The Evidence of the Once-Forbidden Spectra for the Law of $\beta$-Decay

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A series of empirical arguments has developed which leave two alternatives for the law of $\beta$-interaction: an STP (scalar+tensor pseudoscalar) or a VTP ($V$=vector) combination. We offer a new argument, based on an interpretation of the reported "allowed" shapes of once-forbidden spectra, which favors the STP over the VTP form. This new development may be important because it is a possible last step in arriving at a unique law of $\beta$-decay (i.e., all three components of the STP form would become both admissible and necessary).

A critical stage in the new argument is the explanation of the "allowed" shapes. The theory may yield such shapes if the Coulomb potential energy at the nucleus is sufficiently larger than the kinetic energy with which electron and neutrino are emitted. That condition happens to be best fulfilled just for the cases in which the "allowed" shapes are the most accurately observed. However, the dominance of the Coulomb effect cannot be a sufficient condition except with the STP or VA forms of $\beta$-interaction. Other forms make deviations from the allowed shape possible even when the Coulomb energy is supposed indefinitely large.

The last type of deviation has the same origin as the well-known Fierz interference in allowed spectra. The absence of the latter effect has been widely quoted but apparently never heretofore examined in detail. It is important in one of the series of arguments mentioned above. Hence, we investigate the empirical limits on Fierz-type deviations not only in once-forbidden but also in allowed spectra.

1. Introduction

The Fermi theory of $\beta$-decay is not completely definite, even though it has dealt successfully with much that is known concerning $\beta$-processes. The definite part of the theory consists of criteria which are equally well satisfied by several specific formulations. The alternatives can be exhaustively enumerated in more ways than one. The customary way leads to a $\beta$-interaction expressed as an arbitrary linear combination of five "forms": the scalar form, symbolized by $S$; the polar vector, $V$; the tensor, $T$; the axial vector, $A$; and the pseudoscalar, $P$. There has been a continuous effort to find and interpret experimental evidence that some definite combination of the five forms is the correct one.

The following points are currently regarded as having been established by the evidence:

I. The $T$ or $A$ forms must be part of the correct law of $\beta$-interaction. The main evidence is of two kinds. Only the $T$, or the $A$, form yields Gamow-Teller selection rules, which are needed for understanding the short lives of many $\beta$-transitions with a unit spin change. Only these two forms yield the "unique" spectrum shapes, found for $n$-times forbidden transitions in which the spin changes by $n+1$ units.

II. The $S$ or the $V$ form must also be included. The chief evidence for this is the short life found for $\beta$-transitions between nuclear states of zero spin, during the decay of $^{134}$ or $^{136}$. Forms other than the $S$ or $V$ lead to much smaller decay probabilities for these cases, relative to other transitions.

III. The $P$ form is needed to explain the singular RaE spectrum, according to the analysis of Petschek and Marshak. Actually, the $P$ form is needed in combination with the $T$ for this.

Each of the above arguments requires the inclusion of some component form without precluding the presence of other forms. The following argument seems to be the only generally recognized empirical one which tends to eliminate some of the possible combinations.

IV. It is usually regarded as evident that not both the $T$ and $A$ forms, nor both the $S$ and $V$ forms, can be parts of the $\beta$-coupling. If they are, then the spectra of allowed $\beta$-emission should be expected to deviate from the "allowed" (essentially statistical) shape, through the operation of the so-called "Fierz interference." However, no unquestionable deviations have ever been found. A

Perhaps the weakest of the above arguments is III; points for and against it are put forward in Sec. 5. However, even if one accepts this one together with all the other arguments, they still do not lead to a unique law of $\beta$-interaction. They are equally consistent with either an STP or a VTP combination. We shall introduce here a possible method for deciding between those final two possibilities. Like III, and perhaps also IV, our new argument (V) needs further confirmation before it will become completely acceptable, but at present it points to the STP combination as the correct one. This development would be important because one will have finally arrived at a unique law of $\beta$-decay. The law would be unique in the sense that each of its component forms would be necessary, and also the only ones admissible. Formerly, it was not entirely clear that $\beta$-decay data alone would ever be able to lead to a unique law.

The argument V now offered is in one respect an extension of IV. A Fierz type of interference can be identified in the theory of once-forbidden spectra as...
well as for allowed transitions. Whereas for the latter the Fierz interference arises only from $SV$ and $TA$ combinations, in the once-forbidden cases it also arises from $SA$, $VT$, and $AP$ combinations. That such interference should not be admitted for once-forbidden spectra seems to be supported by evidence of the same kind as for allowed spectra: All the normal once-forbidden spectra are reported to have the statistical shape. (The one "abnormal" example, RaE, needs separate consideration, Sec. 5.)

An important preliminary to argument V is an explanation of how a once-forbidden spectrum may have a statistical shape even in the absence of Fierz interference. The various effects which can arise are differentiated in Sec. 2. The condition for a statistical shape then becomes expressible as: a large Coulomb energy, $Ze^2/R$, at nuclear radius $R$, as compared to the kinetic energy which the electron and neutrino finally obtain. This condition is best fulfilled just for the observed cases in which the statistical shape is best confirmed. Whether the condition is sufficiently well fulfilled for all cases reported to have the allowed shape is not easily settled because of the essential occurrence of nuclear matrix elements which can be only crudely estimated. Hence, in Sec. 2 we take the standpoint that the statistical shape is experimentally established and follow out its implications. The result is our argument V.

The importance of the experimental detectability of Fierz interference for both arguments IV and V leads us to make a critical examination of the data: in Sec. 3, the allowed transitions and in Sec. 4, the once-forbidden cases. The empirical limits put on the Fierz type of effect, in both kinds of spectra, are about equally narrow. However, because of the greater complication of the once-forbidden theory, the interpretation there is not as straightforward as it is for the allowed transitions. It becomes clear that measurements of once-forbidden spectra which show positive deviations from the statistical shape (low $Z$, high energy) are needed to check on our explanation of the cases of no deviation, before argument V can be regarded as established on the same footing as IV.

