On the Meson Pair Theory of Nuclear Forces, II.*

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(Received May 30, 1947.)

§ 1. Introduction.

Exploring the possibility of explaining the whole phenomena about the nucleus and the cosmic ray on the basis of the meson pair theory, the author introduced, in the first part of this paper(1) the following interaction between the nucleons and the mesons;

\[
\begin{align*}
P & \rightarrow N + Y^+ + Y^0, \\
N & \rightarrow P + Y^- + Y^0,
\end{align*}
\]

(1)

where \(Y^+, Y^-\) and \(Y^0\) denote a positive, negative, and neutral meson respectively. To account for the fact that the nuclear forces are nearly symmetrical with respect to the charge of the nucleon, it will be further necessary to introduce either or both of the following two kinds of interactions; namely

\[
(P \text{ or } N) \rightarrow (P' \text{ or } N') + Y^+ + Y^-,
\]

(2)

\[
(P \text{ or } N) \rightarrow (P' \text{ or } N') + Y^0 + Y^0.
\]

(3)

It was shown, however, that with the scalar meson we can not obtain the interaction between the neutron and the proton which is in accord with the experience. Moreover the mass \(\mu_0\) of the neutral meson must be assumed to be smaller than the mass \(\mu_e\) of the charged meson in order to obtain a finite lifetime of the charged meson.

Therefore we shall investigate in this paper the next simplest case of spin \(\frac{1}{2}\). Assuming the Dirac equation to hold for the mesons, we first consider the most general interaction between the nucleons and the mesons.

* This paper was read on Nov. 21, 1946, at the Symposium on the Theory of Elementary Particles held at Kyoto University.

(1) S. Noma, Prog. Theor. Phys. 1 (1946) 71. This paper will be referred to as I.
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The mesons with opposite charge are treated symmetrically and the neutral meson is described by the Majorana's abbreviated theory. In order to avoid the divergent interaction between nucleons in a relativistically invariant way, an appropriate source function is introduced (§ 3) and the limit to a point source is taken in the final results. From the interaction of nucleons with mesons thus obtained, the general interaction between two nucleons is derived as the second order effect (§ 5). Finally it is shown that simple qualitative requirements restrict the choice of the interaction to an unique possibility viz. the pseudovector interaction and that it is possible with this interaction alone, to explain qualitatively all known facts about nuclear two body systems (§ 6).

§ 2. Interaction of the Mesons with the Nucleons.

As the Hamiltonian $H_t$ which describes the interaction between the nucleons and the mesons, we first consider the most general one. Corresponding to the three kinds of the interaction mentioned in § 1, $H_t$ consists of three parts

$$H_t = H_1 + H_2 + H_3,$$

each of which is a linear combination of the five possible relativistic invariants which can be formed when no derivatives of the wave functions are involved. It was shown in I that the interaction term in the pair theory should not contain the derivatives of the meson wave functions in order to have a finite interaction.

We denote the wave function of the charged meson by $u$, that of the neutral meson by $v$, and their conjugate complexes by $u^*$ and $v^*$ respectively. Describing the state of the nucleons in their configuration space, $H_t$ can be written in the form (2)

$$H_t = \frac{1}{2} \sum \sum g_{12}(t_1^{(0)} + it_2^{(0)}) u^*(X^{(0)}) O^{(0)} v(X^{(0)}) + \text{Conj.},$$

with

(2) We use $\hbar$ and $c$ as units, so that mass and energy are measured in units of reciprocal length.
\[
\begin{align*}
O_{\alpha} &= \beta^{(\alpha)} \beta, \quad O^{(\alpha)}_\perp = I^{(\alpha)} I - a^{(\alpha)} a, \quad O^{(\alpha)}_\perp = \beta^{(\alpha)} \sigma^{(\alpha)} \cdot \beta \sigma - i \beta^{(\alpha)} a^{(\alpha)} \cdot i \beta \sigma, \\
O^{(\alpha)}_\perp &= \sigma^{(\alpha)} \cdot \sigma - I^{(\alpha)} I, \quad O^{(\alpha)}_\perp = i \beta^{(\alpha)} I^{(\alpha)} I \beta^{(\alpha)} 
\end{align*}
\]  
where $\beta, \sigma$ are the Dirac matrices, $I$ the unit matrix, and $a, \gamma_5$ are related to $a_i$ by $a_i = -i a_i a_5 (i, k, l; \text{cyclic})$, $\gamma_5 = -i a_4 a_5 a_6$. The quantities without indices are for the meson and those with indices for the $s$-th nucleon. $\tau^{(\alpha)}, \tau_5^{(\alpha)}$ and $X^{(\alpha)}$ are the isotopic spin matrices and coordinates which refer to the $s$-th nucleon. $g_{14}$ is an arbitrary real constant with the dimension of area.

In working in the non-relativistic approximation for the nucleons, all terms involving $a^{(\alpha)}$ and $\gamma_5^{(\alpha)}$ can be neglected and $\beta^{(\alpha)}$ can be set equal to $I^{(\alpha)}$. Then the matrices $\sigma^{(\alpha)}$ and $I^{(\alpha)}$ reduce to the two component ones, and we get

\[
O^{(\alpha)}_1 = I^{(\alpha)} \beta, \quad O^{(\alpha)}_2 = I^{(\alpha)} I, \quad O^{(\alpha)}_3 = \sigma^{(\alpha)} \cdot \beta \sigma, \quad O^{(\alpha)}_4 = \sigma^{(\alpha)} \cdot \sigma, \quad O^{(\alpha)}_5 = 0.
\]

Thus the pseudoscalar interaction disappears entirely.

Similarly the remaining parts of the interaction $H_2$ and $H_3$ are of the forms
\[
\begin{align*}
H_2 &= \frac{1}{2} \sum_s \sum_{\alpha=1}^5 g_{\alpha} (\tau^{(\alpha)}_3 + C_{2\alpha}) u_\alpha^* (X^{(\alpha)}_s) O^{(\alpha)}_1 u (X^{(\alpha)}_s), \\
H_3 &= \frac{1}{2} \sum_s \sum_{\alpha=1}^5 g_{\alpha} (\tau^{(\alpha)}_3 + C_{2\alpha}) v_\alpha^* (X^{(\alpha)}_s) O^{(\alpha)}_1 v (X^{(\alpha)}_s).
\end{align*}
\]

Where $\tau^{(\alpha)}_3$ is the third component of the isotopic spin referring to the $s$-th nucleon, $g_{2\alpha}$ and $g_{2\beta}$ are similar quantities as $g_{14}$, and $C_{2\alpha}, C_{2\beta}$ dimensionless real constants.

The commutation rules for $u, u^*$, and $v, v^*$ are as usual
\[
\begin{align*}
[u_\rho (x, x), u_{\rho'}^* (x', x)]_+ &= [v_\rho (x, x), v_{\rho'}^* (x', x)]_+ = \delta_{\rho \rho'} \delta (x - x'), \\
[u_\rho (x, x), u_{\rho'} (x', x)]_+ &= [v_\rho (x, x), v_{\rho'} (x', x)]_+ = 0,
\end{align*}
\]
(10)

$\rho$ and $\rho'$ run from 1 to 4.

