Indefinite-Metric Quantum Theory of Genuine and Higgs-Type Massive Vector Fields

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On the basis of the indefinite-metric vector field theory proposed previously, Johnson's proposition that the physical mass of a vector field tends to zero as its bare mass goes to zero, is shown to be valid if the vector field couples with a charged scalar field in the minimal interaction. In the case of the theory of spontaneously broken gauge invariance, the reason why the vector field acquires a non-zero mass in spite of the above theorem is clarified. The theory of a vector field which is massive owing to the spontaneous breakdown of gauge invariance is consistently formulated in the framework of the indefinite-metric quantum field theory. In this formalism, both renormalizability and the unitarity of the physical S-matrix are self-evident.

§ 1. Introduction

Recently, the present author\textsuperscript{1,2)},\textsuperscript{8) has proposed an indefinite-metric theory of a massive vector field such that as its mass goes to zero the theory smoothly tends to the Landau-gauge quantum electrodynamics.\textsuperscript{3,4)} As one of important consequences of this theory, we can reasonably show the validity of Johnson's proposition\textsuperscript{5,6)} that if the bare mass of the vector field $U_{\mu}$ goes to zero, its physical mass must also tend to zero, provided that there are no other massless physical particles, under the assumption that the current $j_{\mu}$ is conserved and does not explicitly depend on $U_{\mu}$.

On the other hand, in connection with Weinberg's theory of leptons,\textsuperscript{9)} much attention has been paid to the spontaneous breakdown of gauge invariance in the massless vector field theories. Several years ago, Higgs and others\textsuperscript{7)} noted that if gauge invariance of the theory is spontaneously broken, the massless vector field acquires a non-zero mass, but then Goldstone bosons do not appear in the Coulomb gauge because we do not have manifest covariance, which is necessary for the proof of the Goldstone theorem.\textsuperscript{8)} If one reconsiders this situation in a covariant gauge, in which we have to introduce indefinite metric, Goldstone bosons appear but they become unphysical. This interesting phenomenon is now called the Higgs phenomenon. Recently, 't Hooft\textsuperscript{5)} has applied it to the Yang-Mills

\textsuperscript{8) Johnson's reasoning was based on the conventional massive vector field theory whose massless limit is non-existent.
field in order to construct a renormalizable theory of massive charged vector fields.\textsuperscript{a)} B. W. Lee\textsuperscript{10} has made a detailed study of the Higgs phenomenon in the theory of a neutral vector field which couples with a charged scalar field (Higgs model). All these recent investigations are made in the Feynman functional-integral formalism.

The purpose of the present paper is to analyze the Higgs phenomenon on the basis of our indefinite-metric theory of a vector field. We first extend our proof of Johnson's proposition to the case in which the neutral vector field $U_\mu$ couples with a charged scalar field (§ 2). Then we encounter an apparent dilemma between Johnson's proposition and the Higgs phenomenon. In order to resolve this paradox, we investigate a solvable model, which is essentially the zeroth approximation to the Higgs model (§ 3). The reason for the dilemma is found to be a special character of $j_\mu$ in the case of the spontaneously broken gauge theory. Finally, the full Higgs model is studied (§ 4). We clarify why Goldstone bosons become unphysical in such a way that the unitarity of the physical $S$-matrix is not violated. The main results obtained by B. W. Lee\textsuperscript{10} are reproduced in quite a transparent way.

§ 2. Genuine massive vector field

In this section, we consider a neutral vector field $U_\mu$, which couples with a charged scalar field $\phi$. We assume that the interaction between them is the so-called minimal interaction. The Lagrangian density $\mathcal{L}$ of the system is given by

$$\mathcal{L} = -\frac{1}{4} (\partial^\mu U^\nu - \partial^\nu U^\mu) (\partial_\mu U_\nu - \partial_\nu U_\mu) + \frac{1}{2} m^2 U_\mu U^\mu + B \partial^\mu U_\mu + \mathcal{L}_\phi$$

with

$$\mathcal{L}_\phi = (\partial^\mu + ig U^\mu) \phi^* (\partial_\mu - ig U_\mu) \phi + F(\phi^* \phi).$$

Here, $B$ is an auxiliary scalar field having negative norm; $m$ and $g$ denote the bare mass of $U_\mu$ and the bare coupling constant, respectively; $F$ is a quadratic real polynomial; a dagger stands for hermitian conjugation and the Minkowski metric employed is $(1, -1, -1, -1)$.