2. THE SIGNIFICANCE OF ONCE-FORBIDDEN SPECTRA WITH STATISTICAL SHAPES

We use notation in which the $\beta$-interaction energy density is

$$\Sigma X G_X X + \text{Herm. conj.}$$

(1)

with $X=S$, $V$, $T$, $A$, or $P$. Each of these five symbols stands for one of the interaction forms of the list (6a-e) of Konopinski. Each $G_X$ is a Fermi coupling constant, measuring the magnitude of interaction of form $X$ which might be present in the correct law.

We present the theoretically expected spectra in terms of a "shape factor," which multiplies the well-known statistical factor to give the spectrum characteristically of the degree of forbiddenness, $n=0, 1, 2, \ldots$.

For allowed transitions:

$$C_0 = \left[ G_\beta^2 + \kappa^2 G_{\gamma}^2 \pm (2\kappa/W) G_\gamma G_{\beta} \right] \left| \int \beta \right|^2$$

$$+ \left[ G_\gamma^2 + \lambda^2 G_\beta^2 \pm (2\lambda/W) G_\beta G_{\gamma} \right] \left| \int \beta \sigma \right|^2.$$  (2)

The upper sign refers to positron, the lower to negatron, emission. Of course, the intrinsic signs of $G_\gamma G_{\beta}$ and $G_\beta G_{\gamma}$, as well as their magnitudes, are to be treated as unknown. The real parameters $\kappa$ and $\lambda$ are defined by

$$\int 1 = -\kappa \int \beta \quad \text{and} \quad \int \sigma = -\lambda \int \beta \sigma,$$

(3)

which are nuclear matrix elements to be found in reference 5. $W$ is the electron energy, inclusive of rest mass, in units of $mc^2$.

The terms $1/W$ in (2) are the well-known Fierz interference terms. Their presence would cause allowed spectra to deviate systematically from a statistical shape. The apparent absence of such deviations is the basis of argument IV of the introduction. We proceed to show that interference effects of the same type can be identified in the theory of once-forbidden spectra.

The latter was worked out by Uhlenbeck and Konopinski for each of the five interaction forms separately. Recently, Smith and Pursey calculated the contributions from the interference between the various forms. The resulting shape factor $C_1$, if written with the same generality as $C_0$ in (2), would require too formidable an expression for convenience. We therefore take several steps to simplify its exposition.

First, we restrict the name "once-forbidden" here to just the transitions having spin changes $\Delta I=0 \pm 1$, together with parity change. The $\Delta I=\pm 2$ transitions are well understood, as indicated in argument I of the introduction, and need no discussion here.

Second, we accept argument IV, so that we need consider as possible alternatives for the $\beta$-law only the four arbitrary combinations: $STP$, $SAP$, $VT$, or $VAP$.

Third, we separate the shape factor into three parts:

$$C_1 = C_1^{(0)} + C_1^{(1)} + C_1^{(2)}.$$  (4)

$C_1^{(0)}$ is to consist of the terms which contribute to $I_{\text{f}}=0 \rightarrow I_{\text{f}}=0$ transitions. Then the $S$ and $V$ forms do not contribute to $C_1^{(0)}$ at all, and only $C_1^{(0)}(AP)$ and $C_1^{(0)}(TP)$ will be needed [(12) and (13)]. $C_1^{(0)}$ is to consist of terms which contribute to $\Delta I=\pm 1$ but not to $\Delta I=\pm 2$ (or $0\rightarrow 0$) transitions. The $P$ interaction

\begin{itemize}
  \item[\textsuperscript{7}] M. Fierz, Z. Physik 104, 553 (1937). The largest contribution of the $P$ form is $G_\gamma^2 / P_{\beta}$; it requires a nuclear parity change and so is classed as once-forbidden, rather than allowed.
  \item[\textsuperscript{8}] C. Longmire and A. Messiah, Phys. Rev. 83, 464 (1951).
  \item[\textsuperscript{9}] A. Smith, Phys. Rev. 82, 953 (1951).
  \item[\textsuperscript{10}] D. Pursey, Phil. Mag. 42, 1193 (1951).
  \item[\textsuperscript{11}] Evidence for it is examined critically in Sec. 3.
\end{itemize}
makes no contribution to $C_1^{(1)}$ since its only matrix element is $\int \beta \gamma_s$, which vanishes unless $\Delta I=0$. The $C_1^{(2)}$ is the complete shape factor for the $\Delta I=\pm 2$ transitions which we have dismissed; however, $C_1^{(2)}$ also contributes to $\Delta I=0, \pm 1$ transition, although not to $0\rightarrow 0$, $1\rightarrow 0$ or $\frac{1}{2}\rightarrow \frac{1}{2}$ cases. As indicated in argument 1, $C_1^{(2)}$ arises from $T$ or $A$ interactions only.

Fourth, we introduce certain real parameters, $x_i, y_i, z_i$, which help simplify the notation for the nuclear matrix elements which occur. The latter are defined in references 5 and 6. Here we define $x_i$ by means of

$$
\int \beta \gamma = -ix_i(G_T/G_S) \int \beta \sigma \times r,
$$

$$
\int r = -ix_2(G_A/G_V) \int \sigma \times r,
$$

$$
\int \beta r = ix_3(G_A/G_S) \int \sigma \times r,
$$

$$
\int r = ix_4(G_T/G_V) \int \beta \sigma \times r.
$$

Next, the $y_i$ are defined by

$$
\int \beta \alpha = y_1(\alpha |Z|/2R) \int \beta \sigma \times r,
$$

$$
\int \alpha = y_2(\alpha |Z|/2R)(G_A/G_V) \int \sigma \times r,
$$

$$
\int \alpha = -y_4(\alpha |Z|/2R)(G_T/G_V) \int \beta \sigma \times r.
$$

Here, $Z$ is the nuclear charge and $R$ the nuclear radius in units of $h/mc$. Finally, the $z_i$ are defined by

$$
\int \gamma_s = iz_i(\alpha |Z|/2R) \int \sigma \times r,
$$

$$
\int \beta \gamma = -iz_2(\alpha |Z|/2R)(G_A/G_P) \int \sigma \times r,
$$

$$
\int \beta \gamma = iz_3(\alpha |Z|/2R)(G_T/G_P) \int \beta \sigma \times r.
$$

Attempts\textsuperscript{10,12} have been made to evaluate these matrix elements from simplified nuclear models. These estimates give $x_i, y_i, z_i$ the order of magnitude unity when all $G_x/G_x=1$ (except $z_3, z_4 \approx 0$, see below).

