N_2}, \beta_{N_1 - N_2}) = \binom{1}{2}_{\beta_1} \binom{3}{4}_{\beta_2} \cdots \binom{2N_1 - 2N_2 - 1}{2N_1 - 2N_2}_{\beta_{N_1 - N_2}} (2N_1 - 2N_2 + 1)_{a_1} \cdots (2N_1 - N_2)_{a_{N_2}}.
\]

This has the same structure relative to \(R_+(3)\) as does (72). Thus to the \(a\)th labeled tableau of class \((N_1 N_2)\) belongs a dual tableau of \((N_1, N_1 - N_2)\). Making linear combinations in the \(\alpha, \beta\) labels to get a prescribed \(I\)-spin is therefore the same process in either case. With reference to Theorem B this proves the second half of the identities (15). For the case \(N_3 \neq 0\) one sees that the class dual to
(N_1 N_2 N_3) is given by

\begin{align*}
N_1' &= (N_1 + N_3) - N_3, \\
N_2' &= (N_1 + N_3) - N_2, \\
N_3' &= (N_1 + N_3) - N_1.
\end{align*}

Remark. The duality property is characteristic of \( R_+(3) \) but not of its unitary unimodular extension \( SU(3) \) for which the Cauchy–Jacobi identity (an element of an orthogonal determinant equals its principal minor) does not hold. The invariance of the volume product \( x_1(x_2 \times x_3) \) is shared by \( SU(3) \) and \( R_+(3) \) (it is due to unimodularity) but not by \( U(3) \). Thus subshell and duality properties for \( R_+(3) \) are sufficient conditions for the sequence of constraints \( U(3) \rightarrow SU(3) \rightarrow R_+(3) \). We use this fact to calculate the isotopic content.

(d) Back to \( U(3) \). Calculation of \( \psi(I; N_1 N_2 N_3) \)

The method used may be illustrated by a more familiar example. If one composes angular momenta \( j, j' \) the resultant system has angular momentum content \( j + j', \ldots, |j - j'| \) and each value occurs only once. It is well known that a simple proof for this statement is found by decomposing the character of the reducible product representation as a sum of characters of irreducible ones; the coefficients in the sum give the multiplicity of occurrence.

Likewise we start here from the character of the irreducible representation of \( U(3) \) to which \( (N_1 N_2 N_3) \) corresponds and which is given by (see Ref. 9, p. 201, Theorem (7.5B))

\[ \chi^{[N]}(\alpha_1, \alpha_2, \alpha_3) = \xi^{[N]} \cdot \Delta^{-1}, \]

\[ \xi^{[N]} = \sum_p e_p e_1^{N_1} e_2^{N_2} e_3^{N_3}, \]

\[ \Delta = (\epsilon_1 - \epsilon_2)(\epsilon_1 - \epsilon_3)(\epsilon_2 - \epsilon_3), \]

\[ \epsilon_j = \exp i\alpha_j. \]

The three angles \( \alpha_i \) satisfy \( 0 \leq \alpha_i \leq 2\pi \) and are the characteristic roots of a three-dimensional unitary matrix,

\[ U = \text{diag} (\epsilon_1, \epsilon_2, \epsilon_3). \]

The summation extends over the permutations of \( \epsilon_1, \epsilon_2, \epsilon_3 \). \( e_p = +1(-1) \) if the permutation is even (odd).

We now look upon this irreducible representation of \( U(3) \) as reducible under \( R_+(3) \). This means that two classes \([N], [N']\), related by Eq. (71) or Eq. (74) are equivalent reducible representations of \( R_+(3) \). These conditions imply two
constraints between the three $\alpha$'s. Equation (71) yields

$$\alpha_1 + \alpha_2 + \alpha_3 = 0,$$

which in view of Eq. (76) corresponds to the constraint $U(3) \rightarrow SU(3)$. Equation (74) in conjunction with Eq. (77) implies that one of the $\alpha$'s vanish, say

$$\alpha_1 = 0.$$  \hspace{1cm} (78)

This, indeed, brings us to $R_+(3)$ as is seen\(^{16}\) from inserting Eqs. (77), (78) into Eq. (76):

$$U \rightarrow R = \text{diag. } (1, \epsilon, \epsilon^*); \epsilon = \exp i\alpha.$$ \hspace{1cm} (79)

Substituting Eqs. (77), (78) into Eq. (75) gives

$$\chi^{[N]}_u \rightarrow \chi^{[N]}_R(\alpha) = \xi_R^{-1} \Delta_R^{-1},$$

$$\xi_R = (\epsilon^{N_1-N_2+2} - \epsilon^{N_2-N_3+1} - \epsilon^{N_1-N_3+1}) - c.c.,$$

$$\Delta_R = (\epsilon^2 - 2\epsilon) - c.c.$$ (c.c. = complex conjugate). The character $\chi^{I}(\alpha)$ of the representation $I$ of $R_+(3)$ is given by

$$\chi^{I}(\alpha) = 1 + 2 \sum_{k=1}^{J} \cos k\alpha.$$  

In the standard way we get

$$\psi(I; N_1N_2N_3) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \chi^{[N]}_R(\alpha) (\cos I\alpha - \cos (I + 1)\alpha),$$

from which one finds that the function $\phi(I,M)$ of Eq. (14) is given by

$$\phi(I, M) = -\frac{1}{4\pi} \int_0^{2\pi} \frac{\sin M\alpha}{\sin \alpha} \chi^{I}_{R}(\alpha) d\alpha.$$  

With the help of

$$\frac{\sin M\alpha}{\sin \alpha} = 1 + 2 \sum_{j=1}^{J} \cos 2j\alpha, \hspace{1cm} J = \frac{1}{2}(M - 1), \text{ (odd } M),$$

$$= 2 \sum_{j=1}^{J} \cos (2j + 1)\alpha, \hspace{1cm} J = \frac{1}{2}M - 1, \text{ (even } M),$$

Eqs. (16)--(19) follow immediately.

