Proof of the TCP Theorem

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A comparatively simple proof is given for the general theorem that a wide
class of quantized field theories which are invariant under the proper Lorentz
group is also invariant with respect to the product of time reversal (T), charge
conjugation (C), and parity (P). In the proof use is made of an important sim-
plification introduced by Pauli.

I. DEFINITION OF THE OPERATIONS

1. INTERACTION REPRESENTATION

Since we want to present a proof of the theorem which is not restricted to
interactions without derivatives of the field operators, we employ the interaction
representation. In this representation the field operators obey equations of
motion without interaction and free field commutation relations. We list these
equations for the various types of fields we will be using.²

Spin 0:

\[
(\Box^2 - m^2)\phi(x) = 0, \quad (\Box^2 - m^2)\phi^*(x) = 0
\]

\[
[\phi^*(x), \phi(x')] = -i\Delta(x - x'), \quad [\phi(x), \phi(x')] = [\phi^*(x), \phi^*(x')] = 0;
\]

Spin 1:

\[
(\Box^2 - m^2)\phi_\mu(x) = 0, \quad (\Box^2 - m^2)\phi^*_\mu(x) = 0,
\]

\[
\partial^\mu\phi_\mu(x) = 0, \quad \partial^\mu\phi^*_\mu(x) = 0,
\]

\[
[\phi^*_\mu(x), \phi_\nu(x')] = -i\left(\gamma_\mu - \frac{1}{m^2}\partial_\nu\partial^\nu\right)\Delta(x - x'), \quad [\phi_\mu(x), \phi^*_\nu(x')] = [\phi^*_\mu(x), \phi_\nu(x')]
\]

\[
= [\phi^*_\mu(x), \phi^*_\nu(x')] = 0,
\]

¹ Fulbright and Smith-Mundt Grantee on leave of absence from Max-Planck-Institut
für Physik, Göttingen, Germany.

² We put \(\hbar = c = 1\) and use a metric where \(-g_{00} = g_{11} = g_{22} = g_{33} = 1\). It is \(x^0 = t,\)
\(\partial^\mu = \partial/\partial x^\mu,\) \(\partial^\nu = g^{\mu\nu}\partial_\nu,\) \(\Box^2 = \partial^\mu\partial_\mu.\) A star is used to denote the Hermitian adjoint of an
operator and the complex conjugate of a c number (including Dirac’s \(\gamma\) matrices). Especially
for \(4 \times 4\) matrices we use \(T\) to indicate the transposed and \(\dagger\) to indicate the Hermitian ad-
joint matrix.
Spin $\frac{1}{2}$:

\[
(\gamma_a \partial^a + m)\psi(x) = 0, \quad \bar{\psi}(x)(\gamma_a \partial^a - m) = 0
\]

(I.1.6)

\[
\{\psi_a(x), \bar{\psi}_b(x')\} = i(\gamma_a \partial^a - m)_{ab} \Delta(x - x'), \quad \{\psi_a(x), \psi_b(x')\} = 0
\]

(I.1.7)

where

\[
\bar{\psi}(x) = -i\psi^*(x)\gamma_0
\]

(I.1.8)

\[
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}
\]

(I.1.9)

\[
\gamma_k^\dagger = \gamma_k \quad (k = 1, 2, 3), \quad \gamma_0^\dagger = -\gamma_0.
\]

(I.1.10)

For the sake of simplicity we used the same symbol, $m$, for all masses and also used the same $\Delta$ function without indicating the mass. Equations (1) to (5) in the way they have been written hold for non-Hermitian fields (particles and antiparticles different); for Hermitian fields, i.e.,

\[
\psi(x) = \psi^*(x) \quad \text{or} \quad \varphi(x) = \varphi^*(x)
\]

(I.1.11)

one has to drop those commutation relations which vanish identically. The analog of a Hermitian field can also be constructed for a spin one-half field (Majorana field), but we shall not treat this case here. For our later considerations it will be important to realize that $\Delta(x)$ is a real function with the symmetry properties

\[
\Delta(r, t) = \Delta(-r, t) = -\Delta(r, -t) = -\Delta(-r, -t).
\]

(I.1.12)

The equation of motion for the state vector $|t\rangle$ is given by

\[
i |t\rangle' = H(t) |t\rangle
\]

(I.1.13)

where for local interactions $H(t)$ is an integral over the interaction density $\mathcal{G}(r, t)$

\[
H(t) = \int \mathcal{G}(r, t) d^3r.
\]

(I.1.14)

