Weak decays of polarised muonium and polarised pionium

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Abstract. Using the theory of weak interaction, several decay channels of muonium \((\mu^+e^-)_{jm} (J=0, 1)\) and pionium \((\pi^+e^-)_{1/2M}\) with zero orbital angular momentum are calculated systematically. The decay probabilities, the angular distributions, the selection rules and the energy spectra are obtained.

1. Introduction

An exotic atom is a bound state consisting of an electron and a positively charged particle (called a ‘nuclear particle’), for example, \(\mu^+, \pi^+, K^+,\) etc.

From the viewpoint of electromagnetic interaction, these exotic atoms are nothing but different kinds of hydrogen-like atoms. They may decay away due to the instability of the ‘nuclear particles’ \((\tau_\mu = 2.2 \mu s, \tau_\pi = 0.026 \mu s, \tau_K = 0.012 \mu s)\) or they may annihilate through the weak interaction between the nuclear particle and the ‘orbital electron’ within the atom.

To study the annihilation of the atom, we have to analyse systematically the annihilation properties of these atoms in different angular momentum states; this is the purpose of this paper.

In the next section, we discuss two annihilation channels of muonium consisting of an electron and a positive muon with total angular momenta \(J = 0\) and \(J = 1\) by utilising the simple VA theory:

\[
(\mu^+e^-)_{jm} \rightarrow \nu_e \bar{\nu}_\mu, \nu_e \gamma_{\mu}, \quad J = 0, 1. \tag{1}
\]

For pionium composed of a positively charged pion and an electron, we have calculated its two decay channels:

\[
(\pi^+e^-)_{1/2M} \rightarrow \pi^0\nu_e, \nu_e\gamma. \tag{2}
\]

The computations and the results are given in §3.
As to the kaonium \((K^+e^-)\), in addition to dealing with strangeness arising from the strange-particle kaon, there is little difference in its treatment compared with that of pionium. Nevertheless, we leave it to a later work. Finally, §4 is a summary.

The SCHOONSCHIP program [1] is used for symbolic evaluation of algebraic expressions in this paper.

2. Decays of muonium \((J=0, 1)\)

The two decay channels in equation (1) can be evaluated in the framework of the universal Fermi \(\text{VA}\) theory. In fact, some calculations, e.g., \((\mu^+e^-)\mu\rightarrow\nu_e\bar{\nu}_\mu\), have been done previously [2–4]. Here the neutrinos are assumed to be massless Dirac particles, and we only give the calculated results.

2.1. Decay of \((\mu^+e^-)_{00}\)

This is the decay of the scalar muonium. For the \((\mu^+e^-)_{00}\rightarrow\nu_e\bar{\nu}_\mu\) channel, the angular distribution and the decay rate are

\[
W_{00} = 0 \quad \text{(forbidden).} \tag{3}
\]

If \(m_\nu \neq 0\), we have to consider the helicities of the neutrinos involved; the treatment is rather complicated [5]. In order to estimate approximately the effect caused by the neutrino mass, here we simply assume that the neutrinos are Dirac particles with mass \(m_\nu\) or \(m_\nu\), and take [6]

\[
m_\nu = 46 \text{ eV} \quad m_\nu = 0.5 \text{ MeV}. \tag{4}
\]

Therefore, we get

\[
\frac{dW_{00}}{d\cos\theta_\nu} = \frac{1}{2}(1 - V_\nu V_\nu)\tilde{W}
\]

\[
\tilde{W} = \frac{\alpha^2 G^2 E_\nu(E_\nu + V_\nu E_\mu)m_\nu^3 m_\mu^3}{\pi^2(m_\nu + m_\mu)^4} \tag{5}
\]

\[
W_{00} = (1 - V_\nu V_\nu)\tilde{W} = 1.34 \times 10^{-10} \text{ s}^{-1}
\]

where

\[
V_\nu = |p_\nu|/E_\nu \quad V_\mu = |p_\mu|/E_\mu \quad \alpha = e^2/4\pi = \frac{1}{\sqrt{2}}.
\]