The interpretation of the presence of both signs of charge by the Dirac's hole theory is not quite symmetrical with respect to both the particles with opposite charge. Here we follow the formalism which has been proposed by Heisenberg. It expresses the same physical contents of the

(3) W. Heisenberg, Zs. f. Phys. 90 (1934), 209 and 92 (1934), 692.
theory in the more symmetrical way and consists in setting up the following rule for the order of the operators. Let \( A \) be an Hermitian operator, then the operator density
\[
\sqrt{\langle \mathbf{p} \rangle} \langle \mathbf{p} \rangle = u^* Au
\]
shall be replaced by
\[
\frac{1}{2}(u^* A u - u^* A u) = \frac{1}{2}(u^* A u - u^* A u),
\]
(11)
Where \( A^* \) is the conjugate complex matrix of \( A \) defined by
\[
(A^*)_{\mathbf{pp}'} = A_{\mathbf{pp}'} = A_{\mathbf{pp}'},
\]
To complete the formalism, we extend this rule for the term such as \( u^* A v \) or \( v^* A u \). It is evident that all these replacements are relativistically invariant. For the same reason we replace the density \( v^* A u \) by \( (v^* A u - u^* v^*)/2 \).

If \( u_\mathbf{p} \) and \( v_{\mathbf{p}'} \) are commute, \( H_1 \) vanishes identically when these replacements are made. Therefore we interpret that the three kinds of the mesons are different states of the same particle and correspondingly add to (10) the following commutation relations
\[
[u_\mathbf{p}(\mathbf{x}, \mathbf{x}_0), v_{\mathbf{p}'}(\mathbf{x}', \mathbf{x}_0)] = [v_\mathbf{p}(\mathbf{x}, \mathbf{x}_0), u_{\mathbf{p}'}(\mathbf{x}', \mathbf{x}_0)] = [u_{\mathbf{p}'}(\mathbf{x}, \mathbf{x}_0), v_\mathbf{p}(\mathbf{x}', \mathbf{x}_0)] = 0.
\]
(12)

Next we decompose the spinor field \( u \) into two parts putting
\[
u = \frac{1}{\sqrt{2}} (u^{(1)} + iu^{(2)}), \quad u^* = \frac{1}{\sqrt{2}} C(u^{(1)} - iu^{(2)}).
\]
(13)
where \( C \) is a matrix satisfying the conditions
\[
\beta^* = -C\beta C^{-1}, \quad \alpha^* = CuC^{-1}.
\]
It follows from these relations that \( C^* C \) must be a constant and therefore can be put equal to a unit matrix

---

$C^* C = 1$  \hspace{1cm} (14)

For Hermitian matrices $\alpha$, $\beta$, the matrix $C$ is symmetrical:

$$C_{\alpha\beta} = C_{\beta\alpha}.$$  

This decomposition is relativistically invariant and corresponds to that of a scalar field into its real and imaginary parts. Its reciprocal formulas are

$$u^{(1)} = \frac{1}{\sqrt{2}} (u + C^* u^*), \quad u^{(2)} = \frac{1}{\sqrt{2}} \frac{i}{2} (u - C^* u^*). \hspace{1cm} (15)$$

Similarly we can divide the field $v$ into $v^{(1)}$ and $v^{(2)}$. Now we assume that the neutral meson can be described by the Majorana's abbreviated theory, which consists in identifying the charge-conjugate states $v$ and $C^* v^*$ and therefore giving up the part $v^{(2)}$. In view of our interpretation that the three kinds of mesons are three different states of the same particle, this assumption will be natural. Because in this way we have on the whole three meson fields

$$u^{(1)}, u^{(2)}, u^{(3)} = v^{(1)},$$

corresponding to the three kinds of mesons, the index 3 referring to the neutral field, while the indices 1 and 2 refer to the fields which together describe the charged mesons.

For the variables $u^{(a)} (a=1, 2, 3)$ we find the commutation rules

$$[u^{(a)}(\alpha, x_0), u^{(b)}(\alpha', x_0)]_+ = \delta_{ab} \delta_{\alpha\alpha'} C_{\alpha\beta} \delta(\alpha - \alpha'). \hspace{1cm} (16)$$

In the special representation in which $u$ and $i \beta$ are real, we have

$$C = 1, \hspace{1cm} (17)$$

and then $u^{(a)}$s are all real. For simplicity we choose this representation in the following.

Using the variables $u^{(a)}$, we obtain

$$\frac{1}{2} (u^* A u - u A^* u^*) = \frac{1}{4} u^{(1)} (A - A^*) u^{(1)} + \frac{1}{4} u^{(2)} (A - A^*) u^{(2)} + \frac{i}{4} u^{(1)} (A + A^*) u^{(2)} - \frac{i}{4} u^{(2)} (A + A^*) u^{(1)},$$

(5) E. Majorana, loc. cit.
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\[
\frac{1}{2} (u^*Av - vA^*u^*) \pm \frac{1}{2} (v^*Au - uA^*v^*) = \frac{1}{4} u^{(3)} (A \mp A^*) u^{(3)} \\
\pm \frac{1}{4} u^{(3)} (A \mp A^*) u^{(3)} - \frac{i}{4} u^{(3)} (A \pm A^*) u^{(3)} \pm \frac{i}{4} u^{(3)} (A \pm A^*) u^{(3)}, \\
\frac{1}{2} (v^*Av - vA^*v^*) = \frac{1}{4} u^{(3)} (A - A^*) u^{(3)}.
\]

Thus for a real matrix \( A \) as in the cases of the vector and tensor interactions, the terms involving \( A - A^* \) vanish. On the contrary for an imaginary \( A \) the vanishing term are those involving \( A + A^* \). The latter occurs for the scalar, pseudovector and pseudoscalar interactions.

Applying the Heisenberg's rule (11) and using the variables \( u^{(a)} \), the expressions (5), (8), and (9) for the Hamiltonians become

\[
H_1 = \frac{1}{4} \sum_S \sum_{a=1,2,3} \{ u^{(a)}(X^{(3)}) P_1^{(a)} u^{(3)}(X^{(3)}) \\
+ u^{(3)}(X^{(3)}) P_2^{(a)} u^{(a)}(X^{(3)}) - i [u(X^{(3)}) \times T_1^{(a)} u(X^{(3)})]_S \}, \quad (18)
\]

\[
H_2 = \frac{1}{4} \sum_S \{ \sum_{a=1,2,3} \{ u^{(a)}(X^{(3)}) P_2^{(a)} u^{(a)}(X^{(3)}) + i [u(X^{(3)}) \times T_2^{(a)} u(X^{(3)})]_S \} \}, \quad (19)
\]

\[
H_3 = \frac{1}{4} \sum_S u^{(3)}(X^{(3)}) P_3^{(3)} u^{(3)}(X^{(3)}), \quad (20)
\]

where

\[
P_1^{(a)} = \xi_{11} O_1^{(a)} + \xi_{14} O_1^{(a)} + \xi_{15} O_1^{(a)}, \quad T_1^{(a)} = \xi_{11} O_1^{(a)} + \xi_{13} O_1^{(a)}, \\
P_2^{(a)} = \xi_{21} (\tau_4^{(a)} + C_{21}) O_1^{(a)} + \xi_{24} (\tau_4^{(a)} + C_{24}) O_1^{(a)} + \xi_{25} (\tau_4^{(a)} + C_{25}) O_1^{(a)}, \\
T_2^{(a)} = \xi_{22} (\tau_4^{(a)} + C_{22}) O_2^{(a)} + \xi_{23} (\tau_4^{(a)} + C_{23}) O_2^{(a)}, \\
P_3^{(a)} = \xi_{31} (\tau_4^{(a)} + C_{31}) O_1^{(a)} + \xi_{34} (\tau_4^{(a)} + C_{34}) O_1^{(a)} + \xi_{35} (\tau_4^{(a)} + C_{35}) O_1^{(a)},
\]