\(^{16}\) Note the special case $N_1 - 2N_2 + N_3 = 0$ which is the self-dual case. Here one of the $\alpha$'s becomes indeterminate. In view of Eq. (79) we must nevertheless impose the condition (78).
There emerges now a clear picture of the meaning of principal classes. They refer to what goes on "outside the subshells." The latter are sufficiently closed (all in a mathematical sense) to not affect very much such physically important quantities as isotopic content. Also they do not affect the correlations. We now turn to the proof of the latter statement.

(e) Class Property of Charge Correlations; Charge Tableaux

We first define formally what a branching ratio is. Consider a charge partition \( \{m_+, m_-, m_0\} \) which we shall here denote by \([M]\). Define \(X_{[M]}\) by

\[
X_{[M]} = \pi_+(1)\pi_+(2) \cdots \pi_+(m_+)\pi_-(m_+ + 1) \cdots \\
\pi_-(m_+ + m_-)\pi_0(m_+ + m_- + 1) \cdots \pi_0(N),
\]

where \(\pi\) is related to \(\pi\) by Eq. (7), and where the arguments refer to configuration labels. Let \(P\) denote a permutation of the charge labels which brings us from state \(X_{[M]}\) to a distinguishable state. Thus \(P\) may not interchange two \(\pi\)'s with the same charge. The number of such permutations is

\[
\zeta = \frac{N!}{m_+!m_-!m_0!}.
\]

Thus a \(\zeta\)-dimensional invariant subspace in the \(3N\)-dimensional space of states is generated. We call this the \(M\)-space. We proceed likewise for a different charge partition \(M'\) belonging to the same \(Q\), starting from a state \(X_{[M']}\). Let \(\phi_{[M]}^{[N]}\) be a cloud state. Then

\[
\left( \frac{\sum_P PX_{[M]}, \phi_{[M']}^{[N]} \right)^2 \\
\left( \sum_P P'X_{[M']}, \phi_{[M']}^{[N]} \right)^2
\]

is the branching ratio for the state \(\phi_{a}^{[N]}\) into two different charge partitions. That is, \(\lambda\) is the ratio of projections of the state on two distinct subspaces. The class property of branching ratios will have been shown if it is proved that \(\lambda\) is independent of \(a\).

The states \(PX\) are defined by an orthogonal procedure, while the \(\phi_{a}^{[N]}\) do not have orthogonal character. We prove Theorem E by going to an orthogonal representation for \(\phi_{a}^{[N]}\) and use the same orthogonal representation in the spaces \(PX_{[M]}\).

In the Introduction we mentioned the existence of the Young–Yamanouchi orthogonal representations (12). They are generated by an inductive process in which close attention is paid to sequences of subgroups of \(S_N\) that correspond to a given labeled tableau. Orthogonality is achieved by starting from an orthogonal set of base configurations for \(S_{n-1}\) and then incorporating the \(N\)th configu-
ration label in such a way that only an operation on the \((N - 1)\)th and \(N\)th labels is involved. The recipe for this procedure is very clearly stated by Jahn and van Wieringen (13). All we shall need of this representation is a theorem of Thrall (26)\(^\text{17}\) which contains all the algebraic information that is at the root of the Young–Yamanouchi representations.

**Thall's Theorem.** To each class \([N]\) one can construct \(\rho\) operators \(T_a^{[N]}\) which have the property

\[
T_a^{[N]} T_a^{[N']} = \delta_{aa'} \delta([N], [N']) T_a^{[N]},
\]

that is, the \(T_a^{[N]}\) are orthonormal idempotents.\(^\text{18}\) This is clearly a much more potent property than Eq. (54). The importance of the \(T\)-operators lies in the fact that their use instead of \(\Omega_a^{[N]}\) circumvents many complications such as using ordered tableaux (9, 10). We refer to the literature for the explicit construction of the \(T\)-operators. All we shall need is their existence. Clearly we can construct orthogonal states \(\psi_a^{[N]}\) by the prescription

\[
\psi_a^{[N]} = T_a^{[N]} f^{[N]}
\]

for suitably chosen \(f\). We can now also find orthogonal base vectors in the \(M\)-space by means of the \(T\)-operators:

To a given (unlabeled) tableau with \(N\) squares and a given charge partition we adjoin one or more “charge tableaux” as follows.

(a) Fill each square of the tableau with a +, −, or 0 in such a way that there are never two or more like charges in the same column. Only such and all such tableaux are kept for which this is possible. They are called charge tableaux.

