Table 2. Here, the errors are so large (~128%) that BCS is not even a good first approximation. However, it is again evident that the BCS calculation becomes less accurate as N increases from 4 to 16. It is important to note that this state is almost degenerate with the lowest seniority-two state in these ranges of g and that the states cross at some intermediate value of g. This is a marked contrast between the BCS and exact excitation energies which remains true in the strong coupling limit.

The details of the calculation of the energies and occupation probabilities [7] of these and other states of these systems will be presented in a subsequent article.

References
4. R. W. Richardson and N. Sherman, Nuclear Phys. 52 (1964) 221.

AN INVARIANT DERIVATION OF SU(6) SYMMETRY *

Y. NE'EMAN **
California Institute of Technology, Pasadena, California

Received 19 January 1965

A number of interesting results have been derived [1-6] from the recent suggestion of spin-unitary spin independence of the strong interactions [7-9]. Considering that the symmetry is defined in a non-relativistic context, it is tempting to unravel the relativistic foundations, if any. Several suggestions have been made with this aim in mind [10-13], generally based upon a massless quark model [14] and a study of its interaction Lagrangians [13] *** or in connection with a chiral system generated by current matrix elements [11]. We would like to present a different approach, which may throw additional light on the issue.

Imagine at every point of space time a real six-dimensional manifold, orthogonal to the Minkowsky space itself. Wave functions are thus labelled by six \( \gamma^i \) coordinates, in addition to \( x^1, x^2 \text{ and } x^3 \). In fact, we are going to use only a 6-sphere with unit radius in the \( Y \) space, and can therefore replace the \( y^i \) by five angles of a polar coordinate system \( Y' \). The \( Y' \) sphere supports the 15 parameter \( \text{R}(6) \) group of rotations in six dimensions, isomorphic to \( \text{SU}(4) \); since this contains \( \text{U}(3) \), we can identify nine of these generators [15] with the nine \( \lambda^i \) \( (i = 0, 1, \ldots, 8 \), as in ref. 10). For a more detailed analysis of this connection, we refer the reader to an early attempt at a geometrization of particle internal symmetries [16,17].

We can now write an invariant,

\[
\lambda^2 \left( \sum_{i=0}^{6} y^2_i \right) - \lambda^2 \left( \sum_{i=1}^{6} y^2_i \right) = \lambda^2 \left( \sum_{i=1}^{6} y^2_i \right) - \lambda^2 \left( \sum_{i=1}^{6} y^2_i \right) = \lambda^2 \left( \sum_{i=1}^{6} y^2_i \right)
\]

This is invariant with respect to the product group

\[
\text{L}(4)_{1,3} \otimes \text{SU}(4)
\]

where \( \text{L}(4)_{1,3} \) is the homogeneous Lorentz group. We now replace the \( x_i y_a \) by a merged coordinate,

\[
x_{ia} = x_i y_a
\]

and require invariance of the metric

\[
\sum_{a=1}^{6} z_{ia}^2 - \sum_{a=1}^{6} z_{ia}^2 = \lambda^2.
\]

This defines a particular homogeneous Lorentz group in 24 real dimensions, 6 time-like and 18 space-like. An infinitesimal \( \text{L}(24)_{6,18} \) transformation can be written as

* Work supported in part by the U.S. Atomic Energy Commission.
** On leave of absence from Tel Aviv University, Tel Aviv, Israel and the Israel Atomic Energy Commission, Tel Aviv, Israel.
*** Some considerations in connection with the Sakata-type three-field model contain similar results [12].

327
\[ L(a) = 1 + \frac{1}{2} \alpha^a_{ia,jb} \mathcal{Q}_{ia,jb} \] \quad (5)
\[ \alpha^a_{ia,jb} = -\alpha_{jb,ia} \] \quad (6)
\[ \mathcal{O}_{ia,jb} = -\mathcal{O}_{jb,ia} \] \quad (7)

Viewed as 24 × 24 matrices, the \( \mathcal{Q}_{ia,jb} \) are real and antisymmetric when connecting space-like with space-like and time-like with time-like coordinates, and symmetric when connecting space-like with time-like ones.

