Magnetic Charge Quantization and Angular Momentum

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The quantum mechanical problem of an electric charge moving in the field of a magnetic charge is discussed and solved by algebraic methods which exhibit the direct relation between charge quantization and angular momentum. Commutation relations which lead to the Lorentz force equation are assumed. The operators are realized by constructing the irreducible representations of the three-dimensional Euclidean group in an angular-momentum basis. The Dirac quantization formula for the charges is obtained as a necessary and sufficient condition for the realization of all the observables. The construction proceeds by simple angular-momentum coupling calculations without resort to singular strings in x space. It is emphasized that the charge quantization does not follow from the operator equations of motion and rotation invariance alone. This treatment, like all previous ones, uses additional assumptions that may not be physically inescapable.

I. INTRODUCTION AND OUTLINE OF THE CALCULATION

The Dirac quantization condition (I)

\[
\frac{eg}{c} = \hbar \times \text{ (integer or half-odd-integer) } \tag{1.1}
\]

for electric charge e and magnetic (monopole) charge g has been derived from various sets of assumptions of quantum mechanics or of quantum field theory (1–13). It is evident in many derivations that the charge quantization condition (1.1) is intimately involved with the rotation-invariance of the equations of motion. However, the direct relation between the charge quantization and the quantization

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of angular momentum is obscured by consequences of the nonexistence of a vector potential for the field of a magnetic charge. The consequent nonexistence of the canonical momentum makes it impossible to quantize the theory in the usual way, by straightforward use of the canonical commutation relations. Most authors follow Dirac in assuming a Hilbert space defined on the coordinate space $x$, with certain singularities along lines which terminate on the charges. The canonical quantization procedure may then be restored, and the relation between magnetic charge and rotation invariance has been investigated in great detail by Hurst (11), under the assumptions of that approach.

Here, we adopt a procedure which is based on the algebra of the observables, instead of upon the canonical commutation rules. We consider the quantum mechanical problem of a single spinless electric charge $(M, -e)$ moving in the field of a single spinless magnetic charge, the latter being fixed at the origin. We assume that the generator of time displacements is the energy operator

$$\hat{H} = \frac{1}{2M} (\pi)^2,$$  \hspace{1cm} (1.2)

where $\pi$ is the kinetic momentum operator

$$\pi = M \hat{x}.$$  \hspace{1cm} (1.3)

We assume that $x$ and $\pi$ obey the commutation relations

$$[x_i, x_j] = 0$$  \hspace{1cm} (1.4)

$$[x_i, \pi_j] = i \delta_{ij}$$  \hspace{1cm} (1.5)

$$[\pi_i, \pi_j] = -ie_{ijk} \frac{e}{c} B_k(x),$$  \hspace{1cm} (1.6)

where $B$ is the magnetic field

$$B = g \frac{x}{x^3}.$$  \hspace{1cm} (1.7)

(Here and henceforth, we set $\hbar = 1$.) These commutation relations, which have been used by several authors, are formally the same as the ones which would obtain in the usual case, where $B(x)$ is derivable from a vector potential $A(x)$, and where $\pi$ is derivable from a canonical momentum $p = \pi + (e/c) A$. The assumed commutation relations (1.4)–(1.6) and the assumed time-displacement generator $\hat{H}$ may be combined to give Eq. (1.3) and the Lorentz force equation,

$$\pi = i[\hat{H}, \pi] = -\frac{e}{2Mc} (\pi \times B - B \times \pi).$$  \hspace{1cm} (1.8)
In Section 2, we assume that the vectors $\mathbf{x}$ and $\pi$ form a complete and irreducible set of observables for the problem. We then obtain the angular momentum operator $\mathbf{J}$, and we observe that the scalar operator $(\mathbf{x} \cdot \mathbf{J})/x$, which commutes with all observables, is equal to the number $eg/c$. In Section 3, we give a very simple heuristic proof of the charge quantization, based entirely on the algebra of angular-momentum operators and the value of $(\mathbf{x} \cdot \mathbf{J})/x$.