\( u \) denotes a symbolic vector with the components \( u^{(1)}, u^{(2)}, u^{(3)} \), and the symbol \( \times \) indicates a vector product in the symbolic space. We see that the vector and tensor interactions of the kind (3) disappears entirely as a consequence of the application of the abbreviated theory to the neutral meson.

§ 3. Introduction of the Source Function.

It is well known that the interaction as given in the last section, which corresponds to a point source, gives a divergent interaction between nucle-
ons. Following Wigner and others to avoid this difficulty, we introduce a real source function $S(x)$ normalized according to

$$\int S(x)dx = 1,$$  \hspace{1cm} (22)

and replace, in the Hamiltonian, such a term as

$$\nu^{(a)}(X^{(a)}) \propto \mu^{(a)}(X^{(a)}),$$

by its average value

$$\int \nu^{(a)}(x)S(x-X^{(a)})d^3x \int \nu^{(a)}(x')S(x'-X^{(a)})d^3x'.$$  \hspace{1cm} (23)

It will be convenient to pass from the ordinary space to the momentum space with the help of the Fourier transformation

$$\nu^{(a)}(x) = (2\pi)^{-3/2} \int \xi^{(a)}(k)e^{ik \cdot x} d^3k,$$  \hspace{1cm} (24)

$$\xi^{(a)}(k) = (2\pi)^{-3/2} \int \nu^{(a)}(x)e^{-ik \cdot x} d^3x,$$  \hspace{1cm} (25)

$$S(x) = (2\pi)^{-3} \int R(k)e^{-ik \cdot x} d^3k,$$  \hspace{1cm} (26)

The reality of $\nu^{(a)}(x)$ and $S(x)$ demands that the quantity $\xi^{(a)}(k)$ and $R(k)$ have to fulfill the reality conditions

$$\xi^{(a)}(-k) = \xi^{(a)*}(k),$$

$$R(-k) = R^*(k).$$  \hspace{1cm} (27)

The commutation rules (16) with $C=1$ are equivalent to

$$[\xi^{(a)}(k), \xi^{(a)*}(k')] = \delta_{a,a'}\delta_{k,k'}\delta(k-k').$$  \hspace{1cm} (28)

Written in the momentum space, the three parts of the Hamiltonian which corresponds to an extended source (22) are given by

$$H_i = \frac{1}{4} \sum S f(k, k'; X^{(b)}) \sum_{a=1}^{1,2} \sum_{a=1}^{1,2} \{ \xi^{(a)}(k) P^{(b)}(k) \xi^{(a)}(k') \}
+ \xi^{(a)}(k) P^{(b)}(k') - i[\xi^{(a)}(k) \times P^{(b)}(k')]_{a} d^3kd^3k',$$

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\[ H_s = \frac{1}{4} \int \sum \mathbf{S}_s f(\mathbf{k}, \mathbf{k}'; X^{(s)}) \left\{ \sum_{a=1,2} \xi^{(a)}(\mathbf{k}) \xi^{(a)}(\mathbf{k}') \right\} + i [\mathbf{S}(\mathbf{k}) \times T^{(a)}(\mathbf{k}')] d^3k d^3k', \quad (29) \]

\[ H_s = \frac{1}{4} \int \sum \mathbf{S}_s f(\mathbf{k}, \mathbf{k}'; X^{(s)}) \xi^{(a)}(\mathbf{k}) \xi^{(a)}(\mathbf{k}') d^3k d^3k', \quad (30) \]

where

\[ f(\mathbf{k}, \mathbf{k}'; X^{(s)}) = \frac{1}{(2\pi)^4} R(\mathbf{k}) R(\mathbf{k}') e^{i(\mathbf{k} + \mathbf{k}') \cdot X^{(s)}}. \quad (31) \]

To preserve the relativistic invariance of the theory, we adopt the following method similar to the so-called "\( \lambda \)-limiting process" introduced by Wentzel\(^{(7)}\) and Dirac\(^{(8)}\). As is noted by Pauli\(^{(9)}\), models with different source function \( R(\mathbf{k}) \) which belongs to the same function \( G(\mathbf{k}) \) defined by

\[ G(\mathbf{k}) = G^*(\mathbf{k}) = R(\mathbf{k}) R(-\mathbf{k}) \quad (32) \]

are equivalent, and only the latter function has a physical meaning. On account of the special choice of the normalizing factor in (25), we always

\[ G(0) = 1. \quad (33) \]

The particular case \( G(\mathbf{k}) = \text{const.} = 1 \) corresponds to a point source. Another important case is

\[ G(\mathbf{k}) = \cos \left( \lambda_0 \mathbf{k}_0 - \hat{\lambda} \cdot \mathbf{k} \right), \quad (34) \]

where \( \mathbf{k}_0 = \sqrt{\mathbf{p}^2 + \mu^2} \), \( \mu \) denoting the mass of the particle accompanied by the field, and \( \lambda_0, \hat{\lambda} \) form a four vector. The "\( \lambda \)-limiting process", namely \( (\lambda_0, \hat{\lambda}) \to 0 \) eliminates the particular choice of the \( \hat{\lambda} \)-vector from the final results and avoids the classical divergences of the point source model without destroying the relativistic invariance of the theory provided that the four vector \( \lambda_0, \hat{\lambda} \) is always time-like viz.

\[ \lambda_0^2 > \hat{\lambda}^2. \quad (35) \]

\(^{(7)}\) G. Wentzel, Zs. f. Phys. 86 (1933), 479.


\(^{(9)}\) W. Pauli, Phys. Rev. 64 (1943), 332.
In our case, however, the function (34) is of no use. Therefore we use the exponential function instead of it and put

$$G(k) = e^{-(\lambda_0 k_0 - \lambda \cdot k)}$$

where the four vector $(\lambda_0, \lambda)$ is assumed to be time-like in this case too. As long as we have only to consider a single coordinate system, it is permissible to put $\lambda = 0$; hence

$$G(k) = e^{-\lambda_0 k_0}.$$  \hspace{1cm} (37)

As is shown in § 5 the $\lambda$-limiting process with this function avoids the divergent interaction between nucleons.

In contrast to the function (34), it is characteristic of this exponential function that it gives the value zero, in the limit of a point source, for the radius $a$ of the source, which is defined by

$$a^{-1} = \int \frac{S(\mathbf{x}) S(\mathbf{x}') d^3x d^3x'}{|\mathbf{x} - \mathbf{x}|} = \frac{1}{2\pi^3} \int G(k) \frac{d^3k}{k^3}.$$  \hspace{1cm} (38)

and thus brings about results which can be more easily comprehended by intuition.