Fifth, we present the shape factor here only in the approximation $\langle \alpha Z \rangle^2<<1$. This is more than adequate for our purposes. We have actually used the precise expressions to be found in references 6, 9, and 10 for checking various of the smaller effects. The corresponding increase in accuracy is, however, largely illusory, because of uncertainties in nuclear matrix elements, in the nuclear radius, and in the finite nucleus and screening effects.

We are now ready to give $C_1^{(1)}$ for the $ST(P)$ and $VA(P)$ interactions. For both, one can use the single, positive-definite expression:

$$
C_1^{(1)} = G_{A(T)} \left| \int (\beta) \sigma \times r \right|^2 \left[ \frac{\alpha Z}{2R} (1+x_4-y_4) 
            + \frac{(1+x_1+q)}{3W} (1-x_3) \right] + \frac{(1+x_4+q)}{3W} (1-x_3)
            + \frac{(1-2x_2)^2}{18} (1+x_1+q)^2
            + \frac{(1-2x_2)^2}{18} (1+x_1+q)^2 \right].
$$

The upper sign applies to the $VA$ combination, the lower to $ST$. Further, one may change to the case of positron emission by changing the sign of $y_i$, $i=2$ and 1, respectively, for $VA$ and $ST$. $p$ and $q$ are electron and neutrino momenta, respectively, in units of $mc$. $q=W_S-W$ if $W_S$ is the spectrum end-point value of the electron energy $W$.

Expression (8) makes clear an important result: The main Coulomb effect on the electron, $\sim(\alpha Z/2R)^3$, is energy-independent. Dependence on energy here means on the kinetic energy with which the electron emerges, and naturally the Coulomb potential energy is independent of that.

As contrast, we immediately present $C_1^{(2)}$, which constitutes the entire shape factor for the $\Delta I=\pm 2$ transitions:

$$
C_1^{(2)} = G_{A(T)} \Sigma |B_{ij}|^2 (p^2+q^2)/12.
$$

$B_{ij}$ is another nuclear matrix element defined in reference 6. This expression yields the distinctive shapes referred to in argument 1 and arises only from the $T$ or $A$ forms of interaction. Here, the large possible angular momentum change $\Delta I$ makes the role of orbital angular momenta important. As a consequence, energy-dependent centrifugal effects dominate, rather than such Coulomb effects as were described in the preceding paragraph. The energy-dependent terms of (8) are largely centrifugal effects like those of (9).

Not all the constant terms of (8) arise from the Coulomb effect on the electron. The matrix elements...
$\mathcal{F}(\beta)\epsilon$ and $\mathcal{F}(\beta)\gamma_4$ of (6) and (7) contribute to the once-forbidden shape factors in the same way as do the $\mathcal{F}(\beta)\epsilon$ and $\mathcal{F}(\beta)\gamma_4$ to the allowed shape factor (2), i.e., with constant terms. However, as the estimates of (6) and (7) mentioned above indicate, here also the Coulomb energy is a measure of the size of the contribution. This results in the combination of Coulomb terms actually shown in (8). Hence, the size of all the energy-independent contributions to the shape factor can be said to be measured by the Coulomb energy at the nuclear radius, $aZ/R$.

The existence of the constant terms has particular significance in view of the fact that all the measured once-forbidden spectra are reported to have the statistical shape (except RaE, Sec. 5). Thus, the experiments appear to demand an energy-independent shape factor, $C_{1}$. One can now see that it is important for all the observed cases, that the Coulomb energy should dominate over the electron-neutrino kinetic energies: $aZ/R \gg W_0 - 1$. The uncertain parameters $x_i, y_i$ cannot take on values which would tend to cancel the Coulomb terms. The only alternative would be for the kinetic energy terms to preserve a precarious balance against each other. Such a balance should in principle vary from case to case and frequently fail, giving deviations such as have not so far been observed. It is fortunate for the theory that in all the cases for which the statistical shape is reported, the condition $aZ/R \gg W_0 - 1$ is fulfilled. This is shown in Table 1 in the cases in which the spectra are best measured (i.e., unobscured by concurrent transitions).

Supporting evidence for our interpretation that it is the dominance by the Coulomb terms $\sim (aZ/2R)^2$ which is responsible for the statistical shapes comes from the $\Delta I = \pm 1$ transitions. A notable fact about these distinctive transitions is their long comparative half-lives ($\langle \log f \rangle = 9$) as against those for $\Delta I = \pm 1$ ($\langle \log f \rangle = 7$). The main part of this difference must be due to the absence of the large Coulomb contribution $\sim (aZ/2R)^4$ in (9). This implies that if the Coulomb term were suppressed in (8), e.g., by adverse $x_i, y_i$ values, then the $\Delta I = \pm 1$ half-lives would become as long as the $\Delta I = \pm 2$ ones, contrary to what is actually observed.