The field equations for $U_\mu$ and $B$ are

$$\partial^\nu U_\nu = 0,$$

$$(\Box + m^2) U_\mu - \partial_\mu B = j_\mu$$

with $\Box = \partial^\mu \partial_\mu$. Here the current

$$j_\mu = -\delta \mathcal{L}_\phi / \delta U^\mu$$

$$= -ig [\phi^* \partial_\mu \phi - (\partial_\mu \phi^*) \phi] - 2g^2 \phi^* \phi U_\mu$$

\textsuperscript{a)} We note, however, that this theory is not a genuine Lagrangian field theory because one has to introduce \textit{ad hoc} Feynman's fictitious quanta. On the other hand, Weinberg's idea is based on the Lagrangian formalism.
is conserved:
\[ \partial^* j_\mu = 0, \quad (2.6) \]
as is easily confirmed, but it explicitly depends on \( U_\mu \). From (2.4), (2.3) and (2.6), we have
\[ \Box B = 0. \quad (2.7) \]

The canonical conjugates of \( U_l (l = 1, 2, 3), U_0, \phi \) and \( \phi^\dagger \) are
\[ \Pi_l = \partial \mathcal{L} / \partial \dot{U}_l = \dot{U}_l - \partial_t U_l, \]
\[ \Pi_0 = \partial \mathcal{L} / \partial \dot{U}_0 = B, \]
\[ \pi = \partial \mathcal{L} / \partial \dot{\phi} = \dot{\phi}^\dagger + ig U_0 \phi^\dagger, \]
\[ \pi^\dagger = \partial \mathcal{L} / \partial \dot{\phi}^\dagger = \dot{\phi} - ig U_0 \phi, \quad (2.8) \]
respectively, where a dot stands for differentiation with respect to time. The equal-time commutators for canonical variables are
\[ \{ U_\mu, \Pi_\nu \} = i \delta_\mu^\nu \delta (x - y), \]
\[ \{ \phi, \pi \} = \{ \phi^\dagger, \pi^\dagger \} = i \delta (x - y), \quad (2.9) \]
and vanishing commutators for all other combinations. In terms of field variables, the equal-time commutators are rewritten as *
\[ \{ U_k (x), \dot{U}_l (y) \}_{x = y} = i \delta_{kl} \delta (x - y), \]
\[ \{ U_0 (x), B (y) \}_{x = y} = i \delta (x - y), \]
\[ \{ \dot{U}_k (x), B (y) \}_{x = y} = i \delta^* \delta (x - y), \]
\[ \{ B (x), \phi^\dagger (y) \}_{x = y} = 0 \phi (y) \delta (x - y), \]
\[ \{ B (x), \phi^\dagger (y) \}_{x = y} = - g \phi^\dagger (y) \delta (x - y), \]
\[ \{ \phi (x), \phi^\dagger (y) \}_{x = y} = \{ \phi^\dagger (x), \phi (y) \}_{x = y} = i \delta (x - y), \quad (2.10) \]
and vanishing commutators for all other combinations of \( U_k, U_0, \dot{U}_l, B, \phi, \phi^\dagger \) and \( \dot{\phi}^\dagger \). In order to calculate the equal-time commutators involving \( \dot{U}_0 \) or \( \dot{B} \), we have to make use of
\[ \dot{U}_0 = \sum_k \partial_k U_k, \]
\[ \dot{B} = \sum_k \partial_k \dot{U}_k - (\Delta - m^2) U_0 - j_0, \quad (2.11) \]
the relations which follow from (2.3) and (2.4). Since \( \mathcal{L}_\phi \) is of the minimal interaction, we have

\[ * \quad \partial^* = \partial / \partial x_\mu. \]
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\[
\dot{j}_0 = -\frac{\delta L_s}{\delta \phi} - \frac{\delta L_s}{\delta \dot{\phi}} - ig\phi - ig\phi^* \frac{\delta L_s}{\delta \dot{\phi}^*} = ig(\pi \phi - \phi^* \pi^*). \tag{2.12}
\]