(b) If there exists a permutation of charges in a given charge tableau with the properties (a) it brings two charges initially in the same row into the same column; (b) in doing so there are still no two like charges in the same column, (c) in at least two rows the total charge per row has changed, then we adjoin to this permutation a new charge tableau. And so on if there are more permutations satisfying the above conditions.

To each charge tableau belong a set of \(\rho\) states in \(M\)-space generated by the action of \(T_a^{[N]}\) on the configuration labels. Two distinct charge tableaux belonging to the same tableau have no states in common.\(^\text{19}\) To a given charge partition there belong several tableaux satisfying (\(a\)), to a given tableau there may belong several charge tableaux. To the total set of charge tableaux belong a set of \(\xi\) states which span the \(M\)-space.

\(^{17}\) See also the work of Rutherford (27) for a survey of various representations of \(S_N\).

\(^{18}\) See Thrall's paper, Theorem III. He denotes by \(e(ij)\) what here is called \(T_a^{[N]}\), \(j\) enumerates the Young tableaux, \(i\) the labeled tableaux for given \(j\).

\(^{19}\) As there is no permutation in a given charge tableau which belongs to the operator \(PQ\) of the tableau and which brings one charge tableau into another. See Ref. 10, p. 237, Hilfssatz; Ref. 9, p. 123, Lemma (4.2 B).
Example: see Fig. 3 where all charge tableaux are given for the charge partition \(2,2,2\). The \(p\)-values add up as follows

\[
1 + 5 + 5 + 9 + 9 + 10 + 5 + 16 + 16 + 5 = 90 = \frac{6!}{2!2!2!}
\]

as it should be. An orthogonal set of states in \(M\)-space is thus obtained as follows.

Adjoin to a given charge tableau a state \(Y_{M}^{[N]}\), by first writing down \(\pi\)'s in the same sequence as in the first row of the tableau, then behind it the sequence of \(\pi\)'s corresponding to the second row and finally behind that the third row \(\pi\)'s. Give the first \(\pi\) the configuration label 1, the second one 2, \(\ldots\), the last one \(N\).

Note: \([N]\) enumerates the unlabeled tableaux, \(\nu\) the different charge tableaux that may belong to the given tableau. The suffix \([M]\) keeps track of the specific charge partition. The set of states

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span $M$-space. Thus we write for $\lambda$:

$$\lambda = \frac{\sum_{\{N\}} \sum_{b} \sum_{\nu} \left( T_b^{[N]} Y_{[M],\nu}, T_a^{[N]} f^{[N]} \right)^2}{\sum_{\{N\}} \sum_{b} \sum_{\nu} \left( T_b^{[N]} Y_{[M],\nu}, T_a^{[N]} f^{[N]} \right)^2}$$

and now apply Eq. (83):

$$\lambda = \frac{\sum_{\nu} \left( Y_{[M],\nu}, T_a^{[N]} f^{[N]} \right)^2}{\sum_{\nu} \left( Y_{[M],\nu}, T_a^{[N]} f^{[N]} \right)^2}.$$

We have thus arrived at a form for $\lambda$ which contains a projection of the state on only as many orthogonal states as correspond to the range of $\nu$. Next we observe that instead of showing that $\lambda$ does not depend on $a$, we will also achieve our aim if we show that

$$\xi = \frac{\sum_{\nu} \left( Y_{[M],\nu}, T_a^{[N]} f^{[N]} \right)^2}{\sum_{\nu} \left( Y_{[M],\nu}, T_b^{[N]} f^{[N]} \right)^2}$$

is independent of the charge partition $[M]$. And this follows indeed from the following lemma.

**Lemma.** $f^{[N]}$ can be so chosen that it contains all states $Y_{[M],\nu}$ for various $[M]$ and $\nu$ with relative weight factors $\pm 1$.

This lemma will be shown in the next subsection. Accepting the lemma, we note that $T$ acts on the configuration labels only. $T$ can therefore neither intercombine two $Y$'s belonging to a different $[M]$ nor different $\nu$ because these last two labels refer to charge, not to configuration. Thus the lemma implies that

$$\xi = \frac{\sum_{\nu} \left( Y_{[M],\nu}, T_a^{[N]} Y_{[M],\nu} \right)^2}{\sum_{\nu} \left( Y_{[M],\nu}, T_b^{[N]} Y_{[M],\nu} \right)^2}.$$

Finally, observe that by definition the configurational labeling in all $Y$'s is the same. It follows that the value of a given term in the $\nu$-summation in the numerator as well as in the denominator is independent both of $M$ and of $\nu$. For different $[M]$ the range of $\nu$-values will in general be different but this clearly does not affect the ratio $\xi$.

Thus Theorem E has been established. We have consistently worked with ratios so as to circumvent the need of normalizing the $f$'s.
(f) Standard Products; Calculation of Correlation Coefficients

To prove the lemma and derive Eqs. (29)-(38) we introduce the following definition.

**Definition 8.** To a given class \([N]\) we adjoin a function \(f^{[N]}\), the standard product, by the following rules: start from the first labeled tableau and adjoin to the \(N_3(\geq 1)\) columns of length 3 the products

\[
\begin{pmatrix}
1 \\
2 \\
\vdots \\
N_3
\end{pmatrix}
\begin{pmatrix}
\tilde{j}_1 \\
\tilde{j}_1 + 1 \\
\vdots \\
\tilde{j}_1 + N_3 - 1
\end{pmatrix}
\]

see Eq. (65). To the \(N_2 - N_3\) pairs and \(N_1 - N_3\) singles we adjoin the corresponding polar and axial vectors, respectively, labeled according to the precepts of the first tableau, see Definition 7. We couple these \(N_1 - N_3\) vectors together as follows.