The operators $H(t)$, $\mathcal{G}(r, t)$ are Hermitian; for a relativistic field theory $\mathcal{G}(r, t)$ is the 00 component of a tensor (which essentially reduces to a scalar for interactions without derivatives). This tensor is to be constructed from $c$ numbers (essentially coupling constants) and the field operators. Ordinary tensor calculus will suffice for this purpose if we observe that spin one-half operators

3 From now on we shall mostly indicate space coordinates $r$ and time coordinate $t$ separately.

4 This result which is not too surprising has been proved in Appendix 2 of G. Lüders (1).
occur only in the form of the well-known covariant expressions

\[ i\bar{\psi}\gamma_{\mu}\psi, \quad i\bar{\psi}\gamma_{\mu}\gamma_{\nu}\psi, \quad i\bar{\psi}\gamma_{\mu}\gamma_{\nu}\gamma_{\lambda}\psi, \quad i\bar{\psi}\gamma_{\mu}\psi \]  

with

\[ \gamma_{5} = i\gamma_{0}\gamma_{1}\gamma_{2}\gamma_{3} \]  

Here, \( \bar{\psi} \) and \( \psi \) may denote different fields; further, instead of \( \psi \), we may use \( C\bar{\psi} \) and, instead of \( \bar{\psi} \), one can have \( \bar{\psi}C^{-1} \) with the \( 4 \times 4 \) matrix \( C \) defined by Eq. (I.3.4). The tensor in question can then be constructed by contraction of tensor indices and, possibly, introduction of the totally antisymmetric tensor \( \varepsilon_{\alpha\beta\gamma} \) with

\[ \varepsilon_{0123} = +1. \]  

We want to emphasize that we do not require invariance with respect to space reflections so that the prefix “pseudo-” has no meaning.

2. Parity

Under the parity operation (reflection in space) each physical situation is essentially reflected at an arbitrarily chosen space point; to simplify notation we choose this point as the origin. The field operators (which constitute the operator algebra) are defined mathematically by the free field equations and commutation relations (I.1.1) through (I.1.7). If the parity operation is to map the operator algebra into itself, it has to preserve these defining relations. The question of invariance of a theory with respect to reflections in space then reduces to the question of whether or not the interaction Hamiltonian \( H(t) \) is invariant. We define the parity (or reflection) operator by

\[ P\Phi(r, t)P^{-1} = \eta_{\nu}\Phi(-r, t), \quad P\Phi^{*}(r, t)P^{-1} = \eta_{\nu}^{*}\Phi^{*}(-r, t), \]  

\[ P\Phi_{\alpha}(r, t)P^{-1} = -\eta_{\nu}'\Phi_{\alpha}(-r, t), \quad P\Phi^{*}_{\alpha}(r, t)P^{-1} = -\eta_{\nu}'^{*}\Phi^{*}_{\alpha}(-r, t), \]  

\[ P\Phi_{\alpha}(r, t)P^{-1} = \eta_{\nu}\Phi_{\alpha}(-r, t), \quad P\Phi^{*}_{\alpha}(r, t)P^{-1} = \eta_{\nu}^{*}\Phi^{*}_{\alpha}(-r, t), \]  

\[ P\Phi_{\mu}(r, t)P^{-1} = \eta_{\nu}\gamma_{0}\Phi_{\mu}(r, t), \quad P\Phi^{*}_{\mu}(r, t)P^{-1} = -\eta_{\nu}^{*}\Phi^{*}_{\mu}(-r, t). \]  

The \( c \) numbers \( \eta_{\nu}', \eta_{\nu}'' \) associated with different fields need not be the same. These definitions have been made so that the Hermitian adjointness of operators is preserved. This means that \( P \) can be chosen as a unitary operator

\[ PP^{*} = P^{*}P = 1 \]  

which in turn has the consequence that normalization and orthogonality of

\( \Phi_{\mu} \) denotes the space part of the four vector \( \varphi_{\mu} \).
state vectors are preserved. The definition is made unique by postulating

$$ P \mid 0 \rangle = \mid 0 \rangle $$

for the free field vacuum \( \mid 0 \rangle \); the proof that this is possible shall be omitted.\(^6\)

Equations (1) through (3) need some further explanations. It is easily seen for spin zero and one that the free field equations (I.1.1), (I.1.3), (I.1.4), (I.1.6) are unchanged. For spin one-half use has to be made of Eq. (I.1.9). To check the preservation of the commutation relations (I.1.2), (I.1.5), (I.1.7), one has to employ the first symmetry property (I.1.12) of the \( \Delta \) function; for spin one half again properties of the \( \gamma \) matrices have to be used. One gets the result that the commutation relations are unchanged if the \( \epsilon \) numbers \( \eta_p, \eta'_p, \eta''_p \) have modulus one; for Hermitian fields \( \eta_p \) and \( \eta'_p \) can take on only the real values \( \pm 1 \). The theory is invariant with respect to space reflection if the factors \( \eta_p, \eta'_p, \eta''_p \) can be chosen so that

$$ P \psi(r, t) P^{-1} = \psi(-r, t) $$

i.e.