Hence the canonical commutation relations imply that

\[
[U_\phi(x), j_0(y)]_{x_0 = y_0} = 0,
\]
\[
[U_\phi(x), j_\phi(y)]_{x_0 = y_0} = 0,
\]
\[
[B(x), j_0(y)]_{x_0 = y_0} = 0. \tag{2.13}
\]

In order to find four-dimensional commutation relations involving \(B\), we re-write (2.7) as\(^*\)

\[
B(x) = -\int dz [\dot{D}(x - z)B(z) + D(x - z)\dot{B}(z)], \tag{2.14}
\]

where \(z_0\) is a free parameter. Setting \(z_0 = y_0\) in (2.14), with the aid of (2.11), (2.10) and (2.13), it is straightforward to obtain

\[
[B(x), U_\phi(y)] = i\hbar \delta(x - y), \tag{2.15}
\]
\[
[B(x), B(y)] = -i\hbar^2 \delta(x - y) \tag{2.16}
\]

and \([B(x), j_\phi(y)] = 0\). Because of manifest covariance, therefore, we have

\[
[B(x), j_\phi(y)] = 0. \tag{2.17}
\]

Since \(B(x)\) satisfies a free-field equation (2.7), we can consistently define\(^*\) its positive frequency part \(B^{(+)}(x)\). The constraint for the physical states is

\[
B^{(+)}(x) |\text{phys}\rangle = 0. \tag{2.18}
\]

From (2.17) we see that \(j_\phi(y) |\text{phys}\rangle\) is also a physical state.

Let \(|\Omega\rangle\) be the true vacuum. From manifest covariance, local commutativity and the Lorentz condition (2.3), we have a spectral representation:

\[
\langle \Omega | [U_\phi(x), U_\phi(y)] |\Omega\rangle = -i \int ds dp (p_\mu + s^{-1} \partial_\nu \partial^\nu) A(x - y, s) + i m \hbar \delta_\mu \delta_\nu \partial^\mu \partial^\nu D(x - y). \tag{2.19}
\]

The parameter \(\hbar\) is determined as follows. By making use of (2.4), (2.16) and (2.17), we obtain

\[
(\Box^\nu + m^\nu)(\Box^\mu + m^\mu) \langle \Omega | [U_\phi(x), U_\phi(y)] |\Omega\rangle = \langle \Omega | [j_\phi(x), j_\phi(y)] |\Omega\rangle + i m \hbar \delta_\mu \delta_\nu \partial^\mu \partial^\nu D(x - y). \tag{2.20}
\]

Because of (2.6), we should have

\[^*\] To define \(B^{(+)}(x)\), replace \(D\) by \(D^{(+)}\) in (2.14).
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\[ \langle Q| [j_\mu(x), j_\nu(y)] |Q \rangle = -i \int_0^\infty ds \delta(s) (g_{\mu\nu} + s^{-1} \partial_\mu \partial_\nu) \Delta(x-y, s) \]  

(2.21)

with \( a > 0 \), provided that no massless \textit{physical} particles are present. On substituting (2.19) and (2.21) in (2.20), we find

\[ \hbar = 1, \]

\[ (s - m^2) \rho(s) = \tilde{\rho}(s). \]  

(2.22)

From (2.19) together with (2.22) and the first commutator in (2.10), we find Johnson's formulas\(^\text{b}\)

\[ \int_0^\infty ds \rho(s) = 1, \]

\[ \int_0^\infty ds \rho(s)/s = m^{-1} , \]  

(2.23)

where \( b = \min(a, m^2) \). From (2.23), we conclude that the physical mass \( m_{\text{phys}} \) of \( U_\mu \), which is a point spectrum of \( \rho(s) \), must tend to zero as \( m \to 0 \).