Let $a^{(\alpha)}(j: k), \delta^{(\alpha)}(j: -k) (j = 1 \text{ or } 2)$, be the four normalized eigenfunctions of the Dirac equation in the momentum space

$$\begin{cases}
(\alpha \cdot k + \mu^{(\alpha)} \beta) a^{(\alpha)}(j: k) = \kappa^{(\alpha)} a^{(\alpha)}(j: k), \\
(\alpha \cdot k + \mu^{(\alpha)} \beta) \delta^{(\alpha)}(j: -k) = -\kappa^{(\alpha)} \delta^{(\alpha)}(j: -k),
\end{cases}$$

where

$$\kappa^{(\alpha)} = \sqrt{k^2 + \mu^{(\alpha)^2}}, \quad \mu^{(\alpha)} = \mu^{(\alpha)} = \mu, \quad \mu^{(\alpha)} = \mu.$$  \hspace{1cm} (40)

It follows then with the help of the equations (39)

$$\sum_{j=1,2} a^{(\alpha)}(j: k) a^{(\alpha)*}(j: k) = \frac{1}{2\kappa^{(\alpha)}} \left( \kappa^{(\alpha)} + \alpha \cdot k + \mu^{(\alpha)} \beta \right)_{pp'}, \quad \sum_{j=1,2} \delta^{(\alpha)}(j: k) \delta^{(\alpha)*}(j: k) = \frac{1}{2\kappa^{(\alpha)}} \left( \kappa^{(\alpha)} + \alpha \cdot k - \mu^{(\alpha)} \beta \right)_{pp'}.$$  \hspace{1cm} (41)
Further they show that in our special representation where $\alpha$ and $\beta$ are real it is permissible to put

$$\delta^{(\alpha)}_{\beta}(j: k) = a^{(\alpha)*}_{\beta}(j: k). \quad (42)$$

Now we expand $\xi^{(\alpha)}(k)$ in these eigenfunctions:

$$\xi^{(\alpha)}(k) = \sum_{j=\pm} \{ u^{(\alpha)}(j: k) a^{(\alpha)}(j: k) + u^{(\alpha)*}(j: -k) b^{(\alpha)}(j: -k) \}. \quad (43)$$

On account of the reality condition given by the first equation in (26), $u^{(\alpha)*}(j: k)$ must be the Hermitian conjugate operator of $u^{(\alpha)}(j: k)$. Considering the orthogonality relation for $a^{(\alpha)}(j: k)$ and $b^{(\alpha)}(j: -k)$, the commutation rules for these variables are easily found to be

$$[u^{(\alpha)}(j: k), u^{(\alpha)*}(j': k')]_+ = \delta_{\alpha\alpha'} \delta_{jj'} \delta(k-k'),$$
$$[u^{(\alpha)}(j: k), u^{(\alpha)}(j': k')]_+ = \{u^{(\alpha)*}(j: k), u^{(\alpha)*}(j': k')\}_+ = 0.$$

Further we put

$$u^{(1)}(j: k) = \frac{1}{\sqrt{2}} \{ A[(+1): j: k] + A[(-1): j: k] \},$$
$$u^{(1)*}(j: k) = \frac{1}{\sqrt{2}} \{ A^*[(+1): j: k] + A^*[(-1): j: k] \},$$
$$u^{(2)}(j: k) = \frac{1}{\sqrt{2}} \frac{1}{i} \{ A[(+1): j: k] - A[(-1): j: k] \},$$
$$u^{(2)*}(j: k) = -\frac{1}{\sqrt{2}} \frac{1}{i} \{ A^*[(+1): j: k] - A^*[(-1): j: k] \},$$
$$u^{(3)}(j: k) = A(0: j: k), \quad u^{(3)*}(j: k) = A^*(0: j: k). \quad (45)$$

The $A$'s satisfy the same commutation relations as (44);

$$[A(e: j: k), A^*(e': j': k')]_+ = \delta_{\alpha\alpha'} \delta_{jj'} \delta(k-k'),$$
$$[A(e: j: k), A(e': j: k')]_+ = [A^*(e: j: k), A^*(e': j: k')]_+ = 0, \quad e, e' = \pm 1 \text{ or } 0.$$

The operator $N(e: j: k) = A^*(e: j: k) A(e: j: k)$ represents the number of the mesons per unit volume in the momentum space, which are in the state of momentum $k$, charge $ee$, and definite spin characterized by $j$. This can be seen by expressing the total energy, momentum and charge for the case of free mesons in terms of these variables.
Expressed by these normal coordinates, the interactions (28)—(30) become respectively

\[
H_1 = \frac{1}{\sqrt{2}} \sum \sum f(k, k': X(\theta)) \left[ \{ \tau_{\theta}^{(0)} A(0, j; k) A(0, j': k') \right.
+ \left. \tau_{\theta}^{(0)} A(-1, j; k) A(0, j'; k') \right] a^{(0)}(j; k) P_{\theta}^{(0)} a^{(0)}(j'; k')
+ \{ \tau_{\theta}^{(0)} A(+1, j; k) A^*(0, j'; -k') + \tau_{\theta}^{(0)} A(-1, j; k) A^*(0, j'; -k') \} \\
\cdot a^{(0)}(j; k) P_{\theta}^{(0)} a^{(0)}(j'; k') - \{ \tau_{\theta}^{(0)} A(-1, j; k) A(0, j'; k') \right] a^{(0)}(j; k) T_{\theta}^{(0)} a^{(0)}(j'; k')
- \{ \tau_{\theta}^{(0)} A(+1, j; k) A^*(0, j'; -k') - \tau_{\theta}^{(0)} A(-1, j; k) A^*(0, j'; -k') \} \\
\cdot a^{(0)}(j; k) T_{\theta}^{(0)} a^{(0)}(j'; k') a^{(0)}(j; k) + \text{conj.},
\] (47)

\[
H_2 = \frac{1}{\sqrt{2}} \sum \sum f(k, k': X(\theta)) \left[ \{ A(0, j; k) A(0, j'; k') a^{(0)}(j; k) \right.
+ \left. A^*(-1, j; k) A^*(0, j'; -k') a^{(0)}(j'; k') \right] P_{\theta}^{(0)} a^{(0)}(j; k) + A^*(0, j'; -k') a^{(0)}(j; k)
+ \{ A(0, j; k) A^*(0, j'; -k') - A^*(-1, j; k) \} A^*(0, j'; -k')
+ \{ A(-1, j; k) A^*(0, j'; -k') \} a^{(0)}(j; k) T_{\theta}^{(0)} a^{(0)}(j'; k') - \{ A(0, j; k) A^*(0, j'; -k') \} a^{(0)}(j; k) T_{\theta}^{(0)} a^{(0)}(j'; k')
\cdot a^{(0)}(j; k) + \text{conj.},
\] (48)