For the \((\mu^+e^-)_{0}\rightarrow\nu_e\bar{\nu}_\mu\gamma\), there exist three particles in the final state. Considering the fact that it is so difficult to detect neutrinos in experiments directly, we temporarily
ignore the Dalitz plots and only give the differential decay rate and the energy spectrum of the photon and the total annihilation rate of the channel:

\[ \frac{d^2W_{\gamma}}{dx \, d \cos \theta_\gamma} = 2c(1 - x) \]
\[ dW_{\gamma}/dx = 4c(1 - x) \]
\[ W_{00} = \frac{3}{4}c \] (6)

where

\[ x = \omega/\omega_{\text{max}} = 2\omega/(m_e + m_\mu) \]
\[ c = \frac{\alpha^2 G^2 m_e^3 m_\mu^3}{48\pi^3} \left( \frac{1}{m_e^2} + \frac{1}{m_\mu^2} \right). \]

2.2. Decay of \((\mu^+ e^-)_{1M}\)

This is the decay of the vector muonium with \(J = 1\). For the \((\mu^+ e^-)_{1M} \rightarrow \nu_\mu \bar{\nu}_e\) channel, the angular distribution of the antineutrino \(\bar{\nu}_e\) and the decay rates are as follows:

\[ \frac{dW_{1, \pm \frac{1}{2}}}{d \cos \theta_\bar{\nu}_e} = \frac{1}{2} (1 \pm \cos \theta_\nu) W \]
\[ \frac{dW_{1, 0}}{d \cos \theta_\bar{\nu}_e} = (1 - \cos^2 \theta_\nu) W \]
\[ W_{1M} = \frac{3}{4} W = \frac{\alpha^2 G^2 m_e^3 m_\mu^3}{3\pi^2 (m_e + m_\mu)} \]
\[ = 4.02 \times 10^{-6} \text{ s}^{-1}. \] (7)

As to the \((\mu^+ e^-)_{1M} \rightarrow \nu_\mu \bar{\nu}_e\), we have

\[ \frac{d^2W_{1, \pm \frac{1}{2}}}{dx \, d \cos \theta_\gamma} = c x \left( \frac{3}{2} - x \pm \frac{m_\mu^2 - m_e^2}{m_\mu^2 + m_e^2} \cos \theta_\nu + (x - \frac{1}{2}) \cos^2 \theta_\nu \right) \]
\[ \frac{d^2W_{1, 0}}{dx \, d \cos \theta_\gamma} = c x \left[ 1 + (1 - 2x) \cos^2 \theta_\nu \right] \]
\[ \frac{dW_{1M}}{dx} = \frac{3}{4}c(2 - x) \]
\[ W_{1M} = \frac{8}{3}c = 2.24 \times 10^{-5} \text{ s}^{-1}. \] (8)

2.3. Analysis of the results

Compared with the free muon decay \((\nu_\mu = 1/(2.2 \times 10^{-6}) \text{ s}^{-1})\), the numerical decay rates for different channels of muonium in various angular momentum states are shown in table 1.

From the results shown in table 1, we can see that the free muon decay is much faster than the muonium decays due to the direct weak interaction between the positive muon and the electron in the muonium atom.
Table 1. Numerical values for muonium decays.

<table>
<thead>
<tr>
<th>States</th>
<th>(μ⁺e⁻)_{JM}→ν_μ\bar{ν}_μ Decay rate (×10⁻¹²)</th>
<th>(μ⁺e⁻)_{JM}→ν_e\bar{ν}_μ Decay rate (×10⁻¹¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>w_{0}/w_μ</td>
<td>0</td>
<td>3.7</td>
</tr>
<tr>
<td>w_{1M}/w_μ</td>
<td>8.8</td>
<td>4.9</td>
</tr>
<tr>
<td>w_{2M}/w_μ</td>
<td>6.6</td>
<td>4.6</td>
</tr>
</tbody>
</table>

(μ⁺e⁻)_{JM} is a complex scalar particle. The angular distributions of the final-state particles are apparently isotropic. If we do not consider the neutrino masses, then the two-body decay channel (μ⁺e⁻)_{JM}→ν_μ\bar{ν}_μ is forbidden because it cannot satisfy the requirements of both the momentum and the angular momentum conservation laws simultaneously due to the helicity restrictions of the neutrinos.