In Section 4, we introduce angular-momentum basis states which are eigenvectors of the complete commuting set $x^2, J^2, J_z$. We show that the charge quantization formula (1.1) follows directly from angular momentum quantization. We also show, by exhibiting all the matrix elements of $\mathbf{x}$ (Sec. 4) and of $\pi$ (Sec. 5) that the commutation relations can be realized in our angular-momentum basis for all charge values allowed by the Dirac formula (1.1). Mathematically, Sec. 4 consists of the explicit construction of the unitary irreducible representations of the Euclidean group $E_3$, in the angular-momentum basis.

In Section 6, we obtain the radial eigenfunctions of the energy operator directly from the matrices for $\pi$ in the angular-momentum basis. There, we also discuss an apparent contradiction between the Jacobi identity

$$\left[[\pi_1, \pi_2], \pi_3\right] + \left[[\pi_2, \pi_3], \pi_1\right] + \left[[\pi_3, \pi_1], \pi_2\right] = 0 \quad (1.10)$$

and the commutation relations (1.4)-(1.6). The latter, evaluated in the $\mathbf{x}$ representation, give

$$-\frac{eg}{c} \nabla \cdot \mathbf{B} = -8\pi \frac{eg}{c} \frac{1}{x^2} \delta(x) \quad (1.11)$$

instead of zero, for the l.h.s. of (1.10). The contradiction could be removed at the outset by restricting the domain of $\mathbf{x}$ to exclude the origin, that is by excluding the electric charge from a sphere surrounding the magnetic charge. However, it turns out that for finite energy, all radial wave functions vanish at the origin at least as fast as $x^L$, where $L > 1$ for nonvanishing allowed values of the charges. Therefore, the Jacobi identity is satisfied as an operator equation on the Hilbert space formed by those radial wave functions.

Nothing in our calculations appears to us to justify any preference for integral charge values in the Dirac quantization formula.

2. THE ANGULAR MOMENTUM OPERATOR

The angular-momentum operator must obey the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k \quad (2.1)$$

$$[J_i, x_j] = i\epsilon_{ijk} x_k \quad (2.2)$$

$$[J_i, \pi_j] = i\epsilon_{ijk} \pi_k \quad (2.3)$$
It may be verified with the aid of Eqs. (1.4)–(1.6) that a solution is given by

\[ J = \mathbf{x} \times \pi + \frac{eg}{c} \mathbf{\hat{x}}, \tag{2.4} \]

where \( \mathbf{\hat{x}} \) is the unit-vector \( x/x \). That (2.4) is the unique solution of (2.1)–(2.3) follows from the assumption that \( \mathbf{x} \) and \( \pi \) form a complete set of observables. The difference between two solutions would, according to (2.2) and (2.3), commute with all observables. It would therefore be a constant (c-number) vector. However, the addition of such a vector would contradict (2.1).

The second term in (2.4) is equal to the classically calculated angular momentum in the electromagnetic field when both charges are regarded as fixed sources (14). However, (2.4) does not express \( J \) as the sum of two quantum mechanical angular-momentum operators. For instance, the components of \( \mathbf{\hat{x}} \) commute with each other.

It follows directly from (2.4) that

\[ \mathbf{\hat{x}} \cdot J = \frac{eg}{c}, \tag{2.5} \]

and it may also be verified directly from the commutation relations that \( \mathbf{\hat{x}} \cdot J \) commutes with all the observables.

3. HEURISTIC DERIVATION OF THE QUANTIZATION OF CHARGE

Here, we interrupt the detailed discussion to give a heuristic derivation of the quantization of \( (eg/c) \). This derivation leaves several questions unanswered, but it is short and simple, and it sheds some light on the role of angular-momentum components in the charge quantization.