\[
H_3 = \frac{1}{4} \sum \sum f(k, k': X(\theta)) \left[ A(0, j; k) A(0, j'; k') a^{(0)}(j; k) \right.
+ \left. A^*(0, j; k) A^*(0, j'; k') a^{(0)}(j'; k') \right] P_{\theta}^{(0)} a^{(0)}(j; k) + A^*(0, j; k) A^*(0, j'; k') - A^*(0, j; k) A^*(0, j'; k')
+ \{ A(0, j; k) A^*(0, j'; k') \} a^{(0)}(j; k) P_{\theta}^{(0)} a^{(0)}(j'; k') a^{(0)}(j; k) P_{\theta}^{(0)} a^{(0)}(j'; k')
\cdot a^{(0)}(j; k) + \text{conj.},
\] (49)

with

\[
\begin{align*}
\tau_{\theta}^{(0)} &= \frac{\tau_{\theta}^{(0)} + i \tau_{\theta}^{(0)}}{2}, \\
\tau_{\theta}^{(0)} &= \frac{\tau_{\theta}^{(0)} - i \tau_{\theta}^{(0)}}{2}.
\end{align*}
\] (50)

and

\[
a^{(0)}(j; k) = a^{(0)}(j; k), \quad a^{(0)}(j; k) = a^{(0)}(j; k).
\]

\[
P_{\theta}^{(0)}(i = 1, 2, 3), \quad T_{\theta}^{(0)}(i = 1, 2) \quad \text{and} \quad f(k, k'; X(\theta)) \quad \text{are given in (21) and (31)}
\]

respectively.
§ 5. Derivation of the Nuclear Forces.

In this section we deal with the most important application of our theory viz. the derivation of the forces between two nucleons. We work in the non-relativistic approximation for the nucleons. In this approximation the pseudoscalar interaction disappears entirely and \( F_1^{(5)} \), \( T_1^{(5)} \) reduces to

\[
\begin{align*}
F_1^{(5)} &= g_{11} I^{(5)} \beta + g_{16} \sigma^{(5)} \cdot \sigma, \\
T_1^{(5)} &= g_{11} I^{(5)} I + g_{16} \sigma^{(5)} \cdot \beta \sigma,
\end{align*}
\]

\[
\begin{align*}
F_2^{(5)} &= g_{21} \left( \tau_1^{(5)} + C_\nu \right) I^{(5)} \beta + g_{26} \left( \tau_1^{(5)} + C_\mu \right) \sigma^{(5)} \cdot \sigma, \\
T_2^{(5)} &= g_{21} \left( \tau_1^{(5)} + C_\nu \right) I^{(5)} I + g_{26} \left( \tau_1^{(5)} + C_\mu \right) \sigma^{(5)} \cdot \beta \sigma, \\
F_3^{(5)} &= g_{31} \left( \tau_2^{(5)} + C_\mu \right) I^{(5)} \beta + g_{36} \left( \tau_2^{(5)} + C_\nu \right) \sigma^{(5)} \cdot \sigma.
\end{align*}
\]

The method here employed is the usual method of perturbation theory.

A) Interaction due to pairs \((Y^+, Y^0)\) and \((Y^-, Y^0)\).

The scheme of the interaction between a neutron and a proton in this case is given in (1). It is obvious that in this way no force will be obtained, in the second order approximation, between two nucleons of the same kind. The interaction is given by the perturbation formula

\[
V_i = \sum_n \frac{(f | H | n)(n | H | i)}{E_i - E_n}
\]

Here \( i \) and \( f \) are states where two nucleons are found at the positions \( X^{(n)} \) and \( X^{(i)} \) one of them being a neutron and the other a proton, and no mesons are present. In the states \( i \) and \( f \) the positions of the neutron and the proton are just exchanged. \( n \) are all intermediate states in which a pair of mesons \((Y^+, Y^0)\) or \((Y^-, Y^0)\) is emitted by one of the nucleons.

Denoting the momenta and energies of the charged and neutral mesons by \( k, k^\prime \), and \( k, k^\prime \), respectively, we obtain

\[
V_i = \left( \tau_1^{(i)} + \tau_2^{(f)} + \tau_3^{(i)} \right) U,
\]

with

\[
U = -\frac{1}{2} \left\{ \int \frac{f(-k, -k' : X^{(i)}) f(k, k' : X^{(f)})}{k_0 + \omega} \sum_{jj'} |a^{(i)}(j : k)(P_{1}^{(i)} + T_{1}^{(f)})| \times a^{(f)*}(j' : k') (P_{1}^{(i)} + T_{1}^{(f)}) a^{(i)*}(j : k) d^3k d^3k'
\]

\[
-\frac{1}{2} \left\{ \int \frac{f(k, k' : X^{(i)}) f(-k, -k' : X^{(f)})}{k_0 + \omega} \sum_{jj'} |a^{(i)}(j : k)(P_{1}^{(i)} - T_{1}^{(f)})| \times a^{(f)*}(j' : k') (P_{1}^{(i)} - T_{1}^{(f)}) a^{(i)*}(j : k) d^3k d^3k'
\]
The spin summation can be evaluated by using the formulas given in (41). Let $A^{(o)}$ be a linear combination of $F^{(o)}$ and $T^{(o)}$ as given in (51), then

$$\sum_{i\neq j} \sum_{\alpha \beta} a^{(o)}_{\alpha\beta}(j: \mathbf{k}) A^{(o)}_{\alpha\beta} a^{(o)*}_{\alpha\beta}(j: \mathbf{k}') = \sum_{i\neq j} \sum_{\alpha} a^{(o)}_{\alpha}(j: \mathbf{k}') A^{(o)}_{\alpha\alpha} a^{(o)*}_{\alpha\alpha}(j: \mathbf{k})$$

$$= \frac{1}{\frac{1}{2}k_0^{(o)} h_0^{(o)}} \text{Spur} \{ A^{(o)}(k_0^{(o)} + a \cdot \mathbf{k}' + \mu^{(o)} \beta) A^{(o)}(k_0^{(o)} + a \cdot \mathbf{k} - \mu^{(o)} \beta) \},$$

where the spur is to be taken for the meson operators. The last transformation is possible because of the relations $\sigma_{p'p} = \sigma_{p'p}$ and $\beta_{p'p} = -\beta_{p'p}$ in the special representation we have chosen. Considering further that $A^{(o)}$ commute with $\beta$ and that the spur of the terms which contain an odd number of factors $a_4$ or $\beta$ vanishes, we get

$$\sum_{i\neq j} \sum_{\alpha \beta} a^{(o)}_{\alpha\beta}(j: \mathbf{k}) A^{(o)}_{\alpha\beta} a^{(o)*}_{\alpha\beta}(j: \mathbf{k}') = \frac{1}{\frac{1}{2}k_0^{(o)} h_0^{(o)}} \text{Spur} \{ A^{(o)} A^{(o)}(k_0^{(o)} + \mu^{(o)} \beta) A^{(o)} A^{(o)}(k_0^{(o)} + a \cdot \mathbf{k} - \mu^{(o)} \beta) \} + (a \cdot \mathbf{k}) A^{(o)}(a \cdot \mathbf{k}') A^{(o)} \}$$

inserting (54) in (53) and integrating with respect to directions of $\mathbf{k}$, $\mathbf{k}'$, we find