If the neutrino masses are taken to be those of equation (4), the calculations demonstrate that the effects of the neutrino masses on the decay rates and the shapes of the angular distributions are very small. Of course, this is because the mass of the neutrino or the antineutrino is small compared with that of a muon or electron.

(μ⁺e⁻)_{JM} is a complex vector particle. The angular distributions of the antineutrino in (μ⁺e⁻)_{JM}→ν_e\bar{ν}_μ are plotted in figure 1. It is obvious that the antineutrino has maximum angular distributions along the polarised axis of the muonium, whereas in the opposite direction the distributions contract and finally vanish. These results can be easily interpreted from the relationships between the angular momenta of the initial- and final-state particles.

As to the three-body decay channel of the vector muonium, typical angular distributions of the photon are given in figure 2, the differential decay rates of the photon for the (μ⁺e⁻)_{JM} three-body decay are shown in figure 3 and the energy spectra are plotted in figure 4. Of course, these results are also consistent with the kinematical relations of the particles.

![Figure 1](image.png)

**Figure 1.** Angular distribution for the antineutrino in (μ⁺e⁻)_{JM}→ν_e\bar{ν}_μ.
3. Decays of pionium

Pionium \((\pi^+ e^-)_{1\Sigma M}\) is also a complex particle with \(J = \frac{1}{2}\). The decay of pionium is a semileptonic process which can be treated by considering the weak interaction between the hadronic and leptonic currents while the strong interactions are grouped into some form factors which can be determined by either experiment or the \(c\nu\) hypothesis.

Figure 2. Angular distribution for the photon in \((\mu^+ e^-)_{1\Sigma M} \rightarrow \nu_e \bar{\nu}_\mu \gamma\), here \(\omega = \frac{1}{2} \omega_{\text{max}}\).

Figure 3. Differential angular distribution for the photon in \((\mu^+ e^-)_{1\Sigma M} \rightarrow \nu_e \bar{\nu}_\mu \gamma\).
3.1. $(\pi^+ e^-)_{1/2M} \rightarrow \pi^0 \nu_e$ decay

The matrix element is

$$
\langle \pi^0(p')\nu_e(p_c)|\langle \pi^+(p)e^-(p_e)\rangle_{1/2M}\rangle = -\frac{G\cos\theta_C}{\sqrt{2}} \frac{1}{(2\pi)^6} \frac{m_e m_r}{4EE'E_EE_e} \psi(0) \tilde{U}_c(p_e) \gamma_5 (1 + \gamma_5) U_c(p_e)
$$

$$
\times \left[ f_+(q^2)(p + p')\gamma_5 + f_-(q^2)(p' - p)\gamma_5 \right]
$$

where

$$
q^2 = -(p - p')^2 \sim 0 \quad \psi(0) = 1/(\pi a_0^2)^{1/2} = \frac{\alpha^3 m_e^3 m_r^3}{\pi (m_e + m_r)^3} \left( \frac{1}{\pi a_0^2} \right)^{1/2}
$$

$\theta_C$ is the Cabbibo angle, whereas $f_{\pm}$ are two form factors which are

$$
f_+(q^2 \rightarrow 0) = \sqrt{2} \quad f_-(q^2 \rightarrow 0) = 0.
$$
Starting from equation (9), the angular distribution and the decay rate can be derived as follows:

\[
\frac{dW_\pm}{d \cos \theta_{\gamma\nu}} = \frac{1}{2} [1 \pm (B/A) \cos \theta_{\gamma\nu}] \hat{W}
\]

and

\[
\hat{W} = \frac{A^3 G^2 \cos^2 \theta_c m_\pi^2 m_{\pi^0} (m_\pi + m_{\pi^0} - E')}{4 \pi^2 m_\pi^2 (m_\pi + m_{\pi^0})^4} A
\]

\[
= 2.69 \times 10^{-8} \text{ s}^{-1}
\]

where

\[
A = (m_{\pi^+}^2 - m_{\pi^0}^2)(m_\pi + m_{\pi^0}) + 2(m_\pi + m_{\pi^0})E'^2 - 2m_{\pi^+} m_{\pi^0}^2 + 4m_\pi m_{\pi^+} - m_{\pi^0}^2)
\]

\[
B = E'(3m_{\pi^+}^2 - m_{\pi^0}^2 + 2m_\pi m_{\pi^0} - 2(m_\pi + m_{\pi^0})E')
\]

\[
E' = \frac{1}{2} (m_\pi + m_{\pi^0}) + \frac{m_{\pi^0}^2}{2(m_\pi + m_{\pi^0})^2}
\]

The angular distribution is plotted in figure 5.