The above reasoning is applicable to any theory in which \( U_\mu \) couples with its source in the minimal interaction.

§ 3. Boulware-Gilbert model

The Lagrangian density of the Higgs model\(^\text{d}\) is essentially the same as (2.1) with \( m \to 0 \), though we here adopt the Landau-gauge formulation. All field equations and canonical commutation relations remain unchanged. The only difference consists in the non-vanishing vacuum expectation value of \( \phi \):

\[ \langle Q| \phi(x) |Q \rangle = \nu/\sqrt{2} \neq 0 , \]  

(3.1)

which was not used in the proof, presented in § 2, of Johnson's proposition that \( m \to 0 \) implies \( m_{\text{phys}} \to 0 \). Nevertheless, it is known that \( U_\mu \) acquires a non-zero physical mass \( (m_{\text{phys}} \neq 0) \) at \( m = 0 \) in the Higgs model.

As usual, we set

\[ \sqrt{2} \phi(x) = \nu + \phi(x) + i \chi(x) , \]  

(3.2)

where \( \nu^* = \nu, \phi^* = \phi \) and \( \chi^* = \chi \), so that

\[ \langle Q| \phi(x) |Q \rangle = \langle Q| \chi(x) |Q \rangle = 0 . \]  

(3.3)

On substituting (3.2) in (2.2), we have

\[ L_\phi = \frac{1}{2} M^2 U^* U_\mu + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} \partial^\mu \chi \partial_\mu \chi - M U^* \partial_\mu \chi - \frac{1}{2} g U U^* \phi \partial_\mu \phi \]

\[ + g M U^* \phi \phi + g U^* (\chi \partial_\mu \phi - \phi \partial_\mu \chi) + F(\frac{1}{2} (\nu + \phi)^2 + \frac{1}{2} \chi^2) , \]  

(3.4)

where

\[ M = g \nu . \]  

(3.5)
As the zeroth approximation to the Higgs model, we consider the case in which $g \to 0$ (and $F \to 0$) but $M$ is kept finite. Since $\phi$ then decouples from the rest, we have an effective Lagrangian density

$$\mathcal{L}_0 = -\frac{1}{4} (\partial^\mu U^\nu - \partial^\nu U^\mu) (\partial_\mu U_\nu - \partial_\nu U_\mu) + \frac{1}{2} (m^2 + M^2) U^\mu U_\mu + B \partial^\mu U^\nu - MU^\mu \partial_\nu \chi + \frac{1}{2} \partial^\mu \partial_\nu \chi \chi .$$

This is essentially the model considered by Boulware and Gilbert as an example of a gauge-invariant massive vector field. Since this model is exactly solvable, in this section we analyze it in detail in order to see why the proof of $m_{\text{phys}} \to 0$ as $m \to 0$ does not apply.

The field equations are (2.3) and (2.4) together with

$$j_\mu = M \partial_\nu \chi - M^2 U_\mu$$

and

$$\Box \chi = 0 .$$

We may rewrite (2.4) with (3.7) as

$$\Box + m^2 + M^2) U_\mu - \partial_\mu (B + M \chi) = 0 .$$

From (3.9), (2.3) and (3.8), we have

$$B = 0 ,$$

whence

$$\Box (\Box + m^2 + M^2) U_\mu = 0 .$$

The field equations are thus the same as those in the free-field case having a mass squared $m^2 + M^2$ and an auxiliary field $B + M \chi$. But the constraint is still (2.18).