$$U = \frac{-1}{4\pi} \int d^3k d^3k' \frac{\tilde{k}' \cdot \tilde{k}}{\tilde{k}' \cdot \tilde{k}} G(\mathbf{k}) G(\mathbf{k}') \left\{ \left( g_1^{(o)} + g_2^{(o)} \right) + \left( g_3^{(o)} + g_4^{(o)} \right) \right\} \times$$

$$\times \left[ 1 - \frac{\mu \mu}{k_0^{(o)} h_0^{(o)}} \right] \sin \frac{k \cdot \mathbf{r}}{k} - \sin \frac{k' \cdot \mathbf{r}}{k'} + \left( g_1^{(o)} - g_2^{(o)} \right) + \left( g_3^{(o)} - g_4^{(o)} \right) \frac{\left( 2(\sigma^{(o)} \cdot \mathbf{r})(\sigma^{(o)} \cdot \mathbf{r}) \right)}{\mathbf{r}^2}$$

$$- (\sigma^{(o)}, \sigma^{(o)}) \right\} \frac{1}{\mathbf{k}' \cdot \mathbf{k}} \left( \cos \frac{\mathbf{k} \cdot \mathbf{r}}{k} - \sin \frac{\mathbf{r} \cdot \mathbf{k'}}{k'} \right) \left( \cos \frac{\mathbf{k}' \cdot \mathbf{r}}{k'} - \sin \frac{\mathbf{r} \cdot \mathbf{k}}{k} \right)$$

where $\mathbf{r} = X^{(o)} - X^{(o)}$. The terms involving cross products such as $F_i^{(o)} T_j^{(o)}$ do not give any contribution when integrated over momentum space. Putting now $G(\mathbf{k}) = e^{i k \cdot \mathbf{x}}$, $G(\mathbf{k}') = e^{-i k' \cdot \mathbf{x}}$ and taking the limit $k \to 0$, we obtain finally

$$U = -\mu [ (g_1^{(o)} + g_2^{(o)}) + (g_3^{(o)} + g_4^{(o)}) \sigma^{(o)} \cdot \sigma^{(o)} ] F_1(\delta : x)$$

$$- \mu \left[ (g_1^{(o)} - g_2^{(o)}) + (g_3^{(o)} - g_4^{(o)}) \right] \frac{2(\sigma^{(o)} \cdot \mathbf{x})(\sigma^{(o)} \cdot \mathbf{x})}{\mathbf{x}^2} - (\sigma^{(o)} \cdot \sigma^{(o)}) \right\} F_2(\delta : x),$$

(56)
where \( x = \mu x, \delta = \mu_0 / \mu < 1 \) and

\[
F_1(\delta : x) = \frac{1}{8\pi^3 x^2} \int_0^\infty \left( \frac{p}{\sqrt{p^2 - 1}} \left( 1 - \frac{\delta}{p^2 - 1} \right) e^{-(p+\sqrt{p^2 - 1 + \delta})} dp \right) \]

\[
F_2(\delta : x) = \frac{1}{8\pi^3 x^2} \int_0^\infty \left( \frac{1}{x^2} + \frac{1}{x} \left( \frac{p}{\sqrt{p^2 - 1 + \delta}} \right) e^{-(p+\sqrt{p^2 - 1 + \delta})} dp \right),
\]

As can be seen directly \( F_2(\delta : x) \) is always positive.

If we adopt Bethe's "single force hypothesis", the spin dependence of nuclear forces excludes at once the \( g_1^s \) (scalar) and \( g_2^s \) (vector) interactions. Of the remaining two, only the \( g_1^p \) (pseudovector) interaction will give the right sign for the electric quadrupole moment of the deuteron. This reversed relation to that given in Marshak (10) is due to the factor

\[
\tau_{1p+1p}^{(n)} + \tau_{1p+1p}^{(m)} = \frac{\tau_{1p+1p}^{(n)} + \tau_{1p+1p}^{(m)}}{2},
\]

which has the value \(-3/2\) for states symmetrical in space coordinates and spin, as the ground state of the deuteron. The pseudovector interaction between a neutron and a proton can be written conveniently in the form

\[
V_{1p}^{n} = -(\tau_{1p+1p}^{(n)} + \tau_{1p+1p}^{(m)}) \frac{G_s^2 M c^2}{6} [(\sigma^{(n)} \cdot \sigma^{(m)}) \{ 3F_1(\delta : x) \\
+ F_2(\delta : x) \} + 2\theta(\sigma^{(n)} , \sigma^{(m)} : \alpha) F_2(\delta : x) ],
\]

with

\[
\theta(\sigma^{(n)} , \sigma^{(m)} : \alpha) = (\sigma_1, \sigma_2) - \frac{3(\sigma^{(n)} \cdot \alpha) (\sigma^{(m)} \cdot \alpha)}{x^2},
\]

which vanishes for any single state.

**B) Interaction due to pairs \((Y^+, Y^-)\) and \((Y^o, Y^o)\).**

In contrast to A these types of interactions lead to forces, already in the second order of approximation, between two nucleons of any kind. The scheme of the interaction are given in (2) and (3). The interactions thus obtained will be denoted by \( V_5 \) and \( V_3 \) correspondingly. They are calculated

(10) R. E. Marshak, Phys. Rev. 57 (1940), 1101.
in the same way as the case A. The results are

\[ V_s = -\frac{\mu^2}{2} \left( (g_{\text{np}} \bar{F}_1 + C_n) (g_{\text{np}} \bar{F}_2 + C_n) \right) \{ F_1(1 : x) + F_2(1 : x) \} 
+ (g_{\text{np}} \bar{F}_1 + C_n) (g_{\text{np}} \bar{F}_2 + C_n) \{ F_1(1 : x) - F_2(1 : x) \} 
+ (g_{\text{np}} \bar{F}_1 + C_n) (g_{\text{np}} \bar{F}_2 + C_n) \{ \frac{1}{2} (\sigma^{(1)} \cdot \sigma^{(2)}) (3F_1(1 : x) - F_2(1 : x)) 
- \frac{1}{2} \theta(\sigma^{(1)}, \sigma^{(2)} ; \alpha) F_2(1 : x) \} 
+ (g_{\text{np}} \bar{F}_1 + C_n) (g_{\text{np}} \bar{F}_2 + C_n) \times \}
\times \{ \frac{1}{2} (\sigma^{(1)} \cdot \sigma^{(2)}) (3F_1(1 : x) + F_2(1 : x) + \frac{1}{2} \theta(\sigma^{(1)}, \sigma^{(2)} ; \alpha) F_2(1 : x) \} \}
\text{vs} = \{-\frac{\mu^2}{2} \left( (g_{\text{np}} \bar{F}_1 + C_n) (g_{\text{np}} \bar{F}_2 + C_n) \right) \{ F_1(1 : x) + F_2(1 : x) \} 
+ (g_{\text{np}} \bar{F}_1 + C_n) (g_{\text{np}} \bar{F}_2 + C_n) \{ \frac{1}{2} (\sigma^{(1)} \cdot \sigma^{(2)}) (3F_1(1 : x) + F_2(1 : x) + \frac{1}{2} \theta(\sigma^{(1)}, \sigma^{(2)} ; \alpha) F_2(1 : x) \} \}
\text{total interaction.}

For \( \delta = 1 \) the functions \( F_1 \) and \( F_2 \) can be expressed in terms of cylindrical functions, viz.

\[ F_1(1 : x) = \frac{1}{16\pi^2} \left\{ \frac{K_0(2x)}{x} - \frac{2K_1(2x)}{x^2} + \frac{K_2(2x)}{x^4} \right\} , \]
\[ F_2(1 : x) = \frac{1}{16\pi^2} \left\{ \frac{5K_0(2x)}{x} + \frac{2K_1(2x)}{x^2} + \frac{5K_2(2x)}{x^4} \right\} , \]

with \( K_n(2x) = \frac{\pi i}{2} e^{\pm i\pi} H_n^{(1)}(2ix) \),

where \( H_n^{(1)}(2ix) \) is the Hankel function of the first kind. If we put \( g_{\text{np}} = 0 \) in (60) we obtain just the same interaction as given by Marshak (11).