3.2. \((\pi^+ e^-)_{l2M} \to \nu, \gamma\) decay

This is a radiative semileptonic process. Using the LSZ formalism [7] and with the requirement of fulfilling gauge invariance, the matrix element of this decay channel can then be easily written down as

\[
\omega_{\nu}(Q)g_{\nu}(p_\nu)|\langle \pi^+ (p)e^- (p_\nu) \rangle_{l2M}|_{\text{in}} = i(2\pi)^4 \delta^4(p + p_\nu - Q)\psi(0) \hat{T}_{IB} + \hat{T}_{SDV} + \hat{T}_{SDA}
\]

where

\[
\hat{T}_{IB} = T_c f_\pi m_\pi \hat{U}_\nu(p_\nu)(1 - \gamma_5)\left(\frac{Q\hat{\epsilon}}{2p_\nu Q} + \frac{p_\nu}{pQ} - \frac{p_\nu \hat{\epsilon}}{pQ}\right) \hat{U}_\nu(p_\nu)
\]

\[
\hat{T}_{SDV} = -T_c a(s) \epsilon_{\nu} p_\rho Q_\nu \hat{U}_\nu(p_\nu)(1 - \gamma_5)\hat{U}_\nu(p_\nu)
\]

\[
\hat{T}_{SDA} = T_c b(s) \hat{U}_\nu(p_\nu)(1 - \gamma_5)\left((pQ)\hat{\epsilon} - (p\nu)\hat{\epsilon}\right) \hat{U}_\nu(p_\nu)
\]

and

\[
T_c = \frac{eG \cos \theta_c}{\sqrt{2}(2\pi)^{3/2}} \left(\frac{m_\pi}{4E_\gamma E_\nu E_\nu}\right)^{1/2}
\]

The subscripts IB, SDV and SDA stand for 'internal bremsstrahlung', 'structure-dependent vector' and 'structure-dependent axial vector' terms, respectively. \(f_\pi\) is the pion decay constant

\[
f_\pi = \left(\frac{2\pi m_\pi}{\tau(\pi^0 \to \nu\nu)}\right)^{1/2} \left(\frac{2m_\pi}{G \cos \theta_c m_\pi (m_\pi^2 - m_{\pi^0}^2)}\right)
\]

\[
a(s) \text{ and } b(s) \text{ are two form factors } (s = -(p - Q)^2). \text{ Usually, it is accurate enough to take } a(s) = a(0) \text{ and } b(s) = b(0) \text{ in the calculation. According to the CVC hypothesis, } a(0) \text{ is } [7-9]
\]

\[
|a(0)| = \sqrt{2} a(0) \frac{\sqrt{2}}{a(0)} \left(\frac{2m_\pi^2}{\tau(\pi^0 \to 2\gamma)}\right)^{1/2}
\]

\[
= (0.0261 \pm 0.009)m_\pi^{-1}
\]
whereas \( b(0) \) can be fixed by the values of both the \( a(0) \) and the \( \gamma(0) \). Here \( \gamma(0) \) is defined to be the ratio of the axial vector to the vector currents of the structure-dependent terms:

\[
\gamma(0) = \frac{b(0)}{a(0)}.
\]  

(17)

After a long list of calculations the angular distribution for the photon and the decay rate for the process are obtained:

\[
\frac{dW_+}{d \cos \theta_\gamma} = \frac{1}{2} (1 + \cos \theta_\gamma) \tilde{W}
\]  

(18)

and

\[
\tilde{W} = \frac{\alpha^4 G^2 \cos^2 \theta_\gamma m_e^2 m_\mu^2}{2 \pi m_\pi (m_e + m_\mu)^2} \left| f_\pi + \frac{1}{2} m_\pi (m_e + m_\mu) a(0)(1 + \gamma(0)) \right|^2.
\]  