Not only (8), but all the possible $\Delta I = 0, \pm 1$ shape factors contain the Coulomb terms $\sim (aZ/2R)^2$. For the $SA(P)$ and $VT(P)$ combinations, we write $C_{1}(1)$ keeping only these usually dominant terms:

$$C_{1}(SA) = G_d^2 \int \sigma \cdot r \left[ (aZ/2R)^2 \right] \left[ 1 + x z - 2 x z / W \right],$$

(10)

$$C_{1}(VT) = G_p^2 \int \beta \sigma \cdot r \left[ (aZ/2R)^2 \right] \left[ (1 - y)^2 \right] \left[ x z - 2(1 - y)(x - y) / W \right].$$

(11)

The striking fact about these expressions is that they predict deviations from the statistical shape even for large Coulomb energies. Yet we depend on the latter to account for the occurrence of statistical shapes among once-forbidden spectra.

The energy-dependent terms of (10) and (11) are $\sim 1/W$ and arise from the calculations in the same way as do the Fierz interference terms in the allowed shape factor $C_6$ of (2). Thus, an argument develops against the inclusion of $SA$ and $VT$ combinations in the $\beta$-law which is of the same kind as argument IV against $SV$ and $TA$ combinations.

There remains to be considered the shape factors $C_{1}^{(5)}$ which form a part of the total shape factor (4) for $\Delta I = 0$ transitions and constitute the entire shape factor for $\Delta I = 0$ transitions. For either the $(S)TP$ or $(V)TP$ forms of the $\beta$-law,

$$C_{1}(TP) = G_p^2 \int \beta \sigma \cdot r \left[ (aZ/2R)^2 \right] \left[ (1 - y)^2 \right] \left[ 1 + \left( p \lambda \right) / (3W) \right]^2 \left[ (p / 3W) \right] \left[ (p / 3W) \right].$$

(12)

Clearly, this leads again to the statistical shape in the usual case of $aZ/R > W_0 - 1$. However, the $(S)AP$ or $(V)AP$ forms yield

$$C_{1}(AP) = G_d^2 \int \sigma \cdot r \left[ (aZ/2R)^2 \right] \left[ (1 - y)^2 \right] \left[ x z + 2x(1 - y) / W \right].$$

(13)

when $aZ/R > W_0 - 1$. Here, the Fierz type of deviation occurs again.

The complete roster of possible once-forbidden shape factors, (8) to (13), has led to the following conclusions. The reported statistical shapes (energy-independent shape factors) seem best explainable, according to either an $STP$ or $VA$ form of $\beta$-law, as due to the normal dominance of the Coulomb energy. The $VT$ or $SA$ forms are probably incorrect because even for large Coulomb energies they, in general, predict deviations from the statistical shape, of a Fierz type.

Two considerations tend to keep this argument from being completely conclusive, at least for the present. These apply to some degree also to the widely accepted argument IV, which has a similar basis in allowed spectra. They are:

(a) The uncertain quantities $x_i, y_i, z_i$ may have such values, in all the cases observed so far, that the Fierz interferences would be unobservable.

(b) Although the spectra are reported to have sta-
istical shapes, the experiments might have been too inaccurate to detect Fierz deviations.

The adequacy of the experiments is discussed in detail in Sec. 3, for allowed spectra, and in Sec. 4, for once-forbidden spectra.

As regards point (a), one may point out at once that it is unlikely for the matrix elements, in all the cases observed, to take on just such values as would hide the effects of Fierz interference. In principle, the matrix elements may vary markedly, relative to each other, from case to case. Certainly, the objection (a) will become difficult to believe if the reports of statistical shapes of greater accuracy continue to accumulate.

For a less vague consideration of the objection (a), we may rely on the estimates of the nuclear matrix elements due to Pursey and to Ahrens and Feenberg. One has

\[ \kappa \approx 1 \text{ and } \lambda \approx 1 \]  

for the parameters in \( C_0 \), (2). This results immediately from the nonrelativistic approximation for the Dirac \( \beta (\approx -1) \). That approximation is invalid if \( \beta \) does not commute with other operators occurring in the matrix element.

We quote the estimates for the ratios involved in (5), (6), and (7) in terms of the quantities \( \xi_0, \eta_0, \xi_i \) defined exactly like \( x, y, z \), except that the \( G \)'s are omitted, i.e.,

\[ x \rightarrow \xi_0, \quad ye \rightarrow \eta_0, \quad xe \rightarrow \xi_i \]  

as \( G_x/G_x \rightarrow \infty \). (15)

Further, we restrict the quotations for \( \xi_0, \eta_0 \) to those applicable to the first three, better-defined cases in Table I:

\[ \xi_1 \approx \xi_2 \approx \xi_3 \approx \xi_4 \approx 1, \]  

\[ \eta_1 \approx \eta_2 \approx \eta_3 \approx \eta_4 \approx \Lambda, \]  

where \( \Lambda \) is a quantity defined by Ahrens and Feenberg. These authors find \( \Lambda \approx 1 \) for such cases as ours. On the other hand, Pursey gives \( \Lambda \approx 2 \). The discrepancy stems from different assumptions as to the nuclear model. The estimates (16) are probably more reliable than the others since they follow from kinematical considerations whereas \( \Lambda \) requires more presumptions about the nuclear Hamiltonian.

Further, Ahrens and Feenberg give

\[ \xi_1 \approx 1, \]  

while Feenberg and Primakoff estimate

\[ \xi_2 \approx \xi_3 \approx 0. \]  

being of the order of \( m/M \), the ratio of electron to nucleon mass. According to the last result, the \( P \) form will not contribute at all unless \( G_F \) has an unlikely high value.

If one accepts the estimates just reviewed, one sees that objection (a) does not hold for \( C_0 \) and for \( C_1^{(A)}(SA) \). Thus, sufficiently accurate experiments should reveal the presence of Fierz interference if the \( SV, TA, \) or \( SA \) combinations occur in the correct law. The apparent absence of an observable Fierz deviation confirms argument IV and, further, also our new argument insofar as it is directed against the \( SA \) combination.

The consequence of the estimates for the \( VT \) combination (11) depends on whether \( \Lambda \approx 1 \) or \( \Lambda \approx 2 \). If \( \Lambda \approx 1 \), not only is the Fierz interference in the \( C_1^{(A)}(VT) \) shape suppressed but all the Coulomb terms \( (\alpha Z/2R)^2 \).