The quantity \( \mathbf{\hat{x}} \cdot J \) looks like an angular-momentum component, and it seems that its quantization should be demonstrated easily, by standard algebraic methods. However, the usual method relies upon the existence of raising and lowering operators, which commute with \( J^2 \), but which change the value of the component. No such operator exists in this problem, where \( \mathbf{\hat{x}} \cdot J \) is a c-number.\(^1\)

This difficulty can be avoided by enlarging the problem to include angular-momentum components of the usual type. Suppose that there is a neutral spinless particle (superscript 2) which interacts neither with the magnetic charge nor with

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\(^1\) A short algebraic proof of charge quantization, based on an assumed \( \mathbf{\hat{x}} \) representation, has been given by Peres (12). He asserts without proof that half-odd-integral charge values are excluded by his derivation. Our explicit realization of the observables for half-odd-integral charge values disproves that assertion, and agrees with the conclusions of authors who construct wave functions in \( x \) space (4, 5, 11). We thank Amnon Katz for a useful discussion of this point.
the electrically charged particle (superscript 1). The total angular momentum $J$ is the sum

$$J = J^{(1)} + J^{(2)}$$

(3.1)

of the two independent angular-momentum operators.

$$[J^{(1)}, J^{(2)}] = 0$$

(3.2)

We introduce the three mutually orthogonal unit-vectors

$$n_1 = \hat{x}^{(1)}$$

(3.3)

$$n_2 = \frac{x^{(2)} \times n_1}{|x^{(2)} \times n_1|}$$

(3.4)

$$n_3 = n_2 \times n_1$$

(3.5)

and the three scalars

$$K_j = n_j \cdot J.$$  

(3.6)

It follows from

$$[K_i, K_j] = i\epsilon_{ijk}K_k,$$

(3.7)

that the eigenvalues of $K_1$ are integers or half-odd-integers.

$$K_1 = n_1 \cdot J^{(1)} + n_1 \cdot J^{(2)} = \frac{eg}{c} + n_1 \cdot J^{(2)}$$

(3.8)

Let $\phi_0$ be an eigenvector of $(J^{(2)})^2$ with eigenvalue equal to zero.

$$K_1\phi_0 = \frac{eg}{c} \phi_0$$

(3.9)

Then $(eg/c)$ must be an integer or a half-odd-integer.

4. THE EUCLIDEAN GROUP. REPRESENTATIONS OF $x$ AND OF $J$. CHARGE QUANTIZATION

We now return to the problem of one electrically charged particle in the monopole magnetic field. We start with the commutation relations involving $J$ and $x$ alone.

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

(4.1)

$$[J_i, x_j] = i\epsilon_{ijk}x_k$$

(4.2)

$$[x_i, x_j] = 0$$

(4.3)

These are precisely the commutation relations obeyed by the generators, $J$ for the rotations and $x$ for the translations, of the Euclidean group $E3$ in momentum
space (15). The Casimir operators of that group, which commute with all the
generators, are the scalars \( x^2 \) and \( x \cdot J \). Thus, the representations of \( x \) and \( J \) in
our dynamical problem may be found by construction of the irreducible represent-
ations of the Euclidean group. Then the dynamical relation (2.5) appears as a
restriction of the representations\(^2\) to ones for which \( \hat{x} \cdot J = eg/c \).

It is possible to construct the desired representations in a basis of states labelled
by the eigenvalues of \( x \), following Wigner's (16) construction of the unitary
irreducible representations of the Poincaré group. We choose instead to use an
angular-momentum basis, in which the representations can be determined by
elementary algebraic methods. Our procedure is thus parallel to Naimark's (17)
method for the homogeneous Lorentz group. We find this procedure attractive
because it deals very directly with the relation between angular momentum and
charge quantization.