1. \(N_2 - N_3\) even. Make scalar products of the first and the second polar vector, then the 3rd and 4th, \(\cdots\). If \(N_1 - N_2\) is also even, pair the axial vectors likewise. If \(N_1 - N_2\) is odd, leave the last single unpaired.

2. \(N_2 - N_3\) odd. If \(N_1 - N_2\) is even, make scalar products out of the pairs of polar vectors as before, leave the last polar vector free, treat the singles as before. If \(N_1 - N_2\) is also odd, take the vector product of the last unpaired polar with the first axial vector and pair the remaining singles in scalar products. We observe:

(a) The standard product (SP) has the transformation property of the cloud \(I\)-spin that goes with the class in question. (Incidentally, this shows how obvious the connection (symmetry \(\rightarrow I\)-spin of the cloud) actually is.)

(b) By inspection, (use Eq. (7)) one verifies that the SP satisfies all requirements which the lemma of the previous subsection requires it to have.

Now that we have therefore filled in all details about Theorem E, the proof of Theorem F is obvious. Equation (26) is a consequence of the subshell proposition of Section IV (c): change the labeling of \(f^{[N]}\) as prescribed in the derivation of Eq. (71) and take a fixed \(Q\). Consider one particular labeling of the residual \(N\)-3 squares. Symmetrize in all rows but keep the first row fixed. This gives us one configuration of class \([N']\), Eq. (71) with its characteristic branching ratios pertaining to the same \(Q\). To the partition \([M] = \{m_+, m_-, m_0\}\) of the full tableau corresponds the partition \([M'] = \{m_+ - 1, m_- - 1, m_0 - 1\}\) of the tableau of class \([N']\), as

\[
x_1(x_2 \times x_3) \cong \sum_\pi \epsilon_\pi \pi_+ (1) \pi_- (2) \pi_0 (3),
\]

where we sum over the permutation of charges and where \(\epsilon_\pi = +1 (-1)\) for an even (odd) permutation. It is readily seen from the inclusion of the row-
symmetrization with respect to the first column that the transition probabilities of the various \([M]\) corresponding to \([N]\) are in the same *proportion* compared to the probabilities of the various \([M']\) corresponding to \([N']\). From this and Theorem E, Eq. (26) follows by induction. Note: all that is necessary is to relate *one* configuration in \([N]\) to *one* in \([N']\). It is immaterial therefore whether these configurations are generated by a \(T\)-operator. If they are not, we can (if necessary) always achieve a unitary transformation which aligns our singled-out configuration properly.

Similarly Eq. (27) is a consequence of the duality which connects Eq. (27) with Eq. (73). In terms of charge the dual transformation means the correspondence

\[
\pi_+ \leftrightarrow \pi_0 \pi_0 = \pi_0 \pi_0,
\]

\[
\pi_- \leftrightarrow \pi_+ \pi_0 = \pi_0 \pi_+,
\]

\[
\pi_0 \leftrightarrow \pi_+ \pi_- = \pi_- \pi_+.
\]

By applying duality, one verifies Eq. (27) by an entirely similar argument as the one just given for Eq. (26). There is one point to be observed. Let us be in the class \((N_1 N_2)\) with charge \(Q\) fixed to be \(= -1\). The dual transformation brings us into the class \((N_1, N_1 - N_2)\) with charge \(Q = +1\). Thus in first instance we map by duality \(\hat{p}_n\)-annihilation onto \(\hat{n}_p\)-annihilation. To establish Eq. (27) one has then to invoke charge symmetry.

The one remaining task is the proof of Eqs. (29)–(38). Now that all theorems have been established this is quite simple. All we need is one configuration with symmetry which is guaranteed to belong to the class \([N]\); and that for all \([N]\). We have a candidate: the first labeled tableau. Thus all we need are the normalized probabilities for the row-symmetrized SP.

We hope that by this time the contention of the Introduction will be obvious: take any vector (for \(I = 1\)) or scalar (for \(I = 0\)) composed out of \(N x_i\), and then let the group \(S_N\) do the rest. With a little foresight we have avoided all projection into the various \(p\)-dimensional spaces of configuration by choosing the SP as a starting point. The vector composition implied by the SP definition is all we need of the rotation group \(R_{+}(3)\). It is thus entirely superfluous to use the vector addition tables. And all the rest is combinatorics.