$$ P \phi(t) P^{-1} = \phi(t). $$

3. CHARGE CONJUGATION

Under charge conjugation each state is mapped into one where all particles are replaced by their antiparticles, the other properties of the state being essentially unchanged. Mathematically this means that all field operators are transformed into the Hermitian adjoint operators (and Hermitian fields, of course, into themselves). To preserve the free field equations and commutation relations we define the operator \( C \) in the following way

$$ C \phi(r, t) C^{-1} = \eta c \phi^*(r, t), \quad C \phi^*(r, t) C^{-1} = \eta c^* \phi(r, t), \quad (I.3.1) $$

$$ C \phi_{\mu}(r, t) C^{-1} = \eta c' \phi_{\mu}^*(r, t), \quad C \phi_{\mu}^*(r, t) C^{-1} = \eta c' \phi_{\mu}(r, t), \quad (I.3.2) $$

$$ C \psi(r, t) C^{-1} = \eta c'' \psi(r, t), \quad C \psi^*(r, t) C^{-1} = -\eta c'' C^4 \psi(r, t). \quad (I.3.3) $$

Here \( \eta_c, \eta_{c'} \) have to have modulus one (commutation relations!) and are restricted to \( \pm 1 \) for Hermitian fields. The \( 4 \times 4 \) matrix \( C \) is defined as a unitary matrix with the property

$$ C \gamma_{\mu}^T C^{-1} = -\gamma_{\mu}. \quad (I.3.4) $$

\* The remarks in this paper about the action of the various operations on the vacuum are based upon the observation that the free field vacuum by definition has the property that one obtains zero if the positive frequency part of any field operator acts on it. So one has in all cases only to check that the operation in question maps the positive frequency part of field operators again into positive frequency parts.
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C can be shown to be antisymmetric

\[ C^T = -C. \]  \hspace{1cm} (I.3.5)

Also \( \eta_\nu'' \) has to have modulus one. The matrix \( C \) is needed to conserve field equations and commutation relations. For checking the commutation relations use has to be made of the symmetry relation \( (I.1.12) \) (first and last term).

As in the case of the parity operator also here we can postulate that \( C \) is unitary and that the vacuum has eigenvalue +1.

When applying charge conjugation to the Hamiltonian in order to check its invariance one encounters the characteristic difficulty that products of operators are transformed into those of the Hermitian adjoint operators but that the factors appear in an order which is different from the one required by the Hermiticity of \( \mathcal{H}(r, t) \). In order that the transformed \( \mathcal{H}(t) \) be comparable with the original one and possibly equal to it (i.e., invariant under charge conjugation), one has to assume from the very beginning that a process of proper symmetrization has been applied to \( \mathcal{H}(r, t) \). We shall come back to this point later (first postulate in Section II.1).

4. Time Reversal

Under time reversal (in the Wigner sense) each state of a physical system at time \( t \) is mapped into a state at time \(-t\); in a particle picture this state differs from the original one in that the velocities of all particles are reversed. Field operators at time \( t \) are essentially transformed into those at time \(-t\).

\[
\begin{align*}
T\varphi(r, t)T^{-1} &= \eta_T \varphi(r, -t), & T\varphi^*(r, t)T^{-1} &= \eta_T^* \varphi^*(r, -t), \\
T\varphi_k(r, t)T^{-1} &= \eta_T \varphi_k(r, -t), & T\varphi_k^*(r, t)T^{-1} &= \eta_T^* \varphi_k^*(r, -t), \\
T\psi(r, t)T^{-1} &= -\eta_T'' \psi(r, -t), & T\psi^*(r, t)T^{-1} &= -\eta_T''^* \psi^*(r, -t), \\
T\xi(r, t)T^{-1} &= \eta_T'' T\xi(r, -t), & T\xi^*(r, t)T^{-1} &= \eta_T''^* T\xi^*(r, -t). \tag{I.4.3}
\end{align*}
\]

The \( 4 \times 4 \) matrix \( T \) is a unitary matrix defined by

\[ T^{-1}\gamma_\mu T = \gamma_\mu \] \hspace{1cm} (I.4.4)

from which it follows that \( T \) is symmetric

\[ T^T = T. \] \hspace{1cm} (I.4.5)

