On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. II.

— Case of Interacting Electromagnetic and Electron Fields —

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§ 1. Introduction

In a previous paper with the same title\(^1\) one of the authors has proposed a new formalism of the quantum theory of wave fields which reveals explicitly its relativistic invariance. For this purpose the author has generalized the Schrödinger equation of the system into the following form:

\[
\left\{ H_{12}(P) + \frac{\hbar}{i} \frac{\delta}{\delta C_P}\right\} \mathcal{F}[C] = 0. \tag{1.1}
\]

The generalized \(\phi\)-vector \(\mathcal{F}[C]\) in (1.1) is a functional of the independent variable hyper-surface \(C\) in the four-dimensional space-time world, and \(H_{12}(P)\) is the density of the interaction energy between the fields 1 and 2 at the world point \(P\) lying on the three-dimensional surface \(C\). The operation \(\delta/\delta C_P\) is the functional partial differentiation of \(\mathcal{F}[C]\) at the point \(P\), defined by

\[
\frac{\delta \mathcal{F}[C]}{\delta C_P} = \lim_{\delta C \rightarrow 0} \frac{\mathcal{F}[C'] - \mathcal{F}[C]}{\Delta V}, \tag{1.2}
\]

where \(C'\) is a surface overlapping \(C\) everywhere except a small region surrounding the point \(P\), and \(\Delta V\) is the volume of the small world region enclosed by \(C\) and \(C'\).

In our fundamental equation (1.1) the generalized \(\phi\)-vector \(\mathcal{F}[C]\) and its functional derivative are both relativistically invariant concepts. Moreover \(H_{12}\) is assumed to be a scalar function of the field quantities. (The case where \(H_{12}\) is not a scalar will be discussed in a later paper.\(^*\))

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1) S. Tomonaga: Riken Boku, 22 (1943), 525. English translation. Progr. Theor. Phys., 1 (1946), 40. This paper will be cited as I.

These field quantities satisfy, in turn, the field equations for the free fields, which have covariant forms under Lorentz transformations. The commutation relations between these field quantities are expressed in terms of the so-called four-dimensional delta-functions and their derivatives, and are free from any unsatisfactory feature of the ordinary commutation relations. In this way the equation (1·1) has a quite satisfactory four-dimensional space-time form possessed of a definite meaning without referring to any special Lorentz frame.

The equation (1·1) is also integrable when $H_{\mu\nu}$'s at any two world points at a finite as well as at an infinitesimal distance apart, one lying outside the light cone of the other are commutable with each other.

The aim of the present paper is to apply our method of formulation to the quantum electrodynamics which deals with the electromagnetic field interacting with the electron field.

Although we have in the case of quantum electrodynamics a perfectly relativistic formalism, that is, Dirac's many-time theory,\(^2\) it will be nevertheless of some importance to formulate the theory also in our formalism, since the former, in which the states of electrons are described in the configuration space, applies only to the case where the number of the electrons does not change, and is thus incapable of dealing with such phenomena as the emission of $\beta$-rays or the decay of mesons emitting electrons, without introducing a rather unfavourable complication into the theory at the sacrifice of its beautifulness. It is, therefore, desirable to treat the electrons as a quantized field and not as particles, since the former treatment fits the cases better in which the number of the electrons really changes.

It is already nearly known in I how to apply our formalism to the quantum electrodynamics except the well-known complication due to the existence of the so-called auxiliary condition. One of the main tasks in the present paper is thus to find the auxiliary condition in our formalism and to eliminate it in a relativistically invariant way on the lines which has been reported by Hayakawa, Miyamoto and one of the authors in another place.\(^3\)

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3) S. Hayakawa, Y. Miyamoto and S. Tomonaga: J. Phys. Soc. Japan in press. This paper will be cited as A.

When it is not specially remarked, the same notations are used throughout this paper as in I and A. Further, we use the unit system in which the light velocity is unity.