The equal-time commutators involving $\chi$ and/or $\dot{\chi}$ are as follows:

$$[U_\mu, \chi] = [U_\mu, \dot{\chi}] = 0 ,$$

$$[\dot{U}_\mu, \chi] = [\dot{U}_\mu, \dot{\chi}] = 0 ,$$

$$[B, \chi] = [B, \chi] = [\chi, \dot{\chi}] = 0 ,$$

$$[B, \chi] = -i M \delta(x - y) ,$$

$$[\chi, \dot{\chi}] = i \partial(x - y) .$$

Hence, by using (2.14), (2.11) and (3.7), it is easy to confirm (2.16) and (2.17). For completeness, we here write all four-dimensional commutation relations between fields:

$$[U_\mu(x), U_\nu(y)] = -i [g_\nu + (m^2 + M^2)^{-1} \partial_\nu \partial_\nu] \delta(x - y) (m^2 + M^2)

+ i (m^2 + M^2)^{-1} \partial_\nu \partial_\nu D(x - y) ,$$

A contrary statement was erroneously made in Ref. 1).
\[
[U_\mu(x), B(y)] = -i \partial^*_\mu D(x-y),
\]
\[
[B(x), B(y)] = -im^2 D(x-y),
\]
\[
[U_\mu(x), \chi(y)] = 0,
\]
\[
[B(x), \chi(y)] = -i MD(x-y),
\]
\[
[\chi(x), \chi(y)] = iD(x-y).
\]

Therefore, from (3.7) we have
\[
[j_\mu(x), j_\nu(y)] = M^2 \partial^*_\mu \partial^*_\nu [\chi(x), \chi(y)] + M^4 [U_\mu(x), U_\nu(y)]
\]
\[
= -iM^4 [g_{\mu\nu} + (m^2 + M^2)^{-1} \partial^*_\mu \partial^*_\nu] A(x-y, m^2 + M^2)
\]
\[
- im^2 M^2 (m^2 + M^2)^{-1} \partial^*_\mu \partial^*_\nu D(x-y).
\]

The remarkable point of the Boulware-Gilbert model is the existence of the term proportional to \( \partial^*_\mu \partial^*_\nu D(x-y) \) in the current-current commutator. This fact, which contradicts (2.21), is due to the presence of massless physical particles. Indeed, let
\[
\tilde{\chi}(x) = -M(m^2 + M^2)^{-1}[m^2 \chi(x) - MB(x)],
\]

since
\[
\Box \tilde{\chi}(x) = 0,
\]
\[
[\tilde{\chi}(x), B(y)] = 0,
\]
\[
[\tilde{\chi}(x), \tilde{\chi}(y)] = im^2 M^2 (m^2 + M^2)^{-1} D(x-y),
\]
\[
\tilde{\chi}(x) \text{ is massless, physical (i.e., } B^{(+)}(x) [\tilde{\chi}(y) | \Omega \rangle] = 0 \text{) and of positive norm.}^*)
\]

The intermediate states consisting of a \( \tilde{\chi} \) particle gives a non-zero contribution to \( \langle \Omega | [j_\mu, j_\nu] | \Omega \rangle \).

It is important to note that though \( \tilde{\chi}(x) \) is physical, as \( m \to 0 \) it tends to \( B(x) \) so that its norm tends to zero; therefore the \( \tilde{\chi} \) particles become unobservable. As remarked previously,\(^1\) \( B(x) \) is unphysical for \( m \neq 0 \), but it becomes physical for \( m = 0 \) because it then commutes with \( B(y) \). For \( m = 0 \), the massless unphysical field is
\[
X(x) = \chi(x) + \frac{1}{2} M^{-1} B(x);
\]
both \( B(x) \) and \( X(x) \) are of zero norm, but \([B(x), X(y)]\) is non-vanishing.

\section{Higgs-type massive vector field}

In § 3, we have seen that the reason why the physical mass of \( U_\mu \) can be non-zero as \( m \to 0 \) in the theory of spontaneously broken gauge invariance is the existence of massless physical particles, which are not identical with Goldstone

\(^*)\) The normalization of \( \tilde{\chi}(x) \) is chosen so as to account for the massless spectrum of \([j_\mu, j_\nu]\).
bosons. The crucial point is that as \( m \to 0 \) those massless physical particles become unobservable just like the quanta of the Coulomb interaction. Having understood the mechanism of yielding a non-zero physical mass, in this section we study the Higgs model by setting \( m=0 \) from the beginning. We rewrite \( U_\mu \) as \( A_\mu \) in order to stress \( m=0 \), and we consider the general covariant gauge by adding \( \frac{1}{2} \alpha B^2 \) to \( \mathcal{L} \) for the convenience of the comparison with B. W. Lee's work.\(^{10}\)