Let us first analyze the behavior of spin zero and one fields. The equations of motion are easily seen to be preserved. But at first sight it appears as if the commutation relations would not stay unchanged since, according to Eq. \( (I.1.12) \), the \( \Delta \) function changes sign under the substitution \( t \to -t \). But this only means that \( T \) cannot be a linear operator as \( P \) and \( C \) were. Since the right-
hand sides of the commutation relations are purely imaginary (Δ is a real function), one only has to define T as an antilinear operator. Such an operator especially has the property that

$$T\lambda T^{-1} = \lambda^*$$

(1.4.6)

for any c number λ. Again one has to postulate that η and η’ have modulus one. The equations of motion for spin zero and spin one still hold since both the d’Alambertian □ and the mass m are real quantities. To check the equations of motion and anticommutation relations for spin one-half fields under the operation of the antilinear T use has to be made of Eq. (1.4.4); also η’’ has to have modulus one. T maps the free field vacuum into itself; because of the antilinearity it is not reasonable to postulate a particular eigenvalue in this case.

We shall call the theory invariant with respect to time reversal if the quantities η, η’, η’’ can be chosen so that

$$T\h \Phi(r, t)T^{-1} = \Phi(r, -t), \quad TH(t)T^{-1} = H(-t).$$

(1.4.7)

Applying the antilinear operator T on Eq. (1.1.13) one then finds

$$-i(T | t))' = H(-t)(T | t))$$

(1.4.8)

or

$$i(T | -t))' = H(t)(T | - t))$$

(1.4.9)

so that T | -t) behaves like a state vector at time +t. This indicates the action of time reversal on time-dependent processes if H is invariant. Here again the antilinearity of T is important.

Combining Eqs. (1.3.3) and (1.4.4) we notice that

$$\omega T - 1 = \omega T$$

which shows that we may put

$$CT = \gamma_\delta.$$  

(1.4.11)

This relation will be used later, see Section II.3.

II. PROOF OF THE THEOREM

Now we are in a position to prove the theorem that a wide class of field theories invariant under the proper Lorentz group is also invariant with

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7 The mathematics of antilinear operators is summarized in the Appendix to this paper.

8 The theorem has not been stated explicitly in the papers by J. Schwinger (2). But Schwinger appears to have assumed the validity of essentially this theorem and to have derived from it the connection between spin and statistics. Later proofs and discussions of the theorem proceeded in the opposite direction assuming the usual connection between spin and statistics and then deriving the theorem. In the paper by G. Lüders (1), the the-
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respect to the product TCP,9 i.e., that the $c$ numbers $\eta_r$, $\eta_c$, $\eta_p$, etc., can always be chosen in such a way that one has invariance.10 A few further restrictions on the theories will be needed. They will be given at those places where they are required for the proof; they will be summarized at the end of Section II.3.

The proof of the theorem proceeds by explicitly constructing an operation under which the theory is invariant and then showing that this operation is identical with the product TCP. The construction of this operation shall be done in two steps, the strong reflection as proposed by Pauli and subsequent Hermitian conjugation. As we shall see both of these operations can be regarded as mappings of the operator algebra into itself which leave the interaction density invariant apart from changing the space-time coordinates. Since both mappings reverse the order of operators in products,11 they cannot be applied separately upon the Hilbert space, but the result of the consecutive application of both operations is meaningful both for the operator algebra and for the underlying Hilbert space. In fact, it can be achieved by transformation with an antilinear operator, i.e., TCP.

1. Strong Reflection12

This operation essentially consists in a reflection of space and time about some arbitrarily chosen origin, i.e.,

$$r \rightarrow -r, \quad t \rightarrow -t. \quad (11.1.1)$$

It is to be accompanied by a transformation of the field quantities which shall be given in detail presently. Further, it will turn out that an additional prescription regarding the order of factors in products is needed. If space-time had Euclidian metric (and if we disregard questions of noncommutativity of operators), transformation (1) could be achieved by a rotation since the number of

orem is proved in the form that a relativistic field theory describing particles of spin zero, one-half, and one, invariant with respect to $\mathbf{P}$, is also invariant under CT. The first vague conjecture of the existence of some theorem of this kind was stated by Lüders (9) in 1952. The correct formulation of the theorem was suggested to the present author in early 1953 by B. Zumino. I should like to emphasize on this occasion the important role which Zumino played at all stages of my work which led to the final proof of the theorem. Note added in proof: the theorem has also been discussed and proved by J. S. Bell, Proc. Roy. Soc. A231, 79 (1955).

9 It might be worth while emphasizing that the theorem does not claim (which would be entirely incorrect) that the product TCP is equal or equivalent to the unit operator.