**Derivation of Eq. (29).** The SP is \((12)(34)\cdots(2n - 1, 2n)\), hence

\[
\phi^{(2n)}_{\lambda} = \sigma(12)(34)\cdots(2n - 1, 2n)
\]

where \(\sigma\) denotes symmetrization. As \((12) = (21)\), etc., we only need

\[
N(2n) = \frac{2n!}{2^n n!}
\]
terms of $S_{2n}$. Thus $\phi$ consists of $N(2n)$ monomials, the first one of which is the SP itself. Denote by $T_{\text{SP}}[n - m, n - m, 2m]$ the number of distinct ways in which we can let the charge partition in question be distributed over the positions in the SP:

$$T_{\text{SP}}[n - m, n - m, 2m] = \binom{n}{m} 2^{n-2m}, \quad (88)$$

as is easily seen from

$$(12) \cong \pi_{+}(1) \pi_{-}(2) - \pi_{0}(1) \pi_{0}(2) + \pi_{-}(1) \pi_{+}(2).$$

The total number of terms in $\phi^{(2n)}$ corresponding to our charge partition is thus $N(2n) T_{\text{SP}}[n - m, n - m, 2m]$. The total number of distinct ways in which the charge partition is distributed over the configuration labels in any term of $\phi$ is

$$T[n - m, n - m, 2m] = \frac{(2n)!}{(2m)!(n - m)!!}. \quad (89)$$

But every such distinct way must have equal weight $W[n - m, n - m, 2m]$ as our function $\phi$ is totally symmetric. Hence (dropping arguments)

$$NT_{\text{SP}} = WT. \quad (90)$$

The properly normalized correlation coefficients are equal to

$$\frac{W^2 T}{\Sigma W^2 T}, \quad (91)$$

where in the numerator we have a fixed $m$ in mind and in the denominator we sum over all $m$. Define $\gamma_k(n)$ by

$$\gamma_k(n) = \sum_{m=0}^{n} \frac{m^k}{2^{2m}} \binom{2m}{m}. \quad (92)$$

Then Eq. (29) follows from Eqs. (87)-(91) and

$$\gamma_0(n) = \frac{2n + 1}{2^n} \binom{2n}{n}. \quad (93)$$

For further calculations it is useful to note that

$$\gamma_1(n) = \frac{n}{3} \gamma_0(n), \quad (94)$$

$$\gamma_2(n) = \left(\frac{n^2}{5} + \frac{2n}{15}\right) \gamma_0(n), \quad (95)$$

$$\gamma_3(n) = \left(\frac{n^3}{7} + \frac{6n^2}{35} + \frac{2n}{105}\right) \gamma_0(n), \text{ etc.} \quad (96)$$
All other Eqs. (30)-(38) follow by a similar argument: one always works toward equations of the kind (90), (91). Thus it will suffice to indicate which refinements are still needed for other classes.

**Remark.** Clearly $\text{N}(2n)$ drops out in the final answer.

**Equations (30) and (35).** We start from

$$\phi_1^{(2n+1)} = \sigma(12)(34)\cdots(2n-1,2n)(2n+1).$$

Here and in what follows only the $\chi$'s which cause a $\phi$ to be a vector are printed boldface. Note that

$$N(2n+1) = \frac{(2n+1)!}{2^n \cdot n!}, \quad T_{sp} = \binom{n}{m} 2^{n-m}$$

for either the $\delta p$- or the $\delta n$-coefficient of Eqs. (30), (35). In obvious notation,

$$T^{\delta p} = \frac{(2n+1)!}{[(n-m)!][2m+1]!}, \quad T^{\delta n} = \frac{(2n+1)!}{(n-m)! (n-m+1)! (2m)!}.$$ 

The further argument is identical with the foregoing case.

**Equations (31) and (36).**

$$\phi_1^{(2n+1,1)} = \sigma \left( \begin{array}{c} 1 \\ 2n + 2 \end{array} \right) \cdot (23)(45) \cdots (2n, 2n + 1),$$

where $\sigma$ denotes symmetrization with respect to the first row. The vector character of the configuration stems from the fact that the one polar vector is unpaired. We have

$$N(2n + 1, 1) = \frac{(2n + 1)!}{2^n \cdot n!}.$$ 

Let $Q = 0$. In the SP

$$\left( \begin{array}{c} 1 \\ 2n + 2 \end{array} \right) \cdot \pi_+(1) \pi_-(2n + 2) - \pi_-(1) \pi_+(2n + 2).$$

Distinguish the two types of terms in which the position $2n + 2$ is fixed to be either $\pi_-(2n + 2)$ or $\pi_+(2n + 2)$. Denote by $T_{sp}(\pm)$ the number of distinct ways in which the remaining positions can be occupied by the partition $\{n - m + 1, n - m + 1, 2m\}$:

$$T_{sp}(\pm) = \binom{n}{m} 2^{n-m}.$$ 

In similar notation

$$T(+) = T(-) = \frac{(2n + 1)!}{(n-m)! (n-m+1)! 2m!}.$$
We have likewise two weights, \( W(+) \) and \( W(-) \) which satisfy
\[
NT_{sp}(\pm) = W(\pm)T(\pm)
\]
The correlation coefficient is
\[
\frac{W^2(+)T(+) + W^2(-)T(-)}{\Sigma(\text{same})}
\]
which leads to Eq. (31).

Let next \( Q = -1 \). The position \( 2n + 2 \) can be occupied by \( \pi_- \) or \( \pi_0 \). In a similar notation as used previously
\[
T_{sp}(0) = T_{sp}(-) = \binom{n}{m-1}2^{n-m},
\]
while
\[
T(0) = \frac{(2n+1)!}{(n-m+1)!(n-m+2)!(2m-2)!},
\]
\[
T(-) = \frac{(2n+1)!}{[(n-m+1)!]^2(2m-1)!}.
\]
The weights \( W(0) \), \( W(-) \) follow from
\[
NT_{sp}(0) = W(0)T(0), \quad NT_{sp}(-) = W(-)T(-)
\]
and the \( \bar{p}n \)-coefficient is equal to
\[
\frac{W(0)^2T(0) + W(-)^2T(-)}{\Sigma(\text{same})}
\]
which yields Eq. (36).