Hence, a \( VT \) law would predict for \( \Delta I = \pm 1 \) transitions a distinctly nonstatistical shape very similar to the \( \Delta I = \pm 2 \) shape; moreover, the half-lives of the \( \Delta I = \pm 1 \) and \( \Delta I = \pm 2 \) transitions would be expected to be about the same. Thus, Feenberg's evaluation (\( \Lambda \approx 1 \)) would make the argument against the \( VT \) combination very strong.

On the other hand, Pursey's estimate of \( \Lambda \approx 2 \) would make the Fierz interference in the \( VT \) combination observable. If one accepts only the more reliable estimate \( \Lambda \approx 1 \) for the first three cases of Table I, then (11) becomes

\[ C_1^{(A)}(VT) = \left( \beta \right) \left( \lambda \right) \left( \alpha Z/2R \right)^2 (1 - \Lambda)^2 \times \left( G_x + G_\nu - 2G_\nu G_\nu /W \right). \]  

Thus, if the constant terms here are appreciable at all (\( \Lambda \approx 1 \)) then a Fierz deviation should be present unless \( G_F G_T \rightarrow 0 \).

Finally, accepting the estimates (19) as valid destroys our argument as directed against the \( AP \) combination (13). If the \( P \) form does not contribute at all, naturally its effects become unobservable and objection (a) holds. However, the \( RAE \) case, discussed in Sec. 5, may indicate that occasionally at least the estimate (19) fails.

On the basis of the estimates of the nuclear matrix elements, therefore, argument \( V \) can be successfully directed only against including \( VT \) and \( SA \) forms in the \( \beta \)-law. This, together with argument IV, leaves only the \( STP \) and \( VAP \) possibilities.

3. Fierz Interference in Allowed Spectra

The absence of Fierz deviations in allowed spectra is frequently quoted as a basis for argument IV, but no systematic examination seems to have been presented so far. We have special interest in the details because the sensitivity of current measurements to the Fierz effect also determines the effectiveness of argument V, which depends on the absence of the Fierz type of deviation in once-forbidden spectra [see objection (b) in Sec. 2].
The most convenient way to derive information from spectrum shapes is to examine their (Fermi-)Kurie plots. The ordinates of these are, in theory,

$$K_n = (W_0 - W) C_n^4. \quad (21)$$

When the shape factor \(C_n\) is independent of the energy \(W\), the spectrum has a statistical shape and the Kurie plot is linear \(~\sim (W_0 - W)\). Kurie plots of experimental points have an arbitrary ordinate scale, being proportional to the square root of the source intensity. This fact, together with the fact that the plot itself is used to determine \(W_0\), tends to make experimental Kurie plots insensitive to possible deviations from linearity. Nevertheless, the current accuracy of measurement does not seem to warrant more sensitive treatment.

If the Fierz type of effect should be present, then the Kurie plot should deviate from linearity, since it is now proportional to

$$K_n \sim (W_0 - W)(1 + 2\phi/W)^4, \quad (22)$$

where \(\phi\) is some number for each spectrum, given in theory by some one of the shape factors like (2), (10), or (11). The report that the spectra yield linear Kurie plots implies \(\phi = 0\), but some amount of deviation would have been undetected. Thus, we can only expect to set upper limits on \(|\phi|\) from the experiments. If these turn out to be substantially less than unity, their difference from zero will have been rendered unlikely. That conclusion has particular force when a small \(|\phi|\) implies some \(|G_F^2/G_X^2|\) is a small fraction. A theory with one Fermi constant much smaller than another strains credibility. The burden of proof is transferred to the point of view that the small component exists.

The observed spectra which have the best chance to reveal the presence of Fierz interference are those which are measured reliably down to a low electron energy. We therefore look at spectra which are not superposed with concurrent, lower energy, transitions. Table II seems to have the best measured examples.\(^{19-21}\)

To provide an equal basis for judging the consistency with experiment of the theories with \((\phi \neq 0)\) and without \((\phi = 0)\) Fierz interference, we proceed as follows. For each test, we chose a suitable (small) numerical value of \(\phi\). Then we compared the conventional experimental Kurie plot \((\phi = 0)\) with a plot of the same points divided by \((1 + 2\phi/W)^4\). The second type we refer to as a “Fierz plot.” The latter, rather than the conventional plot, should be linear if actually the \(\phi = 0\) is correct, as can be seen from (22). Of course, it turns out that both plots are nearly linear if \(|\phi|\) has been chosen small enough. We then located the straight lines which the experimental points of each plot ought to follow by using the points on the high energy side of the intensity maximum, the corresponding ones of the Kurie and of the Fierz plots. That part of every spectrum is always the most reliably measured. Finally we examined and evaluated the deviations from the constructed lines which are exhibited by the points not used in the constructions.

The conventional Kurie plots usually showed roughly random deviations from linearity. We took the root-mean-square of the percentage deviations, \(\epsilon\), as a measure of the experimental accuracy. Then we found in each case that values of \(\phi\) surpassing certain limits would yield Fierz plots which deviated systematically from linearity. We regarded the limiting \(\phi\)'s to have been reached when the systematic mean percentage deviation \(\delta\) in the Fierz plot just exceeded the random deviation \(\epsilon\) in the Kurie plot. This seemed quite a generous allowance for random error. Of course, we did not try infinite progressions of \(\phi\)'s. From a suitable enough initial choice it can be shown possible to perform a type of linear extrapolation to the limiting \(\phi\): We find \(\delta - \delta_0 \sim \phi\) if \(\delta_0\) is the percentage deviation in the Kurie plot.