10 The theorem has been applied recently by various authors (5) in the discussion of the violation of some conservation laws in weak interactions.

11 Mappings of an algebra into itself which reverse the order of factors in products are, of course, as legitimate mathematically as those which preserve that order.

12 Both this operation and its name have been proposed by W. Pauli (4). It has been applied before by the same author in a paper on the connection between spin and statistics (6).
dimensions of space-time is even. Invariance with respect to some rotations has already been required by the postulate of invariance under the proper Lorentz group. So it appears very plausible that such a theory is indeed invariant under transformation (1). In fact, because of the indefinite metric of relativity the proper Lorentz group contains all possible rotations, and the strong reflection cannot be produced by a rotation. A closer examination will show that the situation is not quite so simple also in other respects and that especially the connection between spin and statistics will play an important role.

First we define the behavior of the field operators under strong reflection

$$\phi(r, t) \rightarrow \phi(-r, -t), \quad \phi^*(r, t) \rightarrow \phi^*(-r, -t), \quad (II.1.2)$$

$$\phi_\mu(r, t) \rightarrow -\phi_\mu(-r, -t), \quad \phi_\mu^*(r, t) \rightarrow -\phi_\mu^*(-r, -t) \quad (II.1.3)$$

$$\psi(r, t) \rightarrow i\gamma_5\psi(-r, -t), \quad \bar{\psi}(r, t) \rightarrow i\bar{\psi}(-r, -t)\gamma_5. \quad (II.1.4)$$

(These are exactly the transformations which one would have if strong reflection could be obtained by a rotation.) The invariance of the field equations (I.1.1), (I.1.3), (I.1.4), (I.1.6) is checked rather easily. But when analyzing the commutation relations (I.1.2) and (I.1.5) for the Bose fields, we find that they are not invariant as a consequence of the change of sign (I.1.12) of $\Delta(x)$ under the substitution $x_\mu \rightarrow -x_\mu$. Rather the two sides of the commutation relation differ by a sign after the substitution. When defining time reversal in Section I.4 we encountered with a similar situation. There we took advantage of the occurrence of the imaginary unit in the commutation relations and defined time reversal as an antilinear operator. This time we resolve the difficulty in a different manner, postulating that strong reflection shall produce a mapping of the operator algebra into itself which instead of conserving the order of factors in products reverses it. This postulate does not affect the anticommutation relations (I.1.7) for spin one-half fields. Indeed, their invariance is easily verified.

So far we have only defined the operation of strong reflection in a mathematically consistent way. We still have to show that a suitably specified relativistic field theory is invariant with respect to this operation; i.e., we have to show that

$$\mathcal{H}(r, t) \rightarrow \mathcal{H}(-r, -t) \quad (II.1.5)$$

if strong reflection is applied. This, of course, means that we have to apply the substitutions (2) through (4) and to reverse all factors in products. In order to make it possible to achieve this invariance, we introduce the additional postu-
late: (7) In the Hamiltonian all products are symmetrized with respect to Bose fields and antisymmetrized with respect to Fermi fields.

For the examination of the interaction density $\Phi(x, t)$ we make use of the observations on the structure of this quantity as presented at the end of Section I.1. If we disregard for a moment spin one-half fields we see that, as a consequence of the defining equations (2) and (3), each tensor of even (odd) rank takes up a plus (minus) sign under strong reflection. This is also true for tensors which have been derived from the original field operators by means of the four-dimensional gradient. The $\epsilon$ tensor as given in Eq. (I.1.17) remains, of course, unchanged; this is in agreement with the general behavior of tensors since $\epsilon_{\mu\nu\rho\sigma}$ has rank four. Contraction of tensor indices changes the rank by an even number; therefore, the rule determining the change of sign of a tensor under strong reflection is not violated by a contraction. If there were no spin one-half fields, the interaction density would stay unchanged under strong reflection since it is the 00 component of a tensor of second rank.

We now have to look into the transformation of scalars, vectors, and tensors which can be formed from spin one-half operators by means of the $\gamma$ matrices [Eq. (I.1.15)]. If their behavior fits into the general rule regarding change of signs, the invariance of the theory (i.e., of the interaction density, Eq. (5)) would indeed hold. But if one now checks e.g. the transformation of the scalars $\overline{\psi}\psi$ and $i\overline{\gamma}\psi$, one finds that they would take up an unwanted minus sign if these expressions were not really interpreted as antisymmetrized with respect to $\psi$ and $\overline{\psi}$ and if strong reflection would not change the order of factors. So it is indeed the usual connection between spin and statistics which leads to the correct transformation properties of the bilinear quantities under strong reflection. To achieve these properties also if $\overline{\psi}$ and $\psi$ represent different fields, we add the further postulate: Kinematically independent Fermi fields anticommute.