We have now exhibited all the degrees of complication. One distinguishes sub-categories, like \( T(0) \), \( T(-) \) in the last instance, which distinguish special occupations of such positions which do not participate in a full symmetrization. Count for each sub-category the \( T_{sp} \) and the \( T \). Within each sub-category there is total symmetry which allows us to define a \( W \) for the sub-category. Then apply \( NT_{sp} = WT \) to each sub-category and compute the correlation coefficient by summing over the sub-categories as in Eq. (97).

Thus it may suffice to write down the “first configuration with symmetry” for a few more instances.
\[
\phi_1^{(2n+2,1)} = \sigma \left[ \binom{1}{2n+3} \times (2) \right] (34)(56) \cdots (2n+1, 2n+2),
\]
\( \sigma \) denotes symmetrization in the first row. The quantity in the square bracket is a vector which is a vector product of a polar and an axial vector.
The quantity in square brackets is the scalar product of two polar vectors.

For N = 4, such states were first obtained by Halpern (14). The present set is equivalent, of course, to Halpern's results.

V. DYNAMICS. CHARGE AND SPATIAL CORRELATIONS

In this section we make some brief comments on the full charge-space-time description of Nπ-systems. The vectors (in I-space) \( \mathbf{x}_i \) may be considered as Fourier amplitudes of the quantized r-meson field \( \Psi(\mu) \). \( \mu \) stands for the space-time coordinates \( X_\mu \). Similarly \( \mathfrak{r}(\mu) \) shall denote a nucleon spinor and \( S(\mu) \), \( V(\mu) \) an isotopic scalar and vector respectively, bilinear in the nucleons:

\[
S(\mu) = \mathfrak{r}(\mu)\mathfrak{s}(\mu); \quad V(\mu) = \mathfrak{r}(\mu)\mathfrak{v}(\mu).
\]

The effective Hamiltonian density operator for 2π-annihilation can symbolically be written as

\[
H_2 = g_1^{(2)} S(0) V(1) \Psi(2) K_1^{(2)} (012) + g_1^{(11)} V(0) \bar{V}(1) \times \Psi(2) K_1^{(11)} (012)
= g_2^{(2)} S(0) Z_1^{(2)} (12) K_1^{(2)} + g_1^{(11)} V(0) Z_1^{(11)} (12) K_1^{(11)}
\]

where the \( g \)'s are effective coupling strengths and the \( K \)'s are Lorentz invariant kernels of the space-time arguments. In this particular instance \( S \) and \( V \) transform like a scalar, vector in the space-time sense. In some of the equations below they may behave like a pseudoscalar, pseudovector. We shall omit this explicit distinction in what follows, these transformation properties will always uniquely be determined by the structure of the effective interactions to be considered.

Equation (98) is, of course, in accordance with the generalized Bose statistics for \( \pi \)-mesons: symmetry (antisymmetry) in I-space goes with symmetry (antisymmetry) in space. Let us in fact consider which 2π-wave functions are gen-

\[\phi^{(2n+3,2)} = \sigma \left[ \begin{pmatrix} 1 \\ 2n+3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2n+4 \end{pmatrix} \right] (34)(56) \cdots (2n+1, 2n+2) ; (2n+3,2) = \sigma \left[ \begin{pmatrix} 1 \\ 2n+4 \end{pmatrix}, \begin{pmatrix} 2 \\ 2n+5 \end{pmatrix} \right] (34)(56) \cdots (2n+1, 2n+2)(2n+3).
\]
erated by $H_2$ acting between the vacuum and a configuration $\psi_1, \psi_2$. Put $\psi(\alpha) = \exp i\hat{p}\cdot x_\alpha$. To the $S$ and $V$ parts of $H_2$ correspond

$$(x_1x_2)\sigma_1(1)\epsilon_2(2) = (x_1x_2)\phi_1^{(2)}(12) = \phi_1^{(2)}(12)\phi_1^{(2)}(12),$$

$$(x_1 \times x_2)\sigma_1(1)\epsilon_2(2) = (x_1 \times x_2)\phi_1^{(11)}(12) = \phi_1^{(11)}(12)\phi_1^{(11)}(12),$$

respectively, where $\sigma(\sigma_-)$ means symmetrization (antisymmetrization) in the configuration labels (the suffixes!) of the space wave functions. The $\phi_1^{(N)}$ are the isotopic wave functions in the notation adopted since Eq. (55), the $\Phi_1^{(N)}$ are the corresponding space wave functions. We observe:

(1) The entire general formalism of Section IV (a) holds equally for $\phi_1^{(N)}$ as for $\Phi_1^{(N)}$. We have a common notion of configuration in both cases, but a distinct notion of content. For the $\Phi_1^{(N)}$ the "variables of content" are the space vectors $x_\alpha$. All formal operations with such quantities as $\Omega_1^{(N)}$, which are "content-independent" hold true for the $\Phi_1^{(N)}$. Example: space wave functions $\Phi_1^{(2N)}$ are given by Eqs. (62), (63) with the translation of $(y_1y_2y_3) = \epsilon_1(1)\epsilon_2(2)\epsilon_3(3)$ and where the permutaions always act on the suffixes.