The field equations are

\[
\partial^* A_\mu + \alpha B = 0, \quad (4.1)
\]
\[
(\Box + M^2) A_\mu - (1 - \alpha) \partial_\mu B - M \partial_\mu \chi = J_\mu, \quad (4.2)
\]

with

\[
J_\mu = j_\mu + M^2 A_\mu - M \partial_\mu \chi
\]
\[
= -g[A_\mu (\psi^2 + \chi^2) + 2MA_\mu \phi + \chi \partial_\mu \phi - \phi \partial_\mu \chi]. \quad (4.3)
\]

Of course, \( \partial^* j_\mu = 0 \) but \( \partial^* J_\mu \neq 0 \). We still have\(^{**} \)

\[
\Box B = 0, \quad (4.4)
\]

but \( \chi \) no longer satisfies a free-field equation. The constraint (2.18) remains unchanged.

The equal-time commutators (2.10) remain valid if \( U_\mu \) and \( \phi \) are replaced by \( A_\mu \) and by \((1/\sqrt{2})(\nu + \phi + i\chi)\), respectively. Hence we have four-dimensional commutation relations

\[
[B(x), A_\mu(y)] = i\partial_\nu^* D(x-y), \quad (4.5)
\]
\[
[B(x), B(y)] = 0, \quad (4.6)
\]
\[
[B(x), \chi(y)] = -i[M + g\phi(y)] D(x-y), \quad (4.7)
\]
\[
[B(x), J_\mu(y)] = -igM \partial_\nu^* [\psi(y) D(x-y)]. \quad (4.8)
\]

From (4.7) and (4.8), we have

\[
\langle \Omega | [B(x), \chi(y)] | \Omega \rangle = -iMD(x-y), \quad (4.9)
\]
\[
\langle \Omega | [B(x), J_\mu(y)] | \Omega \rangle = 0, \quad (4.10)
\]

respectively. The non-vanishing of (4.9) is the important characteristic of the spontaneously broken gauge theory. From (4.9) together with (4.1), we must have

\[
\langle \Omega | [A_\mu(x), \chi(y)] | \Omega \rangle = i\alpha M \partial_\nu^* E(x-y), \quad (4.11)
\]

\(^*\) In his treatment, the Landau-gauge case is ill-defined in contrast with our formalism. For example, the proper self-energy part of \( A_\mu \) is singular at \( \alpha = 0 \) in his formalism.

\(^{**} \) If \( m \neq 0 \) and \( \alpha 
eq 0 \), \( B \) becomes massive; every \( D(x-y) \) appearing in § 2 then is replaced by \( D(x-y, \alpha m^2) \).
because of manifest covariance and the vanishing equal-time commutators, where
\[ E(x) = -(\partial/(\partial m^2))A(x, m^2)|_{m=0} \]
\[ = -(8\pi)^{-1}e(x_0)\theta(x^2), \quad (4.12) \]
\[ \Box E(x) = D(x). \quad (4.13) \]

As seen in § 3, \( A_\mu \) acquires a non-zero mass at least if \( g \) is small. Hence \( A_\mu \) contains no massless transverse components. From manifest covariance, local commutativity, (4.5) together with (4.1) and the equal-time commutators, we have a spectral representation
\[ \langle \Phi | [A_\mu(x), A_\nu(y)] | \Phi \rangle = -i \int_0^\infty ds \rho(s) \left[ (g_{\mu
u} + s^{-1}\partial_\mu \partial_\nu) A(x-y, s) - s^{-1}\partial_\mu \partial_\nu D(x-y) \right] - i\alpha \partial_\mu \partial_\nu E(x-y) \quad (4.14) \]
with \( \epsilon > 0 \).