It was mentioned at the end of Section I.1 that in bilinear expressions of the kind just considered $\psi$ could be substituted by $C\overline{\psi}$ without violating the correct behavior under the proper Lorentz group. The imaginary factor was introduced in Eq. (4) to insure the desired behavior of such quantities with respect to strong reflections.

\[ A \text{ postulate of this type was already required for the discussion of the operation of charge conjugation, see end of Section I.3. So it is not surprising that it also occurs in the proof of the theorem.} \]

\[ B. \text{ Zumino recently reminded me of the fact that Wick's } \cdot \cdot \text{ product would have the same effect on the results.} \]

\[ C \text{ This mathematical observation expresses the fact that for a Euclidian metric strong reflection could be achieved by a rotation.} \]

\[ D \text{ Both postulates are required if one wants the theorem to hold generally without additional restrictions on the type of coupling. In particular cases, e.g., beta-type couplings between four different Dirac fields, the postulates might become partly or completely unimportant.} \]
As indicated in the beginning of Chapter II, strong reflection is defined only for the operator algebra, but cannot be applied upon Hilbert space. Therefore, we have to introduce a second mapping of the operator algebra into itself which also reverses products and which also leaves the interaction Hamiltonian invariant. The result of the consecutive application of both operations is then defined also for the Hilbert space.

2. **Hermitian Conjugation**

Under Hermitian conjugation

\[ \varphi(-r, -t) \rightarrow \varphi^*(-r, -t), \quad \varphi_n(-r, -t) \rightarrow \varphi_n^*(-r, -t), \quad \text{(II.2.1)} \]

\[ \psi(-r, -t) \rightarrow \psi^*(-r, -t) = -i \overline{\psi}(-r, -t) \gamma_0; \quad \text{(II.2.2)} \]

Further, \( c \) numbers go over into the complex conjugate and the order of factors in products is again reversed. Field equations and commutation relations are invariant and also the interaction density stays unchanged since it is a Hermitian operator.\(^{19}\)

If we first apply strong reflection (Section 1) and afterwards Hermitian conjugation (this Section 2), we end up with the following substitutions:

\[ \varphi(r, t) \rightarrow \varphi^*(-r, -t), \quad \varphi^*(r, t) \rightarrow \varphi(-r, -t), \quad \text{(II.2.4)} \]

\[ \varphi_n(r, t) \rightarrow -\varphi_n^*(-r, -t), \quad \varphi_n^*(r, t) \rightarrow -\varphi_n(-r, -t), \quad \text{(II.2.5)} \]

\[ \psi(r, t) \rightarrow \overline{\psi}(-r, -t) \gamma_0 \gamma_0, \quad \overline{\psi}(r, t) \rightarrow \gamma_0 \gamma_0 \overline{\psi}(-r, -t). \quad \text{(II.2.6)} \]

All factors are left in their proper order. The total transformation is antilinear since all \( c \) numbers are to be conjugated. Field equations and commutation re-

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\(^{18}\) The various authors apparently do not agree about the necessity of this second step.

\(^{19}\) The Hermiticity of the Hamiltonian is of importance for the discussion of both charge conjugation and time reversal separately. From the requirement of Hermiticity, it follows that in the Hamiltonian each product of field operators occurs with a product of the Hermitian adjoint operators; charge conjugation essentially transforms just such products into each other. Hermiticity also determines reality conditions on the coupling constants; they are essential for determining the action of an antilinear operator like that of time reversal. So it is almost clear that a Hermitian Hamiltonian should be required also for the validity of the TCP theorem and so has to enter at some place into the proof of that theorem.

\(^{20}\) We interpret remarks by Pauli (in footnotes to his paper (4)) on the role of this requirement in the following way: Pauli apparently understands Hermiticity as adjointness with respect to a positive definite metric in Hilbert space. All our conclusions would also hold if the asterisk denoted adjointness with respect to some not necessarily definite metric. The definiteness is required by the interpretation of the mathematical scheme in terms of the probability of results of observations on the system.
lations stay unchanged under this combined operation; so the combined substitution does not lead to mathematical inconsistencies. Further, for the interaction density one has the result (II.1.5). This means that the theory is indeed invariant with respect to the total substitution. Since the position of operators is unchanged, this operation can also be applied upon the Hilbert space. It maps the free field vacuum into itself.