![Figure 1](image-url)

**Cu**\(^{64}\) (\(\beta^-\))

Ten points with \(W > 1.5\) were used to determine the straight lines for the Kurie plot and two Fierz plots with \(\phi = \pm 0.2\) in

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Fig. 1. For the Kurie plot the remaining twelve points were randomly distributed above and below the straight line with a mean relative deviation of $\delta_b=0.1$ percent; the root mean square of the relative deviations was $\epsilon=0.9$ percent. As for the Fierz plots, the deviations were systematic, too low for $\varphi=+0.2$ and high for $\varphi=-0.2$, the mean value of their relative magnitudes amounting, respectively, to $\delta_+=+0.2$ percent ($\varphi=+0.2$) and $\delta_-=+3.0$ percent ($\varphi=-0.2$). From these we concluded that the values of $\varphi$ capable of giving Fierz deviations, undetectable within the accuracy of the measurements ($\epsilon=0.9$ percent), were limited by $-0.06<\varphi<0.08$.

A disturbing factor in the course of the reasoning here is that the deviation in the $\varphi<0$ Fierz plots resembles somewhat the effects of too thick a source being used in the measurement. Actually there is no serious danger that this is so; thick source effects tend to set in with distinctly greater abruptness than the theoretical deviations here. Furthermore, it is easy to put the whole question to rest by looking also at positron spectra, which are expected to have deviations of the opposite sign according to (2).

$\text{Cu}^{64} (3^+)$

The procedure used for the electron spectrum of Cu$^{64}$ gave similar results when applied to the positron spectrum. Here seven points with $W>1.6$ determined the straight lines, and eleven points with $W<1.6$ were used to measure the deviations. With the notation used for the previous case we found $\delta_+ = -3.9$ percent, $\delta_- = 3.1$ percent, $\epsilon = 1.1$ percent, $\delta_b = -0.7$ percent, with the consequent limit on unresolved Fierz deviations of $-0.1<\varphi<0.03$.

It now seems possible to say that $|\varphi| \leq 0.1$ for allowed transitions with $\Delta I = \pm 1$. In such cases, $C_0$ of (2) is simplified because $\int \beta' = \int \bar{\beta} = 0$. Then, theoretically,

$$\varphi = \pm \lambda G_A/G_T/(G_p^2 + \lambda G_s^2). \quad (23)$$

Hence, we can say $|\lambda G_A/G_T| \leq 0.1$ or $|G_T/G_A| \leq 0.1$. Considering that probably $\lambda = 1$, see (14), we have here the implication that one of the $G$'s is less than 0.1 of the other. This is about as closely as the present experiments can prove that not both $G_T$ and $G_A$ differ from zero. The significant limits are

$$G_A^2/G_T^2 < 1 \text{ percent or } G_T^2/G_A^2 < 1 \text{ percent.}$$

We hope to come to a similar conclusion about the pair $G_A, G_T$ from the $\Delta I = 0$ experiments.

$\text{N}^{12}$

The measurement of the deviations of thirteen points with $W<2.5$ from the straight line drawn through ten points with $W>2.5$ gave $\delta_+ = -3.6$ percent; $\delta_- = 4.3$ percent, $\epsilon = 1.2$ percent, $\delta_b = +0.5$ percent, and consequently $-0.04<\varphi<0.09$.

$\text{S}^{35}$

Six points with $W>1.16$ and nine points with $W<1.16$ were used giving $\delta_+ = -2.2$ percent, $\delta_- = 1.7$ percent, $\epsilon = 1.2$ percent, $\delta_b = -0.4$ percent, from which $-0.12<\varphi<0.12$.

The $\text{N}^{12}$ experiment seems good enough to conclude that $|\varphi| \leq 0.1$ for $\Delta I = 0$ transitions, also. Theoretically, see (2),

$$\varphi = \pm \lambda G_A/G_T \left[ G_p^2 + \lambda G_s^2 \right]$$

$$+ (\lambda^2) G_T(G_A)^2 \left( \frac{\int \bar{\beta}^2}{\int \beta^2} \right). \quad (24)$$

depending on whether the near vanishing of (23) is taken to imply that $G_T$ or $G_A$ is to be neglected. For $\text{N}^{12}$, $|\lambda G_A/G_T|^2/|\lambda G_T/G_A|^2 = 1/3$. The Fierz interference is expected to be less detectable when $\Delta I = 0$ than it was for $\Delta I = \pm 1$. Nevertheless, it is clear that $|G_T/G_A|$ or $|G_A/G_T|$ must be substantially less than unity and presumably zero.

4. Fierz-Type Interference in Once-Forbidden Spectra

We wish to compare argument V, against SA and VT combinations, with the argument IV, against SV and TA combinations, by showing that the once-forbidden spectra limit the amount of the Fierz-type deviation as much as do the allowed spectra.

We probably must forego directing our argument also against AP combinations. Not only does the estimate (19) of $\int \beta' \beta_\delta$ make it possible that $P$ effects are unobservable anyway, but also there seem to exist no suitable measurements to implement the argument for this case. Many spectra have been observed which are expected to have $\Delta I = 0$ (Zr$^{97}$, Ag$^{111}$, Cd$^{114}$, In$^{114}$, La$^{143}$, Dy$^{166}$, Er$^{149}$, Eu$^{157}$, Ce$^{144}$, Pr$^{145}$). However, all these are overlaid with concurrent transitions or other obscuring effects. The unobscured portion of each spectrum is reported to have the statistical shape, but the obscurations may have hidden considerable Fierz interference.

Table I lists all the once-forbidden spectra which seem to be well-measured down to low energies. Only for the first three cases does the shell model define the spin change fairly well, as $\Delta I = 1$. For these reasons we confined our examination to Pm$^{147}$, W$^{185}$, and Pr$^{145}$.

$\text{Pm}^{147}$

Figure 2 shows the conventional Kurie plots and also the Fierz plots for $\varphi = \pm 1/5$. For each, the points with $W>1.2$ were used to locate the presumptive straight lines. The deviations of the points with $W<1.2$ were random for the Kurie plot ($\delta_b = 0.2$ percent), having a root-mean-square percentage deviation, $\epsilon = 0.9$ percent. The $\varphi = \pm 1/5$ plot has a systematic mean percentage deviation, $\delta_+ = -1.9$ percent. This reduces to $-0.9$ percent if $\varphi = 0.1$. The $\varphi = \pm 1/5$ plot has a systematic mean deviation $\delta_+ = +4.5$ percent. That reduces to $+0.9$ percent if $\varphi = -0.03$. We conclude $-0.03<\varphi<0.1$.