C) Total interaction.

Thus we find that the total interaction energy between two nucleons is in general given by

\[ V = V_1 + V_2 + V_3, \]

where \( V_1, V_2 \) and \( V_3 \) are given in A and B. The dependence of the inter-

(11) R. E. Marshak, loc. cit.
teraction on the distance $x$ is different according as the value of $\delta=\mu_0/\mu_r$.

If $\mu_0$ is nearly equal to $\mu_r$ and $\delta$ can be put equal to 1 approximately, we may use the formulas (62) and express $V$ in terms of Hankel functions. If $\mu_0$ is vanishingly small on the contrary and $\delta$ can be put equal to 0 in approximation, we may use the following formulas:

$$F_1(0 : x) = \frac{1}{384\pi^2}\left\{ \left(\frac{12}{x^5} + \frac{12}{x^3} - \frac{4}{x^2} - \frac{1}{x} + 1\right)e^{-x} - \left(\frac{6}{x^4} - x\right)E_i(-x) \right\},$$

$$F_2(0 : x) = \frac{1}{384\pi^2}\left\{ \left(\frac{60}{x^5} + \frac{60}{x^3} + \frac{12}{x^2} - \frac{8}{x} + 1\right)e^{-x} - \left(\frac{6}{x^4} + x\right)E_i(-x) \right\},$$

(64)

$$\lim_{\delta \to 0} \delta F_1(0 : dx) = \frac{1}{32\pi^2 x}, \quad \lim_{\delta \to 0} \delta F_2(0 : dx) = \frac{5}{32\pi^2 x^3},$$

(65)

where $E_i(-x)$ is the exponential integral. Whatever the value of $\delta$ may be, we can use, in a very rough approximation, the simple expressions for $F_1$ and $F_2$ at small distances:

$$F_1(\delta : x) \simeq \frac{1}{32\pi^2 x^3}, \quad F_2(\delta : x) \simeq \frac{5}{32\pi^2 x^5} \quad (x < 1)$$

(66)

At the moment we do not know nothing about the value of $\delta$ except that it must be taken to be smaller than 1. It will be determined from the lifetime of the charged meson provided that the interaction of mesons with light particles is known.

§ 6. Choice of a Special Form of Interaction.

If we demand of our theory that one type of coupling alone should be capable of explaining all known facts about nuclear two-body systems and that it should be symmetrical in all three kinds of mesons, the only possible case may be the pseudovector interaction with the interaction constants $g_{14}=g_{24}=g$, and $C_{24}=C_{34}=g_{34}=0$ (cf., § 5. A). With these assumptions, the interaction between two nucleons becomes

$$V = V' + V''$$

(67)

---

(12) Cf., e.g., Jahnke-Emde, Tables of Functions, 2nd Ed. p. 78.
where
\[ \nu' = -\frac{1}{6} \delta \mu^2 \left( \sigma^{(1)} \cdot \sigma^{(2)} \right) \left[ \left( \tau^{(1)}_1 \tau^{(2)}_1 + \tau^{(1)}_2 \tau^{(2)}_2 \right) \left\{ 3 F_1(\delta : x) + F_2(\delta : x) \right\} \right] + \left( \tau^{(1)}_3 \tau^{(2)}_3 \right) \left\{ 3 F_1(1 : x) + F_2(1 : x) \right\}, \]
\[ \nu'' = -\frac{1}{3} \delta \mu^2 \left( \sigma^{(1)} \cdot \sigma^{(2)} \right) \left[ \left( \tau^{(1)}_1 \tau^{(2)}_1 \right) \left( \tau^{(1)}_2 \tau^{(2)}_2 \right) F_2(\delta : x) \right] + \left( \tau^{(1)}_3 \tau^{(2)}_3 \right) F_2(1 : x). \]

For \( \delta \equiv 1 \) we get the approximate expressions (cf. (62))
\[ \nu' = -\left( \tau^{(1)} \cdot \tau^{(2)} \right) \left( \sigma^{(1)} \cdot \sigma^{(2)} \right) \frac{\delta \mu^2}{24\pi^2} \left\{ \frac{2K_0(2x)}{x^3} - \frac{K_1(2x)}{x^2} + \frac{2K_1(2x)}{x^4} \right\}, \]
\[ \nu'' = -\left( \tau^{(1)} \cdot \tau^{(2)} \right) \left( \sigma^{(1)} \cdot \sigma^{(2)} \right) \frac{\delta \mu^2}{48\pi^2} \left\{ \frac{5K_0(2x)}{x^3} + \frac{2K_1(2x)}{x^2} + \frac{5K_1(2x)}{x^4} \right\}. \]

We shall now show that it is possible, with this interaction alone, to explain qualitatively all the empirical facts about two-body systems.

In the following we assume \( \delta = 1 \) for simplicity. The same argument will be hold for any value of \( \delta \).

The quantum states of the two-body system with such an interaction as is given by (67) and (68) or (69) are fully discussed by Bethe(13). The quantum numbers of the system are; the total angular momentum \( J \), its \( z \) component \( M \), the total spin \( S \), and the parity which is characterized by the orbital angular momentum \( L \).

For any singlet states \( \nu'' \) is identically zero. For even \( L \) the remaining term \( \nu'' \) becomes
\[ \nu' = \frac{\delta \mu^2}{24\pi^2} \left\{ \frac{2K_0(2x)}{x^3} - \frac{K_1(2x)}{x^2} + \frac{2K_1(2x)}{x^4} \right\}, \]
which is attractive for large \( x \), reaches the minimum at \( x = 2.9 \), its absolute value remaining always very small, and then becomes repulsive for smaller \( x \). Thus it is essentially repulsive. For odd \( L \), the potential changes its sign and three times as large.

For the triplet states with \( L = J \), \( \theta(\sigma_1, \sigma_2 : \pi) \) has the value \(-2 \). Thus for even \( L \), the central interaction \( \nu' \) is repulsive, and the directional interaction \( \nu'' \) is attractive. The total interaction is

(13) H. A. Bethe, Phys. Rev. 57 (1940), 390.
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\[ V = -\frac{3g^2 \mu^2}{8\pi^3} \left\{ \frac{K_0(2x)}{x^3} + \frac{K_1(2x)}{x^2} + \frac{K_1(2x)}{x^4} \right\} \]  

(71)
i.e., attractive at all distances. For odd \( L \), \( V \) is repulsive and three times smaller.