The Lagrange density of the free electromagnetic field is then given by

\[ L_I = -\frac{1}{4} (\text{Curl } A)^2 - \frac{1}{2} (\text{Div } A)^2, \quad (2\cdot 1) \]

and that for the free electron field is

\[ L_{II} = i\hbar \psi^* \{ (a, \text{Grad}) + i\hbar \beta \} \psi, \quad (2\cdot 2) \]

where \( a \) denotes the four-vector \((1, \vec{a})\), \( \vec{a} \) and \( \beta \) being the Dirac matrices. The constant \( \kappa \) in (2·2) is the reciprocal of the Compton wave length of the electron.

In the presence of the interaction between electron and electromagnetic field one has to replace Grad in (2·2) by Grad \(-\frac{i\kappa}{\hbar} A\), so that for the Lagrange density of the total system we have

\[ L = -\frac{1}{4} (\text{Curl } A)^2 - \frac{1}{2} (\text{Div } A)^2 + i\hbar \psi^* \{ (a, \text{Grad}) + i\hbar \beta \} \psi \]

\[ + \varepsilon(\psi^* \alpha \psi, A). \quad (2\cdot 3) \]

Through the usual procedure we obtain from (2·3) the Hamiltonian density of the system

\[ \begin{cases} H = H_I + H_{II} + H_{I,II} \\ H_I = \frac{1}{2} \left\{ (\frac{\partial \vec{A}}{\partial t})^2 - (\text{grad } A_0)^2 + (\text{Div } A)^2 - (\frac{\partial A_0}{\partial t})^2 + (\text{curl } A)^2 \right\} \\ H_{II} = -i\hbar \psi^* \{ a, \text{Grad} \} + i\hbar \beta \psi \\ H_{I,II} = -\varepsilon(\psi^* \alpha \psi, A) \end{cases} \quad (2\cdot 4) \]

in which \( H_I \) is the energy density for the electromagnetic field, \( H_{II} \) that for the electron field, and \( H_{I,II} \) the energy density of the interaction of
these two fields. Having thus obtained the Hamiltonian, we can write down the Schrödinger equation for our system:

$$\left( \hat{H} + \frac{\hbar}{i} \frac{\partial}{\partial t} \right) \Psi = 0.$$  

According to the general scheme given in I, we introduce the unitary operator $U$ defined by

$$U = \exp \left\{ \frac{i}{\hbar} (\hat{H}_f + \hat{H}_i) t \right\}$$

and transform the field quantities $\mathcal{A}$, $\phi$ and $\phi^*$ as follows:

$$\mathcal{A} \rightarrow U \mathcal{A} U^{-1}$$
$$\phi \rightarrow U \phi U^{-1}$$
$$\phi^* \rightarrow U \phi^* U^{-1}.$$  

We further transform the Schrödinger functional by means of $U$:

$$\Psi \rightarrow U \Psi.$$  

In the following the letters $\mathcal{A}$, $\phi$, $\phi^*$ and $\Psi$ are used to denote these transformed quantities.

These transformed quantities depend on $t$, while the original ones did not, since there we were dealing with the Schrödinger form of the quantum theory. It can be shown that the transformed field quantities depend on $t$ as if each field were free. Thus they satisfy the field equations for the free fields:

$$\begin{align*}
\{\Box \mathcal{A} = 0 \\
\{ (\alpha, \text{Grad}) + i \beta \} \phi = 0 \\
\{ \phi^* \{ (\alpha, \text{Grad}) - i \beta \} = 0. \}
\end{align*}  \quad (I)$$

The time dependence of the transformed $\Psi$-vector, on the other hand, is determined by

$$(\hat{H}_{1,\Pi} + \frac{\hbar}{i} \frac{\partial}{\partial t}) \Psi = 0  \quad (2.5)$$

in which $\hat{H}_f$ and $\hat{H}_{1,\Pi}$ appear no longer, while the field quantities contained in $\hat{H}_{1,\Pi}$ have to be considered as satisfying (I).
The field equations (I) together with the ordinary commutation relations, which give the commutator between field quantities at the same instant of time, enable us to calculate the four-dimensional commutation relations, which give the commutators between field quantities at any two world points $X$ and $X'$:

$$
\begin{align*}
[A^\mu(X), A^\nu(X')] &= \frac{\hbar}{i} g^{\mu\nu} D_1 (X - X') \\
[\phi_\rho(X), \phi_\sigma^*(X')]_+ &= W_{\rho\sigma}(X - X') \\
\text{with} \quad W_{\rho\sigma}(X - X') &= \left\{ \frac{\partial \phi_\rho}{\partial x^\mu} \right\}_{\partial \phi_\sigma}/(\partial \phi_\nu \text{ grad}_X) - i\hbar \beta_{\rho\sigma} \right\} D_{II} (X - X') \quad (II) \\
[\phi_\rho(X), \phi_\sigma(X')]_+ &= [\phi_\rho^*(X), \phi_\sigma^*(X')]_+ = 0 \\
[A^\mu(X), \phi_\rho^*(X')] &= [A^\mu(X), \phi_\rho^*(X')] = 0.
\end{align*}
$$

In (II) $D_1$ and $D_{II}$ are the four-dimensional delta-functions belonging respectively to the electromagnetic field and the electron field. They are both invariant with respect to Lorentz transformations. When $X$ and $X'$ are two world points, one of which lies outside the light cone of the other, they have the properties:

$$
\begin{align*}
(D(X - X')) &= 0 \\
(\{\partial/\partial x_i \ D(X - X')\}_{\nu \mu} &= 0 \quad \text{for} \quad i = 1, 2, 3 \\
\{\partial/\partial x_0 \ D(X - X')\}_{\nu \mu} &= \delta(x - x') \quad (2 \cdot 6) \\
\{\partial^2/\partial x_0^2 \ D(X - X')\}_{\nu \mu} &= 0
\end{align*}
$$

From these properties it follows that, when $X$ and $X'$ refer to the same instant of time, the commutation relations (II) reduce to the ordinary ones:

$$
\begin{align*}
[A^\mu(X), A^\nu(X')] &= 0, \\

\left[ A^\mu(X), \frac{\partial A^\nu(X')}{\partial x_0} \right] &= i\hbar g^{\mu\nu} \delta(x - x') \\
[\phi_\rho(X), \phi_\sigma^*(X')]_+ &= W_{\rho\sigma}(X - X') \quad (II') \\
\text{with} \quad W_{\rho\sigma}(X - X') &= \delta_{\rho\sigma} \delta(x - x').
\end{align*}
$$

We now generalize the Schrödinger equation (2.5) according to the general scheme developed in I. This gives

$$
(\hat{H}_{I,II}(P) + \frac{\hbar}{i} \frac{\partial}{\partial C_F}) \Psi(C) = 0. \quad (III)
$$

The meanings of $\hat{H}_{I,II}(P)$, $\partial/\partial C_F$, and $\Psi[C]$ have been already given.
in § 1. The interaction energy density $H_{1,1}(X)$ is here of the form\footnote{We use the letters $X, X', X''$, \ldots to denote arbitrary world points whereas the letters $P, P', P''$, \ldots are used to denote world points on $C$.}

$$H_{1,1}(X) = -\varepsilon(J(X), A(X)) \quad (2 \cdot 7)$$

with

$$J(X) = \phi^*(X) a \phi(X) \quad (3 \cdot 8)$$

which has the physical meaning: four-current density at the point $X$. Since $J(X)$ is a four-vector, $H_{1,1}$ of (2 \cdot 7) is evidently an invariant, so that our fundamental equation (III) is relativitically invariant. Further, as we shall prove in the following, $H_{1,1}(X)$ has the property:

$$[H_{1,1}(X), H_{1,1}(X')] = 0 \quad (2 \cdot 9)$$

for any two world points $X$ and $X'$, one lying outside the other's light cone. These points may lie a finite distance apart or may be two adjacent points. The property (2 \cdot 9) guarantees that our equation (III) is integrable when $C$ is space-like.

We shall now show that the integrability condition (2 \cdot 9) is really satisfied.