We introduce angular-momentum basis states \( | r, \mu, j, m \rangle \), which obey

\[
\begin{align*}
\mathbf{x}^2 | r, \mu, j, m \rangle &= r^2 | r, \mu, j, m \rangle \\
\mathbf{x} \cdot \mathbf{J} | r, \mu, j, m \rangle &= \mu r | r, \mu, j, m \rangle \\
\mathbf{J}^2 | r, \mu, j, m \rangle &= j(j + 1) | r, \mu, j, m \rangle \\
\mathbf{J}_z | r, \mu, j, m \rangle &= m | r, \mu, j, m \rangle
\end{align*}
\]

with the normalization\(^3\)

\[
\langle r', \mu', j', m' | r, \mu, j, m \rangle = \delta_{\mu\mu'} \delta_{jj'} \delta_{mm'} \delta(r - r').
\]

We use the standard phase conventions for angular-momentum eigenfunctions.\(^4\)
There remain arbitrary phases which depend upon \( \mu, j, \) and \( r \). The dependence
upon \( \mu \) is irrelevant because of the restriction to \( \mu = eg/c \). The phase conventions
for \( j \) and for \( r \) are defined below in Eqs. (4.31) and (5.8), rps.

We assume that the eigenvalues \( r, \mu, j, m \) completely specify the basis state (19).
In our dynamical problem, this corresponds to the assertion that \( \mathbf{x}^2, \mathbf{J}^2, \) and \( \mathbf{J}_z \)
form a complete set of commuting observables, while \( \hat{x} \cdot \mathbf{J} \) is a c-number. We now
proceed to compute the matrix of \( x \) in the angular-momentum basis. The matrix
elements of \( \mathbf{J} \) are well known from the rotation group alone. We shall hereafter
suppress the eigenvalue \( \mu \) in the notation for basis states, because it is effectively

\(^2\) Zwanziger (13) has previously noted that the charge quantization formula can be obtained
from the values of the Casimir operator \( \hat{x} \cdot \mathbf{J} \) of the Euclidean group.

\(^3\) The corresponding completeness relation is

\[
1 = \sum_{\mu, j, m} \int_0^\infty dr | \mu, r, j, m \rangle \langle \mu, r, j, m |.
\]

\(^4\) We follow the conventions used by Rose (18) for angular-momentum quantities.
a fixed number. For the irreducible representations of $E_3$, the eigenvalue $r$ is equally a fixed number, but we retain $r$ in the notation for convenience in calculating the matrix elements of $\pi$.

According to the Wigner–Eckart theorem, the matrix elements of a scalar operator $S$, or of a vector operator $V$, may be expressed in the form

$$
\langle \rho', j', m' | S | \rho, j, m \rangle = \delta_{j', j} \delta_{m', m} \langle \rho', j | S | \rho, j \rangle.
$$

$$
\langle \rho', j', m' | V^{(u)} | \rho, j, m \rangle = C(j, 1, j'; m, u, m') \langle \rho', j' | V | \rho, j \rangle,
$$

where $C(j, 1, j'; m, u, m')$ is the orthogonal vector-coupling coefficient for the angular-momentum coupling scheme $j + 1 = j'$; $m + u = m'$. That coefficient vanishes unless $j' = j, j \pm 1$. For Hermitean $S$ or $V$, the reduced matrix elements obey

$$
\langle \rho', j' | S | \rho, j \rangle = \langle \rho, j | S | \rho', j' \rangle^*.
$$

The reduced matrix elements of the vector $\mathbf{J}$ are given by

$$
\langle \rho', j' | J | \rho, j \rangle = \delta_{j', j} \langle \rho, j | J | \rho', j' \rangle^*.
$$

We shall make repeated use of the following two expressions for the reduced matrix elements of products of Hermitean vector operators $U$ and $V$.

$$
\langle \rho', j' | U \cdot V | \rho, j \rangle = \sum_{j''} \int d^2 \rho' \langle \rho', j' | U | \rho'' \rangle \langle \rho'' | V | \rho, j \rangle^*.
$$

$$
\langle \rho', j' | U \times V | \rho, j \rangle = \sum_{j''} W(j', j'', j) \int d^2 \rho' \langle \rho', j' | U | \rho'' \rangle \langle \rho'' | V | \rho, j \rangle^*.
$$

The coefficient $W$ is related to the Wigner $6-j$ symbol through

$$
W(j', j'', j) = i^j 6(2j + 1)^{1/2} \begin{vmatrix} j \cr 1 \cr j' \cr 1 \cr j'' \cr 1 \end{vmatrix}.
$$