(2) Because of the content properties of the $\phi_1^{(N)}$ there exist "spurious symmetries" (from the point of view of the group $S_N$) like Eqs. (60), (61) for these functions. These have no counterpart for the $\Phi_1^{(N)}$, the general $f$-function

$$f(y_1, \cdots, y_N) = \epsilon_1(1)\cdots\epsilon_N(N)$$

is totally unsymmetric (see Section IVa).

(3) The $N\pi$-wave functions are generated by a product of $N$ operators $\Psi$ which is uniquely determined and which will be denoted by

$$Z_1^{(N)}(1, \cdots, N), Z_2^{(N)}(1, \cdots, N),$$

respectively, depending on whether the class has $I = 0$ or 1. We recall that the latter distinction is dictated by the class $[N]$ itself. The $Z^{(N)}$ have the same structure as the $\phi_1^{(N)}$, that is, $\phi_1^{(N)}$ generates $Z_1^{(N)}$ if we replace in the former each $x_i$ by $\Psi(i)$. A set of $N\pi$-wave functions is formally obtained as the elements of the $Z$-operators between the vacuum and the configuration $(\psi_1, \cdots, \psi_N)$ and they are automatically symmetric with respect to the simultaneous interchange of space and isotopic spin variables.

(4) We can define dynamical triples, doubles, singles with the help of Eq. (65) by reading $\Psi(j)$ for $(j)$. Note. Dynamical triples, doubles, singles transform space time wise like pseudoscalar, scalar, pseudoscalar operators, respectively.

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For definiteness we consider momentum representations.

For nondegenerate configuration.
(5) Halpern's problem (14) (see Section II (a)) is formally solved by a multiple expansion of the $Z_a^{[N]}$ in spherical harmonics. This generates a complete but nonorthogonal set of partial waves classified in terms of the group $S_N$.

(6) The most general form $^2$ of the effective Hamiltonian density operator $H_N$ for a nucleon-antinucleon-$N\pi$ system is

$$H_N = \sum_{[N]}^{(e)} \sum_a g_a^{[N]} S(0) Z_a^{[N]} K_a^{[N]} + \sum_{[N]}^{(o)} \sum_a g_a^{[N]} V(0) Z_a^{[N]} K_a^{[N]}$$

(100)

where $\sum_{[N]}^{(e)}$ denotes a summation over classes for which $N_1 - N_3, N_2 - N_3$ are even, while $\sum_{[N]}^{(o)}$ is the summation over the remaining classes. The terms for fixed $[N]$, variable $a$ are linearly independent, they generate nonorthogonal $N\pi$-wave functions.

(7) If $N_3$ is even (odd) the $S(0)$-operator multiplied into $Z_a^{[N]}$ transforms like a scalar (pseudoscalar). If $N_3$ is even (odd) the $V(0)$-operator multiplied into $Z_a^{[N]}$ transforms like a pseudovector (vector).

Equation (100) has been arrived at without regard for the connection between the true dynamical interaction and the various effective interactions $H_N$. It is physically evident that without a detailed theoretical apparatus we cannot say a priori which of our classes if any will appear with larger or smaller weight in $H_N$. The different Young tableaux correspond to physically distinct types of effective coupling; the preference (if any) for one or the other effective coupling could well be energy dependent.

What about the $a$-dependence for a given class? We here come to a point that was not necessary to consider earlier, namely the magnitude of the configuration labels $p$. Different $a$ for given $[N]$ means that different groupings of configuration labels participate in a different way in a coupling in triples, pairs and singles. It is again physically evident that there is no a priori reason why a set of momentum values should participate with the same weight in coupling schemes characterized by given $[N]$, varying $a$. The conclusion is, therefore, that there are no general dynamical arguments which entitle one to simplify Eq. (100) in an a priori way. We need both more theory and more experiment to bridge the gap between true and effective interaction.

In this connection we note that the general structure of the $Z_a^{[N]}$ implies that if a charge correlation were particularly enhanced, this would carry with it a corresponding possibility of spatial correlations becoming marked. Unfortunately, the detailed magnitude of a spatial correlation effect depends on the structure of the kernels $K$. All one can say is that the existence of a charge correlation of given symmetry implies the existence of a spatial correlation of given symmetry but of unknown magnitude. For example a statistical distribution of charge correlations does not exclude nonstatistical space correlations and vice versa. This is merely a paraphrasing of the independent nature of the two assumptions underlying the statistical model; see Section III.
Thus, as was stated already in the Introduction, the present analysis cannot in itself provide predictions as to the magnitudes of correlation effects. It is hoped, however, that the considerations given here may have provided some tools which seem appropriate for a further study of such effects. In first instance, experiment may decide the extent to which the concept of class, which is at the root of the present work, is physically useful.

APPENDIX. ORTHOGONAL ISOTOPIC SPIN WAVE FUNCTIONS FOR N = 3, 4

The first wave function of each class is of the type used in the calculation of correlation coefficients. \( \sigma \) denotes symmetrization in the first row.