3. Analysis of the Result

The combined substitution defined by (II.2.4) through (II.2.6) and the prescription of conjugation of c numbers leaves invariant relativistic field theories in the sense specified in this paper. These were theories with local interaction invariant with respect to the proper Lorentz group. They had to fulfill the two postulates formulated in Section II.1. One easily recognizes that the substitutions given at the end of the last section indeed have the characteristics of the product TCP.

1. \( t \to -t \), c numbers conjugated \((T)\)

2. field operators into Hermitian adjoint operators \((C)\)

3. \( r \to -r \) \( (P) \).

It is easily seen that for spin zero and one the above operation really is the product TCP if

\[
\eta_T \eta_C \eta_P = 1, \quad \eta_{T'} \eta_{C'} \eta_{P'} = 1. \tag{II.3.1}
\]

This is exactly true if first \( P \), then \( C \) and then \( T \) is applied; for a different order of operations Eq. (II.3.1) would be slightly modified. It might be worthwhile to check more explicitly the product TCP for spin one-half fields. Omitting the space-time coordinates, the various transformations act as follows:

\[
\begin{align*}
\bar{\psi} & \xrightarrow{P} - \eta_p^{\gamma_0} \gamma_0 \bar{\psi} \xrightarrow{C} - \eta_c^{\gamma_0} \gamma_0 \gamma_0^{\gamma_0} C \gamma_0^{\gamma_0} \psi \xrightarrow{T} - \eta_T^{\gamma_0} \gamma_0^{\gamma_0} C T \psi.
\end{align*} \tag{II.3.2}
\]

Making use of Eqs. (I.1.10) and (I.4.11) and comparing the result with (II.2.6) one sees that strong reflection and Hermitian conjugation together are indeed equal to the operation TCP if

\[
\eta_T^{\gamma_0} \gamma_0^{\gamma_0} \gamma_0^{\gamma_0} C T \psi. \tag{II.3.3}
\]

If the operations \( T, C, P \) are applied in a different order, this condition is again somewhat modified.

This observation completes the proof that a quantized field theory constructed from fields of spin zero, one-half, and one by local interactions which are invariant under the proper Lorentz group is invariant also with respect to the product of time reversal, charge conjugation, and space reflection if the connec-
tion between spin and statistics is the usual one (with kinematically independent Dirac fields anticommuting) and if products occurring in the interaction Hamiltonian have been symmetrized and antisymmetrized in the proper way.\textsuperscript{21} Pauli has given a more general proof that also theories of the same general character containing fields of higher spin are invariant with respect to strong reflection; his proof uses the theory of finite representations of the proper Lorentz group.

\textbf{APPENDIX}

\textbf{Antilinear Operators in Hilbert Space}

An antilinear operator $A$ is defined by\textsuperscript{22}

\begin{align}
A\psi_1 + A\psi_2 &= A\psi_1 + A\psi_2, \\
A\lambda \psi &= \lambda^* A \psi
\end{align}

or

\begin{equation}
A\lambda = \lambda^* A. \tag{A.2a}
\end{equation}

$\psi_1, \psi_2, \psi$ denote vectors in Hilbert space; $\lambda$ is a number, $\lambda^*$ its complex conjugate. Equation (2) characterizes the operator as antilinear. From the definition it follows that a product of any number of linear operators and an even (odd) number of antilinear operators is itself a linear (antilinear) operator.

If $A$ induces a one-to-one mapping of the Hilbert space into itself, i.e., if

\begin{equation}
A\psi = 0 \tag{A.3}
\end{equation}

is true only for

\begin{equation}
\psi = 0 \tag{A.4}
\end{equation}

then the inverse operator $A^{-1}$ is defined uniquely (this holds in a strict sense only for vector spaces of finite number of dimensions). The inverse has the well-known properties

\begin{equation}
AA^{-1} = A^{-1}A = 1. \tag{A.5}
\end{equation}

\textsuperscript{21} In the interaction representation TCP transforms the interaction Hamiltonian $H(t)$ into $H(-t)$; in the Schrödinger representation it simply commutes with the total Hamiltonian. The Hamiltonian of the Schrödinger representation is the sum of the interaction representation at $t = 0$ and the free field Hamiltonian. TCP commutes with $H(0)$. It also commutes with the free field Hamiltonian since this operator is represented by a space integral over the 00-component of a tensor at $t = 0$.

\textsuperscript{22} Dirac's notation for Hilbert vectors, scalar products in Hilbert space, etc., is not too well suited for antilinear operators. Therefore, we use a different notation in this Appendix. Note: $\langle \psi, \lambda \phi \rangle = \lambda \langle \psi, \phi \rangle$ and not $\lambda^* \langle \psi, \phi \rangle$ for any number $\lambda$. 
A is an antilinear operator: Multiply
\[ \lambda = \lambda \mathbf{A} \mathbf{A}^{-1} = \mathbf{A} \lambda^* \mathbf{A}^{-1} \] (A.6)
from the left by \( \mathbf{A}^{-1} \); this gives
\[ \mathbf{A}^{-1} \lambda = \lambda^* \mathbf{A}^{-1} \]. (A.7)
This proves only "anti-"; the proof of "-linear" proceeds as usual.