The other two triplet states with \( L=J\pm 1 \) are mixed together. The exceptional case is \( J=0 \), for which there is a pure state \(^3\!P_0\). It is found that the diagonal part of the interaction is repulsive for all triplet states of odd \( J \), attractive for all triplet states of even \( J \). The coupling between the two states \( L=J+1 \) and \( L=J-1 \) tend to lower the lower of the two.

If the interaction constant is small, the lowest state will be a mixture of \(^3\!S \) and \(^3\!D_1 \) with the former predominating. We can make it plausible by the semiquantitative argument as follows. Since the centrifugal potential increases with \( J \). The competition will be among the \(^1\!P \), \(^3\!P_0 \) and the mixed state \(^3\!S+^3\!D_1 \). The attractive potential for \(^1\!P \) and \(^3\!P_0 \) are respectively

\[ V = -\frac{g^2 \mu^2}{4\pi^3} \left\{ \frac{2K_0(2x)}{x^3} + \frac{K_1(2x)}{x^2} + \frac{2K_1(2x)}{x^4} \right\} \]  

(72)
\[ V = -\frac{g^2 \mu^2}{4\pi^3} \left\{ \frac{2K_0(2x)}{x^3} + \frac{K_1(2x)}{x^2} + \frac{2K_1(2x)}{x^4} \right\} \]  

(73)

Since the potential for \(^1\!P \) is lower at small distances, it will be sufficient to consider only the two states \(^1\!P \) and \(^3\!S+^3\!D_1 \). The simultaneous differential equations for the radial wave functions \( u_0 \) and \( u_2 \) of the states \(^3\!S \) and \(^3\!D_1 \) are

\[ \begin{align*} 
\frac{\mu^2}{M} \frac{d^2 u_0}{dx^2} + E u_0 &= D_0 u_0 - N u_2, \\
\frac{\mu^2}{M} \frac{d^2 u_2}{dx^2} + E u_2 &= D_2 u_2 - N u_0, 
\end{align*} \]  

(74)

where \( M \) is the mass of the nucleon, \( E \) the energy of the system and

\[ \begin{align*} 
D_0 &= \frac{g^2 \mu^2}{8\pi^3} \left\{ \frac{2K_0(2x)}{x^3} - \frac{K_1(2x)}{x^2} + \frac{2K_1(2x)}{x^4} \right\}, \\
D_2 &= \frac{6\mu^2}{Mx^2} + D_0 + 2D_1, \\
D_1 &= \frac{g^2 \mu^2}{16\pi^3} \left\{ \frac{5K_0(2x)}{x^3} + \frac{2K_1(2x)}{x^2} + \frac{5K_1(2x)}{x^4} \right\}, \\
N &= 2\sqrt{2} D_1.
\]  

(75)
The two simultaneous equations are almost equivalent to two separate Schrödinger equations with the potentials

\[ W = \frac{1}{2} (D_0 + D_2) \pm \left( \frac{1}{4} (D_0 - D_2)^2 + N^2 \right)^{\frac{1}{2}} \]  

(76)

The lower potential \( W_\ell \) of these two is to be compared with the potential \( W_{1p} \) of the \(^1P\) state. For the latter we must take the potential (72) added to the centrifugal potential \( 2\mu_e^2/Mx^2 \). In order that the difference

\[ W_\ell - W_\ell = -\frac{\mu_e^2}{Mx^2} - 4D_0D_1 + \left( \frac{3\mu_e^2}{M^2} + D_1 \right)^{\frac{3}{2}} + 8D_1 \]  

may be positive, it will be necessary that

\[ \frac{25}{24} \geq \left( \frac{M^2}{\mu_e^2} D_1 \right)^2 + \frac{5}{36} \frac{M^2}{\mu_e^2} D_1 \]  

(78)

This follows at once from (77) using the relation \( 4D_1 > 5D_0 \). As our potential behaves as \( 1/x^2 \) for small \( x \), it is necessary to cut it off at a small distance \( x_0 \) which should be taken to be greater than \( \mu_e/M \). If the above inequality is fulfilled for the cut-off distance \( x_0 \), it is evidently true for greater distances.

Using \( M \approx 10\mu_e \), the condition (78) thus reduces approximately to

\[ 16x_0^2 \geq 5\mu_e^2 \]  

(79)

Though the value of \( x_0 \) is not known at the moment, this condition seems to be very plausible. If it is satisfied, the difference \( W_{1p} - W_\ell \) becomes positive and the mixed state will lie lower than the \(^1P\) state.

Since the repulsion is larger for \(^3D_1\) than for \(^3S\) (cf (75)); the lower of the two \(^3S + ^3D_1\) states will be mainly \(^3S\) with some \(^3D_1\) mixed in. Assuming the validity of the condition (78) or (79), therefore, the ground state of the deuteron will be the one which is composed of \(^3S\) and \(^3D_1\) with the former predominating, and the sign of the directional interaction \( V'' \) will be in favour of the positive quadrupole moment of the deuteron in agreement with the experimental result.

Investigations by Breit and his collaborators\(^{(15)}\), on the Bethe's neutral...
meson theory, of the experimental data on proton-proton scattering and neutron-proton scattering, point to a range of nuclear forces of about $(2\mu_e)^{-1}$. It would seem that this is a definite advantage for the meson pair theory, through the interpretation may be altered with our potential, it being repulsive in the $^1S$ state and attractive in the $^3P$ state.

§ 7. Conclusion.

We have seen that the pseudvector interaction between the nucleons and the mesons leads to a theory of nuclear forces which will be capable of explaining the experimental facts about nuclear two-body systems. It contains three unknowns viz. the ratio $\delta$ of the mass of the neutral meson to that of the charged one, strength of the interaction $g$ and the cut-off distance $x_0$. The interaction constant $g$ will be conveniently determined from the cross section for the proton-proton scattering or the scattering of charged mesons by nucleons without changing their sign and the mass ratio $\delta$ from the lifetime of the charged meson with the known interaction between the mesons and the light particles. When these two constants are known, the remaining constant $x_0$ will be determined from the binding energy the deuteron (triplet state). The two constants $g$ and $x_0$ thus determined must fulfill the inequality (79), which will provide a test of the theory.

Of course there are many possibilities to be considered, if we give up the requirements given in the last section. One can for instance develop a theory which will lead to an attraction in the $^1S$ and still give the right sign of the quadrupole moment (by a linear combination of the scalar and pseudvector interactions). This procedure is not also excluded.

In conclusion, I should like to express my deep gratitude to Professor H. Yukawa and to Professor M. Kobayasi not only for their kind interest and encouragement in this work but also for their valuable discussions.

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**Note:** Modified Meson Pair Theory. By S. Noma [Progr. Theor. Phys. 2 (1947), 159].

The same interpretation of the experiment of the Roman group as given in this paper has been given by V. F. Weiskopf [Phys. Rev. 72 (1947), 155].