For \( N = 3 \),

- Class (3) \( \phi^{(3)}_1 = \sigma(12)(3) \),
- Class (21) \( \phi^{(21)}_1 = \sigma \left( \begin{array}{c} 1 \\ 3 \end{array} \right) \times (2) \),
- \( \phi^{(21)}_2 = \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \times (3) \),
- Class (111) \( \phi^{(111)}_1 = \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \).

For \( N = 4 \),

- Class (4) \( \phi^{(4)}_1 = \sigma(12)(34) \),
- Class (22) \( \phi^{(22)}_1 = \sigma \left( \begin{array}{c} 1 \\ 3 \\ 4 \end{array} \right) \),
- \( \phi^{(22)}_2 = \left( \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \right) \),
- Class (31) \( \phi^{(31)}_1 = \sigma \left( \begin{array}{c} 1 \\ 4 \end{array} \right) \),
- \( \phi^{(31)}_2 = \sigma \left( \begin{array}{c} 1 \\ 2 \\ 4 \end{array} \right) - \sigma \left( \begin{array}{c} 2 \\ 1 \\ 4 \end{array} \right) \),
- \( \phi^{(31)}_3 = 3 \cdot \sigma \left( \begin{array}{c} 3 \\ 2 \\ 4 \end{array} \right) + \sigma \left( \begin{array}{c} 3 \\ 4 \\ 12 \end{array} \right) \).
- Class (211) \( \phi^{(211)}_1 = \left( \begin{array}{c} 1 \\ 3 \\ 4 \end{array} \right) \),
- \( \phi^{(211)}_2 = \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \).
Note the useful identity

\[
\binom{1}{2}^{(4)} = \binom{1}{2}^{(34)} + \binom{2}{3}^{(14)} + \binom{3}{1}^{(24)}.
\]

TABLE I

**STATES AND CLASSES OF N_T-CLOUDS**

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<th>N</th>
<th>States: ( \rho(N) )</th>
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TABLE II

**STATES FOR N = 8**

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<td>(31)</td>
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<td>0</td>
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<td>2/3</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>(221)</td>
<td>(1)</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
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<td>4</td>
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<td></td>
<td>15</td>
<td>3/7</td>
<td>36/7</td>
<td>66/7</td>
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599
<table>
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<tr>
<th>C</th>
<th>PC</th>
<th>I</th>
<th>( \rho )</th>
<th>( \tilde{p} \tilde{p} )</th>
<th>( \eta )</th>
<th>( \tilde{p}_n )</th>
<th>( \eta )</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(006)</td>
<td>(114)</td>
<td>(222)</td>
<td>(330)</td>
</tr>
<tr>
<td>(6)</td>
<td>(6)</td>
<td>0</td>
<td>1</td>
<td>1/7</td>
<td>6/35</td>
<td>8/35</td>
<td>16/35</td>
</tr>
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<td>(51)</td>
<td>(51)</td>
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<td>5</td>
<td>0</td>
<td>3/35</td>
<td>8/35</td>
<td>24/35</td>
</tr>
<tr>
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<td>(42)</td>
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<td>9</td>
<td>0</td>
<td>4/15</td>
<td>7/15</td>
<td>4/15</td>
</tr>
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<td>(411)</td>
<td>(3)</td>
<td>1</td>
<td>10</td>
<td>0</td>
<td>3/5</td>
<td>2/5</td>
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<td>(3)</td>
<td>1</td>
<td>5</td>
<td>0</td>
<td>2/5</td>
<td>3/5</td>
<td>6.5</td>
</tr>
<tr>
<td>(321)</td>
<td>(21)</td>
<td>1</td>
<td>16</td>
<td>0</td>
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<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(222)</td>
<td>(0)</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
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</tbody>
</table>

\( \tilde{p}_n \) and \( \tilde{p}_n \) values for different values of \( C \), \( PC \), \( I \), and \( \rho \) are given in the table. The values are computed using the formulae provided in the original text, which are not displayed here. The table continues with values for \( N = 7 \) and \( N = 8 \) in a similar format.
TABLE IV
Two-Prong Star Table

<table>
<thead>
<tr>
<th>Class</th>
<th>$P_z(N)$</th>
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<tr>
<td></td>
<td>$N$ even</td>
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<tr>
<td></td>
<td>$N(N - 1)$</td>
</tr>
<tr>
<td></td>
<td>$3(N - 1)$</td>
</tr>
<tr>
<td>$(N)$</td>
<td>$N(N + 1)$</td>
</tr>
<tr>
<td>$(N - 1, 1)$</td>
<td>$N(N - 2)$</td>
</tr>
<tr>
<td>$(N - 2, 2)^a$</td>
<td>$N(N - 2)$</td>
</tr>
<tr>
<td>$(N - 2, 1, 1)^b$</td>
<td>$N(N - 2)$</td>
</tr>
<tr>
<td></td>
<td>$N - 2$</td>
</tr>
</tbody>
</table>

* For $N \geq 4$ (N even), $N \geq 5$ (N odd).

b For $N \geq 4$ (N even), $N \geq 3$ (N odd).

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References


19. K. M. Watson, Phys. Rev. 85, 852 (1952), especially Section II.