Now, we consider the asymptotic fields. Since in-fields and out-fields can be discussed in the same way, for definiteness we consider in-fields alone. In order to avoid gauge complication, we first discuss the Landau-gauge case (\( \alpha = 0 \)). Suppose that
\[ A_\mu(x) \rightarrow A_\mu^{\text{in}}(x), \quad B(x) \rightarrow B^{\text{in}}(x), \]
\[ \psi(x) \rightarrow \psi^{\text{in}}(x), \quad \chi(x) \rightarrow \chi^{\text{in}}(x) \quad (4.15) \]
as \( x_0 \rightarrow -\infty \). Each in-field has to satisfy a free-field equation. Hence (4.5), (4.6), (4.9) and (4.11) with \( \alpha = 0 \) yield
\[ [B^{\text{in}}(x), A_\nu^{\text{in}}(y)] = i\partial_\mu \chi^{\text{in}}(x-y), \quad (4.16) \]
\[ [B^{\text{in}}(x), B^{\text{in}}(y)] = 0, \quad (4.17) \]
\[ [B^{\text{in}}(x), \chi^{\text{in}}(y)] = -iMD(x-y), \quad (4.18) \]
\[ [A_\mu^{\text{in}}(x), \chi^{\text{in}}(y)] = 0, \quad (4.19) \]
respectively. From (4.18) we see that \( \chi^{\text{in}}(y) \) must satisfy the d'Alembert equation, that is, the \( \chi \) field is massless. This fact represents that \( \chi \) is the Goldstone field. Hence \( \chi^{\text{in}}(x) \) has to satisfy
\[ [\chi^{\text{in}}(x), \chi^{\text{in}}(y)] = i\gamma D(x-y), \quad (4.20) \]
where \( \gamma \) is some real dimensionless constant. The constraint for the physical in-states is
\[ [B^{\text{in}}(x)]^{(+)}|_{\text{phys}} = 0. \quad (4.21) \]

From (4.18) we see that Goldstone bosons are unphysical.

Since
\[ [B^{\text{in}}(x), V^{\text{in}}_\mu(y)] = 0, \quad (4.22) \]
where
\[ V_{\mu}^{\text{in}}(x) = A_{\mu}^{\text{in}}(x) - M^{-1} \partial_\mu \chi^{\text{in}}(x), \]  

the physical-state subspace is generated by the hermitian conjugates of \( \star \)
\[ [V_{\mu}^{\text{in}}(x)]^{(+)}, \quad [\phi^{\text{in}}(x)]^{(+)}, \quad [B^{\text{in}}(x)]^{(+)} \]  
from the vacuum. Since \( V_{\mu}^{\text{in}}(x) \) should satisfy a Klein-Gordon equation, it cannot contain a massless component. From (4.14) with \( \alpha = 0 \), the massless spectrum of \( [A_{\mu}^{\text{in}}(x), A_{\nu}^{\text{in}}(y)] \) is \( i K \delta_{\mu\nu} D(x-y) \) with
\[ K = \int_0^\infty ds \rho(s)/s. \]  

Therefore, using (4.20) and (4.19), we find
\[ \gamma = M^2 K. \]  

Thus we should have
\[ <\Omega| [\chi(x), \chi(y)]|\Omega> = i M^2 K D(x-y) + i \int_0^\infty ds \rho(s) A(x-y, s). \]  

For \( \alpha \neq 0 \), we have to be careful of the invalidity of (4.15), as was noted by Källén in quantum electrodynamics. Indeed, the right-hand side of (4.11) is inconsistent with any free-field equation. The appearance of \( E(x-y) \) implies that there should exist dipole-ghost states. As is well known, however, the Gupta-Bleuler theory, which corresponds to \( \alpha = 1 \), involves no dipole ghosts. This dilemma is due to the breakdown of the operator manifest covariance of a non-Landau-gauge theory, as has been pointed out recently. As far as two-point functions are concerned, however, this trouble can be bypassed. Following Lautrup, we define an operator
\[ A(x) = \frac{1}{4} A^{-1} [x_\mu \partial_\mu B(x) - \frac{1}{2} B(x)], \]  

where \( A \) denotes the Laplacian. Though \( A(x) \) is not a Lorentz scalar, it satisfies
\[ \Box A(x) = B(x), \]  
\[ [B(x), A(y)] = [A(x), A(y)] = 0, \]  
\[ [A_{\mu}(x), \partial_\nu A(y)] + [\partial_\mu A(x), A_{\nu}(y)] = i \delta_{\mu\nu} \partial_\tau D(x-y). \]  