One way of writing the \( \mathcal{Q}_{ia,jb} \) is in terms of the \( L(4) \) generators \( \gamma_{ij} \) and a symmetric tensor \( \mathcal{Y}_{ab} \) in \( Y' \) space,
\[ \Lambda^{(1)}(a) = 1 + \frac{1}{2} \gamma_{ia,jb} A_{ij} \mathcal{Y}_{ab} \] \quad (8)
and a symmetric tensor \( \theta_{ij} \) in Minkowski space coupled with the \( \mathcal{R}_{ab} \) generators of rotations in \( Y' \),
\[ \Lambda^{(2)}(a) = 1 + \frac{1}{2} \alpha^a_{ia,jb} \mathcal{R}_{ij} \mathcal{Y}_{ab} \] \quad (8')
(The \( \theta_{ij} \) are \( 4 \times 4 \) symmetric matrices if \( i,j = 0 \) or \( i \) and \( j = 0 \), and antisymmetric if only \( i \) or \( j = 0 \).)
\[ G_{ia,jb} \] is the Hermitian generator (incorporating a \( -i \)) in the infinite dimensional representation; \( \rho_i \) is the Hermitian translation operator.

We can now define
\[ \omega^a_{ab} = \frac{1}{2} \varepsilon_{ikl} \mathcal{G}^a_{ia,jb} \rho^l \] \quad (9)
where we have in fact used only those \( G_{ia,jb} \) which are antisymmetric in \( (i,j) \) and symmetric in \( (a,b) \), i.e., of form (8). The \( \omega^a_{ab} \) is an axial vector in Minkowski space and a symmetric tensor in \( Y' \). It is orthogonal to the momentum four-vector and has the following commutation relations,
\[ \left[ M_{ij}, \omega^a_{ab} \right] = i \left( \varepsilon_{jk} \omega^a_{ib} - \varepsilon_{ik} \omega^a_{jb} \right) \] \quad (10)
\[ \left[ \omega^a_{ab}, \rho_i \right] = 0 \] \quad (11)
\[ \left[ \omega^a_{ab}, \rho^k \right] = 0 \] \quad (12)

From (12) we note that the \( \omega^a_{ab} \) is a constant of the motion in the physical world. Explicitly, it will be \( \rho(p_1, p_2, p_3) \), \( M(M_{23}, M_{31}, M_{12}) \), \( N(M_{14}, M_{24}, M_{34}) \),
\[ \omega_{0,ab} = p \cdot M y^{ab} \]
\[ w^{ab} = p_0 M y^{ab} - p \times N y^{ab} \] \quad (13)

In the rest frame, we are left with a three-vector

\[ \omega^{ab} = m S y^{ab} \] \quad (14)

where \( S \) is the total spin with the commutation relations
\[ [S_i, S_j] = i \varepsilon_{ijk} S_k \] \quad (15)

For \( a = 0, b = 0 \) our \( \omega^{00} \) is indeed proportional to the spin. We can now more be precise in our choice of \( a,b \). In the \( Y' \) model we used, the second-order tensor reduces
\[ 6 \times 6 = 15(-) + 21(+) \]
and the \( SU(3) \) content (as \( 6 = 3 + \bar{3} \)) is
\[ 15(-) = 8(-) + 3 + \bar{3} + 1(-), \]
\[ 21(+) + 8(+) + 6 + \bar{6} + 1(+) \].

The \( 8(+) \) and \( 1(+) \) are formed from the direct sum of two reductions \( 3 \times 3 \) and \( \bar{3} \times 3 \) (they become symmetric matrices in the real coordinates). They are thus \( \lambda \oplus \lambda' \), and fulfill all commutation and anti-commutation rules of the \( \lambda_i \) set\[15\]. As a result, we replace our \( (a,b) \) indices \( (a, b = 1 \ldots 6) \) by the appropriate \( c = 0 \ldots 8 \) of the \( \lambda_c \oplus \lambda'_c \):
\[ \omega^c = m S y^c \] \quad (16)

We now find that these 27 operators, together with the antisymmetric \( \nu^c \) \( \lambda \oplus -\lambda' \) and represents \( F \) spin
\[ \nu^c = m \nu^c \] \quad (17)
genenerate the algebra of \( SU(6) \) of Gürsey and Radicati\[7\], Sakita\[8\] and Zweig\[9\].