$\text{W}^{185}$

The results here were much the same as for Pm$^{147}$. The points near the end point showed some scattering; hence, they were not

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of

relied on to determine the straight lines. It was necessary to use instead points more nearly at the middle of the spectrum. This naturally cut down the range in which the Fierz deviations could show themselves. In conclusion, it was then possible to say that $-0.09 \leq \phi \leq 0.14$.

**Pr**

The accuracy here was found to be about the same as for W. The result was $-0.13 \leq \phi \leq 0.09$.

Thus, with some indication of concurrence from W and Pr, the most stringent limits on the amount of Fierz interference is imposed by the particularly carefully measured Pm$^{147}$ spectrum. From it we can conclude that $|\phi| \leq 0.1$.

We interpret the result $|\phi| \leq 0.1$ by supposing, in turn, that the Fierz interference arises from the $S4$ combination as in (10), then from the VT combination, as in (11) or (20). According to (10) and (15),

$$|\phi| = \frac{|x_4|}{|1+2x_4^2|} = \frac{|\xi_4 G_4 G_A|}{|G_4^2 + \xi_4^2 G_A^2|}.$$  

We can put $\xi_4 = 1$ according to (16) without introducing more uncertainty than was done for argument IV when $x = \lambda = 1$ was supposed. Then $|\phi| \leq 0.1$ implies with the present interpretation:

$G_4^2/G_A^2 \leq 1$ percent or $G_A^2/G_4^2 \leq 1$ percent.

When the VT combination is being tested,

$$|\phi| = \frac{|1-\eta_i| (\xi_4 - \eta_i) G_4 G_T|/[(1-\eta_i)^2 G_T^2 + (\xi_4 - \eta_i)^2 G_T^2]|$$  

according to (11) and (15). When (16) is used,

$$|\phi| \approx \frac{|G_T G_T G_T|}{|G_T^2 + G_T^2|}.$$  

Hence, $|\phi| \leq 0.1$ makes

$G_T^2/G_4^2 \leq 1$ percent or $G_4^2/G_T^2 \leq 1$ percent.

One must remember, however, that the terms under consideration here are multiplied with $(1-\eta_i)^2$, which may nearly vanish. It has already been pointed out, in Sec. 2, how this would only strengthen the case against the VT combination.

We have thus carried out argument V to the same kind of result as IV. Ratios of the type $G_4^2/G_4^2$ were shown to be small enough for the presumption that they vanish to be likely. However, such a presumption is not necessary to complete argument V, as it was for argument IV. This is because we have an independent measure for the possible values of the ratios in question. One can show$^{24}$ that the comparative half-lives of $H^0$ (or the neutron) and $He^8$ can be consistent with each other only if

$$G_T^2/G_4^2 = 0.6 \text{ to } 2.4.$$  

Here $G_T$ may be $G_S$ or $G_V$, whereas $G_4$ is $G_T$ or $G_A$. The

able. Their presence might not be evident in deviations from the statistical shape because their effect is just cancelled by the energy-sensitive terms we have so far ignored. There is, of course, the immediate objection that it is unlikely for the statistical shape to depend on such accidental cancellations. We have nevertheless explored the possibility.

The efforts to save the \( S A \) and \( V T \) interactions, by supposing the constant shape factor to be a resultant of accidental cancellations of energy-sensitive terms, are easiest to describe for the \( S A \) case. This is because only the single uncertain parameter \( x_3 \) is involved. We did find that indeed \( x_3 \approx +0.08 \) yielded a more constant shape factor for \( Pm^{167} \) than did \( x_3 = 0 \) when a conventional value \( R = 1.54 \times 10^{-13} \) cm was used for the nuclear radius. However, \( x_3 = \xi G_{S}/G_{A} = 0.08 \) makes \( \varphi \leq 0.1 \), within the limits which were regarded as an argument against the \( S A \) combination above. Moreover, if one wanted to save the \( Pm^{167} \) shape in the same way, the larger energy-dependent terms here make it necessary to adopt an \( x_3 \) approximately 3 times as large as for \( Pm^{167} \). The similarity of the nuclear states involved in the two cases (see Table I) makes it implausible that \( \xi \) could be responsible for such a threefold variation. It does not seem likely that this method of obtaining the observed statistical shapes should be taken seriously.

Similar efforts to save the \( V T \) interaction met with similar objections. The greater number of uncertain parameters which is here involved makes the description of the details too lengthy to be given here.

5 CONCLUDING DISCUSSION

The arguments IV and V by themselves leave only the alternatives \( STP \) and \( VAP \) with which to reproduce the observed statistical shapes of the once-forbidden spectra. These forms are supposed to do that by way of the large constant Coulomb terms in the shape factors (8). The energy-sensitive terms of (8) are supposed to be negligible, at least within the accuracy of the observations. The uncertain factors in all these terms make it impossible to prove that they indeed are so negligible; all that can be done at present is to show that it is possible that they are.

A typical situation arises when a detailed effort is made to fit \( C_{i}^{(T)}(S) \) of (8) to \( Pm^{167} \). The only appreciable perturbation to the constant \((\alpha Z/2R)^{2}\) term is the addition to it of the terms

\[
\frac{\alpha Z}{3R} \left\{ \frac{G_{T} + \xi G_{S}}{G_{T}(1 - \lambda) + \xi G_{S}} \right\} + \frac{G_{T} - \xi G_{S}}{G_{T}(1 - \lambda) + \xi G_{S}} \right\}.
\]

If we take \( G_{T} = G_{S} \), \( \xi = 1 \), \( \lambda = 1 \), and \( R = 1.54 \times 10^{-13} \) cm, this causes \( \sim +3 \) percent mean deviation from the statistical shape. This deviation can be reversed if \( \lambda = 2 \) and \( G_{T} > G_{S} \) instead. Such conclusions were little changed when we used the more precise expressions of references 6 and 9, and when we considered finite nucleus and screening effects. One sees that there is little point in pursuing this study in view of the uncertainties involved. Probably such uncertainties will be best resolved when cases with \( W_{0} - 1 \rightarrow \alpha z/2R \) are found and measured. Certainly, deviative effects are better evaluated from definite deviations than through their apparent absence, as in the cases now available.