(19)

3.3. Analysis of the results

The numerical values for the decay rates of the decay channels are

\[
W[(\pi^+ e^-) \rightarrow \pi^0 \nu_e] = 2.98 \times 10^{-8} \text{ s}^{-1}
\]

\[
W[(\pi^+ e^-) \rightarrow \nu_e \gamma] = \begin{cases} 
2.1 \times 10^{-7} \text{ s}^{-1} & \gamma(0) = 0.26 \\
1.98 \times 10^{-7} \text{ s}^{-1} & \gamma(0) = -1.98
\end{cases}
\]

\[
W(\text{free } \pi^+ \text{ decay}) = \tau_{\pi^+}^{-1} = \frac{1}{2.603 \times 10^{-8} \text{ s}^{-1}}.
\]  

(20)

**Figure 5.** Angular distribution for the \( \pi^0 \) in \((\pi^+ e^-)_{\mu2M} \rightarrow \pi^0 \nu_e\).
Table 2. Decay probabilities for \((\pi^+e^-)_{\mu \pi} \rightarrow \nu_\gamma\) when \(\gamma(0)\) takes different possible values.

\[
\begin{array}{ccc}
\gamma(0) & W[\gamma(0)] \times 10^{-7} \text{s}^{-1} & \text{Reference} \\
-2.36 & 1.957 & [9] \\
-1.98 & 1.978 & [10] \\
0.15 & 2.098 & [11] \\
0.26 & 2.105 & [10] \\
0.44 & 2.115 & [9] \\
\end{array}
\]

From the results shown above, it is easily seen that the decay probabilities of these two channels are much smaller than that of the free pion decay. However, one thing is worth noticing: the ratio of these two channels is nearly the same as that of the two corresponding free pion decay channels:

\[
\frac{W[(\pi^+e^-)_{\mu \pi} \rightarrow \nu_\gamma]}{W[(\pi^-e^-)_{\mu \pi} \rightarrow \nu_\gamma]} = \frac{W(\pi^+ \rightarrow e^+\nu_\gamma)}{W(\pi^- \rightarrow \pi^0 e^-\nu_\gamma)} = 8.
\]

The calculations show that the internal bremsstrahlung term is the leading part in the pionium radiative decay process, whereas the structure-dependent terms \((\text{SDV} + \text{SDA})\) contribute only \(3 \times 10^{-4}\) times that of IB term. Also, in this process, only the left-handed photon has non-vanishing effect.

How to determine the \(\gamma\) value remains an unsolved important problem. No theories or experiments so far have given a satisfactory answer. In table 2, some of the decay values for the radiative channel \((\pi^+e^-)_{\mu \pi} \rightarrow \nu_\gamma\) with several groups of \(\gamma(0)\) values are listed. The photon angular distributions for the \((\pi^+e^-)_{\mu \pi} \rightarrow \nu_\gamma\) decay are plotted in figure 6 with the \(\gamma(0)\) taken from [10]. Because of the smallness of the

![Figure 6](image-url)
resultant $W$, we cannot simply decide which $\gamma(0)$ should be taken as the most feasible and exact value.

4. Summary

In summary, we have calculated systematically the decays of muonium with $J=0$, 1 and of pionium with $J=\frac{1}{2}$. We have also discussed the angular distributions, the energy spectra and the forbidden causes from the point of view of the particle kinematics. In this paper we have also discussed the relationship between the $\gamma(0)$ parameter and the radiative pionium decay channel.

The effects of neutrino masses are only estimated very roughly here. When the mass of neutrino is not negligible, the neutrino has both right- and left-hand helicities which result in some more difficulties [5]. This problem is worth further probing.

The method of treating a complex particle as a simple particle is a useful technique. In the treatment of the weak interaction processes of nuclei, the elementary particle method developed by Kim [12] has been used to avoid complicated nuclear structures and has yielded some interesting results during the past ten years [13].

Though we have computed the decays of some exotic atoms with $J=0, \frac{1}{2}$ and 1, the general decay rule for these atoms still remains unknown today and how to find it and what method to be used comprise later goals of the authors.

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