From (4.14), (4.30) and (4.31), we see that the vacuum expectation value of
\[ \hat{\Lambda}_\mu(x) = A_{\mu}(x) + \alpha \partial_\mu A(x) \]  
has no \( \alpha \)-dependent term, that is, it equals (4.14) without the last term. With
\( V_{\mu}^{\text{in}} \) and \( \phi^{\text{in}} \) are of positive norm and \( B^{\text{in}} \) is of zero norm. The zero-norm unphysical field
\[ X^{\text{in}}(x) = X^{\text{in}}(x) + \frac{1}{4} M^{-1} B^{\text{in}}(x). \]
the aid of (4.5) and (4.9), we can show that
\[
\langle \mathcal{O} | M[A_\mu(x), A(y)] + [\partial_\mu A(x), \chi(y)] | \mathcal{O} \rangle = -iM \partial_\mu E(x-y).
\]
(4.33)
Hence if we define
\[
\tilde{\chi}(x) = \chi(x) + \alpha MA(x),
\]
(4.34)
then we have
\[
\langle \mathcal{O} | [\tilde{A}_\mu(x), \tilde{\chi}(y)] | \mathcal{O} \rangle = 0.
\]
(4.35)
Therefore, by defining \( A_\mu^{in}(x) \) and \( \chi^{in}(x) \) as the asymptotic fields of \( \tilde{A}_\mu(x) \) and \( \tilde{\chi}(x) \), respectively, the discussion of the in-fields reduces to that in the Landau-gauge case. Since from (4.9)
\[
\langle \mathcal{O} | [\chi(x), A(y)] + [A(x), \chi(y)] | \mathcal{O} \rangle = -iME(x-y),
\]
(4.36)
we have
\[
\langle \mathcal{O} | [\tilde{\chi}(x), \tilde{\chi}(y)] | \mathcal{O} \rangle = \langle \mathcal{O} | [\chi(x), \chi(y)] | \mathcal{O} \rangle - i\alpha M^2 E(x-y).
\]
(4.37)
Since the left-hand side of (4.37) should be identified with (4.27), we finally find
\[
\langle \mathcal{O} | [\chi(x), \chi(y)] | \mathcal{O} \rangle
\]
\[
= iM^2 KD(x-y) + i \int_{+\infty}^{0} ds \sigma(s) A(x-y, s) + i\alpha M^2 E(x-y)
\]
(4.38)
in the general covariant gauge. The Green's function counterparts of (4.14), (4.11) and (4.38) were given by B. W. Lee10 by calculating the proper self-energy parts by means of the Ward-Takahashi identities.

To sum up, we have shown that the neutral vector field theory of spontaneously broken gauge invariance can be consistently formulated in the framework of the indefinite-metric quantum field theory. We can avoid the use of complicated functional-integral technique completely. Our theory is manifestly renormalizable, and the unitarity of the physical S-matrix is self-evident because the constraint (2.18) persists at all time. The Goldstone field \( \chi \) is massless and unphysical, while the massless \( B \) field is physical but unobservable because of its zero norm just like the quanta of the Coulomb interaction.

Extension of our formalism to the non-Abelian gauge field will be formally straightforward,11 but we then encounter the difficulty that the constraint no longer persists.

References
4) B. Lautrup, Kgl. Danske Videnskab. Selskab, Mat.-fys. 35 (1967), No. 11.
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See also, B. W. Lee and J. Zinn-Justin, Phys. Rev. D5 (1972), 3121, 3137, 3155.