There is one case of definite deviation from the statistical shape which may be a once-forbidden transition: the singular \( RaE \) spectrum. This was analyzed by Petschek and Marshak, who concluded that the shape could be reproduced only by \( C_{i}^{(M)}(TP) \) and so must be a \( 0 \rightarrow 0 \) transition. Rather than formula (12) they used its more precise equivalent, to be found in references 6 and 9. In order to make the application successful, it was necessary to assume that

\[
\xi = \xi G_{P}/G_{T} = +1.
\]

The main objection against the Marshak-Petschek treatment is Ahrens, Feenberg, and Primakoff's estimate (19) that \( \xi \approx 0 \) and hence \( \xi \rightarrow 1 \) is unlikely. This objection does indeed make the \( RaE \) analysis; hence argument III, doubtful. On the other hand, the \( RaE \) analysis may perhaps be regarded as empirical evidence that at least for some nuclear states the estimate \( \xi \approx 0 \) may fail. The fact that the \( RaE \) case is the only one of its kind, even though deliberate searches have been directed toward finding similar cases, perhaps adds plausibility to the occurrence of the failure as a singular instance.

There is an additional, qualitative piece of evidence which supports the Marshak-Petschek result. \( \xi \approx 1 \) means that the deviation from a constant shape factor, in spite of \( \alpha Z/2R \approx (W_{0} - 1) \) as seen in Table I, is achieved through the approximate cancellation of the large term \( \sim (\alpha Z/2R)^{2} \) in (12). This has the consequence that the half-life of \( RaE \) is lengthened; a highly uncertain estimate is a lengthening by a factor of about \( 10^{3} \). Now, the study of comparative half-lives reveals that \( RaE \) does indeed have a life \( \sim 10^{3} \) times as long as that of otherwise similar nuclei. \( RaE \) has \( \log ft \approx 8 \), and involves in its transition an 83rd proton and a 127th neutron. The \( Pi^{290} \) transition also converts a 127th neutron into an 83rd proton. It has \( \log ft \approx 5.6 \). \( Hg^{298} \) and \( RaE'' \) involve 81st protons and 125th neutrons, one below magic numbers in each case. These nuclei have \( \log ft \approx 5.4 \) and 5.5, respectively.

The Marshak-Petschek result provides the argument III, which is the only one favoring the \( STP \) over the \( VAP \) combination. Without it, our new argument V leaves both the latter alternatives, having only eliminated the \( SAP \) and \( VTP \) forms.

Our results contradict the Critchfield-Wigner hypothesis.
Variations in the Relative Abundances of the Isotopes of Common Lead

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Mass spectrometer measurements of the relative isotopic abundances of samples of lead ores from Archean-type rocks showed larger variations than reported by Nier. The isotopic constitution of one lead was within the limits calculated by Holmes and by Bullard and Stanley. From the new measurements combined with existing data estimates of the time of formation of the earth’s crust of 3.5 billion years and of a maximum time of formation of the elements of 5.5 billion years have been calculated. These values are in reasonable agreement with previous estimates.

Nier reported the first, and still the only, thorough investigation of the variations of the isotopic constitution of ordinary and radiogenic leads. Several authors have since used Nier’s data to calculate the time of solidification of the earth’s crust (Holmes, 3.35 billion years, Bullard and Stanley, 3.29 billion years) and a limit on the time of formation of the elements (Alpher and Herman, 5.3 billion years). All these estimates have been based upon extrapolations from the abundances of ordinary leads ranging in age from 25 to 1400 million years. More analyses of ordinary leads dated at 2000 million years or older have been needed to check these estimates. We have therefore analyzed a number of common lead samples, dating as many as possible from Nier’s ages and from analyses of radiogenic leads in our own laboratory. Calculations of the age of the earth’s crust and of a maximum age of the elements have been made, using the new measurements listed in Table I combined with Nier’s data.

The experimental work was carried out with a 180° direction-focusing Nier-type mass spectrometer with a resolution of about 1/300 of an atomic mass unit. Analyses were made using lead tetramethyl as previously reported by the authors and by Dibeler and Mohler. The isotopic lead abundances listed in Table I are believed to be accurate to within 1 percent on an absolute basis, although comparative measurements are good to 0.1 percent.

Since Pb° is not believed to be of radiogenic origin, the total amount of this isotope is assumed to have remained constant from the time of formation of the elements to the present. The abundances of Pb°, Pb° and Pb°° are therefore given with respect to Pb°°.

A value of 5.5±0.2 billion years for the maximum limit on the time of formation of the elements has been calculated from the data given in Table I combined with Nier’s measurements of the isotopic constitution of common leads of known age. The method of calculation and the assumptions made were essentially the same as those of Alpher and Herman, and all values of constants were the same except the half-life of U°°°, which was taken as 7.07×10⁹ years.

Using Nier’s data, Alpher and Herman reported a figure of 5.3 billion years. The average abundances of the lead isotopes at different times have been listed in Table II.

The calculated relative number of atoms of U, Th, and Pb in the earth’s crust at the present time are U—1.00, Th—3.83, and Pb—7.35.

The isotopic abundances of those lead minerals (Table I) which could be assigned an age were combined with Nier’s measurements on other dated lead minerals and used to calculate a value for the age of the earth’s crust after the method of Bullard and Stanley. This method assumes that the isotopic constitution of lead