The values needed for our purposes are

$$
W(j, j + 1, j) = -i^j [j/(j + 1)]^{1/2}
$$

$$
W(j, j, j) = [j/(j + 1)]^{1/2}
$$

$$
W(j, j + 1, j + 1) = +i[j(2j + 3)/((j + 1)(2j + 1))]^{1/2}
$$

$$
W(j, j - 1, j) = -i^j [j/(j + 1)]^{1/2}
$$

$$
W(j, j, j - 1) = -i^j [j/(j + 1)]^{1/2}
$$

$$
W(j, j + 1, j - 1) = +i[j(2j + 3)/((j + 1)(2j + 1))]^{1/2}
$$

$$
W(j, j - 1, j + 1) = -i^j [j/(j + 1)]^{1/2}.
$$
Equations (4.14), (4.15), and (4.17) may be derived directly from the Wigner–Eckart theorem and the values of the needed vector-coupling coefficients, or more quickly by the angular-momentum recoupling technique indicated in (4.16).

Since $\mathbf{x}$ commutes with $\mathbf{x}^2$, the reduced matrix elements of $\mathbf{x}$ must have the form

$$
\langle r', j \| \mathbf{x} \| r, j \rangle = r\delta(r - r') R_{r',r}.
$$

(4.18)

The matrix $R$, which may depend upon $r$, $\mu$, and $j$, is determined by applying (4.14) and (4.15) to the equations

$$
\langle r', j \| \mathbf{x} \cdot \mathbf{x} \| r, j \rangle = r^2\delta(r - r')
$$

(4.19)

$$
\langle r', j' \| \mathbf{x} \times \mathbf{x} \| r, j \rangle = 0,
$$

(4.20)

with $j' = j, j - 1$, to obtain

$$
| R_{j,j} |^2 + | R_{j,j+1} |^2 + | R_{j,j-1} |^2 = 1
$$

(4.21)

$$
| R_{j,j} |^2 - j | R_{j,j+1} |^2 + (j + 1) | R_{j,j-1} |^2 = 0
$$

(4.22)

$$
R_{j,j+1}[\sqrt{j} R_{j,j} - \sqrt{j + 2} R_{j+1,j+1}] = 0.
$$

(4.23)

The case $j' = j + 1$ gives nothing new.

For given $r$ and $\mu$, the eigenvalues $j$ are certain nonnegative integers or half-odd-integers. Let $j_0$ be the least of them. For $j = j_0$, the last terms in (4.21) and in (4.22) disappear, and those equations may be solved to obtain

$$
| R_{j_0,j_0} |^2 = \frac{j_0}{j_0 + 1}.
$$

(4.24)

For $j \geq j_0$, Eq. (4.23) gives

$$
R_{j,j} = \left[ \frac{j_0(j_0 + 1)}{j(j + 1)} \right]^{1/2} R_{j_0,j_0}.
$$

(4.25)

These results combined with (4.21), (4.22), and the Hermitean condition (4.12), to give

$$
| R_{j,j+1} |^2 = \frac{(j + 1)^2 - j_0^2}{(j + 1)(2j + 1)}
$$

(4.26)

All these results are independent of $r$.

Finally, we must compute the Casimir operator $\mathbf{x} \cdot \mathbf{J}$.

$$
\langle r', j \| \mathbf{x} \cdot \mathbf{J} \| r, j \rangle = \mu r\delta(r - r')
$$

(4.27)

Application of (4.13) and (4.14) gives

$$
[j(j + 1)]^{1/2} R_{j,j} = \mu.
$$

(4.28)
Comparison with (4.24) then gives
\[ j_0 = | \mu | = | \operatorname{eg} |. \] (4.29)
This is the Dirac charge quantization rule, since the angular-momentum quantum number \( j_0 \) must be an integer or a half-odd-integer. For the ordinary case where \( g = 0 \), this constitutes an algebraic proof that the quantum numbers for orbital angular momentum are the nonnegative integers.