We note the following points:
1) The \( \omega^c \) are constants of the motion. Their general explicit structure is complicated; in the rest frame, however, they reduce to the total angular momentum operators and \( SU(3) \) tensors, enabling us to derive their commutation relations in a simple way. The \( \nu^c \) are also constants of the motion, since an ideal \( SU(3) \) commutes with the Lorentz group.
2) In this way, the \( U(6) \) quantum numbers are well-defined quantities and we are allowed to use them in relativistic situations, including the labelling of incoming and outgoing states of the \( S \)-matrix.
3) This derivation does not depend upon a definite field model, e.g., quarks; neither it is linked to any particular representation of the Lorentz group as is the \( U(12) \) group of refs. 10 and 11. This is a theoretical advantage; on the other hand, since it does not fit into the three-field structures, we may be losing information due to these models. In particular, the spin operators...
we use here correspond to what has been named the total angular momentum in a recent paper by
Gell-Mann [18], i.e., quark-spin plus the orbital
momenta of the quarks making up the physical
particle, in a quark model.
4) In a bootstrap approach, with no fundamental
hadron, our system has definite advantages.
5) Our derivation can also be realized in the geo-
metrical picture of the symmetry we have re-
cently studied [19, 20].
6) We do not know if L(24) is a symmetry of the
strong interactions. An invariant mass term can
be written in this system, but no kinetic energy
term could be added as long as one does not introduce
orbital momenta and translations in the \( y^i \) con-
tinuum. Such a step would imply a reformulation
of physics in terms of \( L(24) \)-invariant field equa-
tions, provided these equations are made so as
to ensure that wave functions do not spread out-
side of Minkowski space. As presented here, we
have not introduced such a formalism and the
\( y^i \) substrate was only used in an abstract sense
to define the group \( L(24) \). (We could have derived
the same result from a vector meson \( \phi^a_{\mu} \) trans-
forming like \( 3 + \bar{3} \) in SU(4), since it would have
formed an \( L(24) \) vector like \( z_{\alpha} \).
7) There may be other groups whose \( \omega \) generates
\( SU(6) \) at rest in the same fashion. The conclusion
we wish to emphasize is that \( SU(6) \) can be co-
vARIANTLY defined in the abstract, independently
of any explicit expressions in terms of specific
fields.

After completion of this study, a preprint by
R. Delburgo, A. Salam and J. Stathdee has reached
us, in which a similar redefinition of \( SU(6) \) is
presented through the use of Lubanski's \( \omega_{\mu} \),
superimposed upon the \( U(12) \) system mentioned
in ref. 11. Both \( U(12) \) and \( L(24) \) [18] are rank 12
groups, but it is conceivable that \( U(12) \) may turn
out to be nearer the true hadron symmetry. An-
other preprint by the same authors introduces a
\( U(12) \) invariant field equation - which seems pos-
sible only if momenta in a substrate of the \( y^i \)
type are added to the physical world. We would
like to thank the authors for acquainting us with
their results prior to publication.

References
Letters 13 (1964) 299.
415.
(1964) 416.
(1964) 509.
Letters 13 (1964) 514.
(1964) 173.
9. G. Zweig, "Lettres Intern. School of Phys.," "Ettore
Rev. Letters 13 (1964) 575.
11. K. Bardakci, J. M. Cornwall, P. G. O. Freund and
(1961) 1226, Appendix.
15. M. Gell-Mann and Y. Ne'eman, The Eightfold Way,
Gables Conf. on Symmetries at high energy (W. H.
17. A. Pais, Physica 19 (1953) 869, introduced such a
substrate for the isospace, in order to use "orbi-
tal" isospins, and redefined all observables (total
momentum, total energy, etc.) as averages over this
compact space. This need not be done here,
as we are only using the substrate as an auxiliary
feature to define \( L(24) \), which could have been also
done in the abstract.
18. M. Gell-Mann, California Institute of Technology
preprint CALT-68-18, to be published.
19. Y. Ne'eman and J. Rosen, Ann. Phys., to be pub-
lished.
20. Y. Ne'eman, Revs. Modern Phys., to be published.