The anti-Hermitian adjoint \( \mathbf{A}^* \) of an antilinear operator \( \mathbf{A} \) is defined by postulating
\[ (\psi, \mathbf{A} \varphi) = (\varphi, \mathbf{A}^* \psi) \] (A.8)
to hold identically in \( \psi \) and \( \varphi \). Notice the difference compared with the definition of Hermitian adjoints of usual linear operators. A proof of the unique existence can be given in a strict sense only for vector spaces of finite dimension. For fixed \( \psi \), the left-hand side of (8) is antilinear in \( \varphi \), i.e., if we write
\[ (\psi, \mathbf{A} \varphi) = (\varphi, \mathbf{A}^* \psi) \] (A.9)
we have
\[ A_\psi(\varphi_1 + \varphi_2) = A_\psi(\varphi_1) + A_\psi(\varphi_2), \quad A_\psi(\lambda \varphi) = \lambda^* A_\psi(\varphi). \] (A.10)
Consequently, for fixed \( \psi \) there is a uniquely defined vector \( \alpha_\psi \) so that
\[ A_\psi(\varphi) = (\varphi, \alpha_\psi). \] (A.11)
Varying now \( \varphi \), one sees that there exists a uniquely defined antilinear (!) operator \( \mathbf{A}^* \) so that
\[ \alpha_\psi = \mathbf{A}^* \psi. \] (A.12)
This completes the proof and, furthermore, shows that also \( \mathbf{A}^* \) is antilinear. From the definition (3), it follows that one has
\[ (\mathbf{A}^*)^* = \mathbf{A}. \] (A.13)

For an antilinear operator \( \mathbf{L} \mathbf{A} \) which is the product of a linear operator \( \mathbf{L} \) and an antilinear operator \( \mathbf{A} \), one has
\[ (\mathbf{L} \mathbf{A})^* = \mathbf{A}^* \mathbf{L}^* \] (A.14)
where \( \mathbf{A}^* \) is the anti-Hermitian adjoint of \( \mathbf{A} \) and \( \mathbf{L}^* \) the ordinary Hermitian adjoint of \( \mathbf{L} \). Proof: \( (\mathbf{L} \mathbf{A})^* \) is defined by
\[ (\psi, \mathbf{L} \mathbf{A} \varphi) = (\varphi, (\mathbf{L} \mathbf{A})^* \psi) \] (A.15)
identically in $\psi$ and $\varphi$. The derivation of (14) then goes through the following steps:

$$ (\psi, L A \varphi) = (L^* \psi, A \varphi) = (\varphi, A^* L^* \psi). \quad (A.16) $$

Similarly one shows that if $A_1$ and $A_2$ are both antilinear operators, the ordinary Hermitian adjoint of the product is given by

$$ (A_1 A_2)^* = A_2^* A_1^*. \quad (A.17) $$

Similar formulas hold for any number of factors.

The concept of Hermitian and unitary linear operators can easily be generalized to antilinear operators. An antilinear operator is called anti-Hermitian if

$$ A^* = A \quad (A.18) $$

and antiunitary if

$$ A^* = A^{-1}. \quad (A.19) $$

The operator $T$ and the product $TCP$ are both antiunitary operators in this sense. The proof of this assertion goes as follows. We start with the defining relations, e.g.,

$$ T \varphi(r, t) T^{-1} = \eta \varphi(r, -t), \quad T \varphi^*(r, t) T^{-1} = \eta^* \varphi^*(r, -t). \quad (A.20) $$

Without questioning the mathematical existence of the operator $T$, we want to show that

$$ T^* T = 1. \quad (A.21) $$

Let us form the adjoint to the second Eq. (20); applying some generalization of (14) and (17), we find

$$ T^{-1} \varphi(r, t) T^* = \eta \varphi(r, -t). \quad (A.22) $$

Combining this with the first Eq. (20) we immediately find that the linear (!) operator $T^* T$ commutes with $\varphi(r, t)$ and, as shown similarly, also with all other independent operators of the operator algebra. According to Schur's lemma, one has

$$ T^* T = \alpha \quad (A.23) $$

where, as one easily shows, the $c$ number $\alpha$ is real and positive and can be chosen to be equal to $+1$.

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