The diagonal reduced matrix elements of \( x \) are now determined. For \( j \geq \mu \),
\[ R_{j,j} = \frac{\mu}{[j(j + 1)]^{1/2}} \] (4.30)
The off-diagonal values of \( R \) contain an arbitrary phase, which depends upon the relative phases of the basis states for the same \( r \) but successive values of \( j \).
We choose those phases to give, for \( j > | \mu | \),
\[ R_{j,j+1} = - \left[ \frac{(j + 1)^2 - \mu^2}{(j + 1)(2j + 1)} \right]^{1/2}. \] (4.31)
For \( j > | \mu | \), it follows that
\[ R_{j,j-1} = + \left[ \frac{j^2 - \mu^2}{(j + 1)(2j + 1)} \right]^{1/2}. \] (4.32)
The phase convention for the basis states still contains an arbitrary dependence upon \( r \). We settle that in Eq. (5.8) below.

In this section, we have used only the properties of angular momentum, the value of \( \frac{\hbar}{\mu} \), and the fact that the components of \( x \) commute with each other.

5. THE MOMENTUM MATRIX

To complete the solution of the dynamical problem, we determine the matrix elements of \( \pi \) in the space spanned by the angular-momentum basis which we used for the Euclidean group, with fixed \( \mu = \operatorname{eg}/c \) and different values of \( r \). In view of the Wigner–Eckart theorem, we have only to evaluate the reduced matrix elements \( \langle r', j' || \pi || r, j \rangle \).

The commutation relation (1.5) gives
\[ \langle r', j' || [\pi, x^2] || r, j \rangle = -2i\langle r', j' || x || r, j \rangle. \] (5.1)
\[ (r^2 - r'^2)\langle r', j' || \pi || r, j \rangle = -2i\hbar\delta(r - r') R_{j',j} \] (5.2)
Therefore,
\[ \langle r', j' || \pi || r, j \rangle = iR_{j',j}\delta(r - r') + \frac{1}{r} Q_{j',j}\delta(r - r') \] (5.3)
The matrix \( Q \) is yet to be determined. It may depend upon \( r \), and it obeys the Hermitean condition
\[
Q_{j',j}^* = (-y')^{-j} \left[ \frac{2j + 1}{2j' + 1} \right]^{1/2} Q_{j,j'}.
\] (5.4)

An additional restriction on \( Q \) can be obtained directly from the commutation relation (2.3). It is simpler and equivalent to use
\[
\langle r', j' \parallel x \parallel r, j \rangle = \langle r', j' \parallel J - \mu \hat{x} \parallel r, j \rangle
= [\sqrt{j(j + 1)} \delta_{j',j} - \mu R_{j',j}] \delta(r - r').
\] (5.5)

Reduction of the l.h.s. with the aid of (4.15) and (5.4) gives
\[
\sum_{j'} W(j', j^*, j) R_{j',j}^* \left\{ -i\delta'(r' - r) R_{j,j'} + \frac{1}{r} Q_{j,j'}^* \delta(r - r') \right\}
= [\sqrt{j(j + 1)} \delta_{j',j} - \mu R_{j',j}] \delta(r - r').
\] (5.6)

The coefficient of \( \delta' \) vanishes in consequence of \( x \times x = 0 \), so that
\[
\sum_{j'} W(j', j^*, j) R_{j',j}^* Q_{j,j'}^* = \sqrt{j(j + 1)} \delta_{j',j} - \mu R_{j',j}.
\] (5.7)

The solution of Eqs. (5.7) for \( Q \) is arbitrary to the extent that the quantity \( [f(r) R_{j',j}] \), where \( f(r) \) is any real function, may be added to \( Q_{j',j} \). This ambiguity reflects the remaining arbitrariness in the phase of \( \langle r, m, j \rangle \). It can be removed by the transformation \( | r, m, j \rangle \rightarrow | r, m, j \rangle \exp[iF(r)] \), where \( r^{-1}f(r) = F'(r) \). We choose the phase to give
\[
Q_{j,j} = 0.
\] (5.8)

Then Eqs. (5.7) can be solved to obtain
\[
Q_{j,j+1} = -i(j + 1) R_{j,j+1}.
\] (5.9)

\[
Q_{j,j-1} = +i j R_{j,j-1}.
\] (5.10)

Our result for \( \pi \) can then be summarized as
\[
\langle r', j' \parallel \pi \parallel r, j \rangle = i R_{j',j} \left\{ \delta'(r - r') + \frac{j'^* - j}{2r} (j + j' + 1) \delta(r - r') \right\}.
\] (5.11)
6. EIGENFUNCTIONS OF THE ENERGY

Let \( |E, j, m\rangle \) be the eigenvector of \( \mathfrak{H} \) with energy \( E \) and angular-momentum quantum numbers \( j, m \). The radial wave function \( \langle r, j | E, j \rangle \) does not depend upon \( m \).

\[
E\langle r', j | E, j \rangle = (2M)^{-1} \langle r', j \| \pi^2 \| E, j \rangle
\]

\[
= (2M)^{-1} \int dr \langle r', j \| \pi^2 \| r, j \rangle \langle r, j | E, j \rangle
\]

\[
\langle r', j \| \pi^2 \| r, j \rangle = \sum_{j'} \int dr'' \langle r', j \| \pi \| r'', j'' \rangle \langle r, j \| \pi \| r'', j'' \rangle^* 
\]

Use of the explicit formula (5.11) for the reduced matrix elements of \( \pi \) gives

\[
\langle r', j \| \pi^2 \| r, j \rangle = -\delta''(r - r') + \frac{j(j + 1) - \mu^2}{r^2} \delta(r - r').
\]

\[
\left[ -\frac{d^2}{dr^2} + \frac{j(j + 1) - \mu^2}{r^2} - 2ME \right] \langle r, j | E, j \rangle = 0.
\]

The solution of (6.4) which is regular at the origin is given by the Bessel function,

\[
\langle r, j | E, j \rangle = (kr)^{1/2} J_\alpha(kr).
\]

\[
k = (2ME)^{1/2}
\]

\[
\alpha(j) = [(j + \frac{1}{2})^2 - \mu^2]^{1/2}. 
\]

This radial wave function was obtained previously by Tamm (2).

Near the origin, the radial wave function for finite energy states goes to zero at least as rapidly as \( r^L \), where

\[
L = \frac{1}{2} + \alpha(j_0) > 1.
\]

Therefore, all matrix elements of the delta function in (1.11) vanish, and the apparent contradiction with the Jacobi identity (1.10) is removed.

7. DISCUSSION

That the Dirac charge quantization formula (1.1) is necessary for consistency with the properties of angular momentum follows from formula (2.5) for \( \hat{\mathbf{x}} \cdot \mathbf{J} \) and the commutation relations (4.1)–(4.3) for \( \mathbf{x} \) and \( \mathbf{J} \). We have shown, by construction of the matrices for all the observables, that our assumptions can be satisfied for all integral and half-odd-integral charge values allowed by (1.1). Theories with additional assumptions (10) may of course eliminate certain charge values.
It has sometimes been argued in a general way that half-odd-integral charge values make no sense because when the electric charge coincides with the magnetic charge, the angular momentum should be equal to zero, a forbidden value for half-odd-integral charge. It appears to us that the vanishing of all radial functions at the origin in this model refutes that general argument.

Most experimental searches (20) for magnetic charges have relied heavily upon $g$ having the order of magnitude ($g \sim 70e$) suggested by the quantization condition. We call attention to the fact that we have not derived the charge quantization from rotation invariance alone, even if the equation of motion (1.8) is accepted as an inescapable feature of magnetic charge. We have additionally assumed the commutation relations (1.4)-(1.6) for $x$ and $\pi$. It appears to us that all other treatments have used at least equally restrictive assumptions.

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19. Ref. (15), p. 188.