Since $H_{1,1}$ is an invariant and the commutation relation is also invariant, the left-hand side of (2 \cdot 9) can be most conveniently calculated in a reference system in which $X$ and $X'$ refer to the same instant of time. Let the components of various quantities in such a coordinate system be denoted by barred suffixes. Then, according to (II') we have $[A_{\overline{F}}(X), A_{\overline{E}}(X')] = 0$, so that we obtain first

$$[H_{1,1}(X), H_{1,1}(X')] = \varepsilon [\overline{J}(X), \overline{J}(X')] A_{\overline{F}}(X) A_{\overline{E}}(X') \quad (2 \cdot 10)$$

In order to calculate $[\overline{J}(X), \overline{J}(X')]$ we use the following theorem which can be proved by a straightforward calculation:

**Theorem I.** Let $U$ and $V$ be any two four-row-four-column matrices. Then we have

$$[\phi^*(X) U \phi(X), \phi^*(X') V \phi(X')] = \phi^*(X) U W(X - X') V \phi(X')$$

$$-\phi^*(X') V W(X - X') U \phi(X). \quad (2 \cdot 11)$$

Now, to calculate $[\overline{J}(X), \overline{J}(X')]$, we have only to put $U = V = a$ in this formula. Then we obtain, noting $W_{\overline{\gamma} \overline{\tau}} = \delta_{\overline{\gamma} \overline{\sigma}} \delta(\overline{x} - \overline{x'})$,
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\[ [f^2(X), f^2(X')] = \phi^*(X)(u^T a^\xi - a^\xi u^T) \phi(X) \delta(x-x'). \]

Substituting this commutator into (2.10), we get

\[ [H_{1,\Pi}(X), H_{1,\Pi}(X')] = \phi^*(X)(u^T a^\xi - a^\xi u^T) \phi(X) A_\xi(X) A_\xi(X) \delta(x-x') = 0, \]

what is to be proved.

§ 3. Auxiliary Condition for the Electromagnetic Potential.

It is well known that our Lagrange function (2.1) gives rise to \( \nabla A = 0 \) but does not yield the Maxwell equation. In order that the canonical equations derived from (2.1) have the Maxwell form, one must impose the auxiliary condition

\[ \{ \text{Div} \, A(X) \} \mathcal{F} = 0 \quad (3.1) \]

on \( \mathcal{F} \), and admit only such \( \mathcal{F} \) to represent actual states. When the field is interacting with electrons, however, the condition (3.1) is not compatible with our fundamental equation (III), so we must replace (3.1) by

\[ \mathcal{E}_x[C] \mathcal{F}[C] = 0, \quad (3.2) \]

where \( \mathcal{E}_x[C] \) has the form

\[ \mathcal{E}_x[C] = \text{Div} \, A(X) + f_x[C]. \quad (3.3) \]

The added term \( f_x[C] \) is a function of the world point \( X \) on one hand and a functional of the variable surface \( C \) on the other hand. It must be so determined that the condition (3.2) is compatible with the equation (III). It is thus required that \( f_x[C] \) satisfies

\[ \left[ H_{1,\Pi}(P) + \frac{\mathcal{E}}{i} \frac{\delta}{\delta C_P}, \text{Div} \, A(X) + f_x[C] \right] = 0 \quad (3.4) \]

for all points \( X, P \) being any point lying on \( C \).

Now, since we have

\[ [H_{1,\Pi}(P), \text{Div} \, A(X)] = \frac{\mathcal{E}}{i} \epsilon(J(P), \text{Grad}_P D_1(P-X)), \]

\[ \left[ \frac{\mathcal{E}}{i} \frac{\delta}{\delta C_P}, \text{Div} \, A(X) \right] = 0, \]
our problem of finding the solution of (3.4) is solved when we can find $f_x[C]$ such that

$$
\begin{align*}
\left[ [H_{i,n}(P), f_x[C]] \right] = 0 \\
\left[ \frac{\mathbf{A}}{t}, \frac{\partial}{\partial C_P}, f_x[C] \right] = -\frac{\mathbf{A}}{t}(J(P), \text{Grad}_P D_1(P-X)).
\end{align*}
(3.5)
$$

The second relation of (3.5) gives rise to the differential equation

$$
\frac{\partial f_x[C]}{\partial C_P} = -s \text{Div}_P (J(P)D_1(P-X))
(3.6)
$$
on account of $\text{Div} \mathbf{J} = 0$.

The differential equation (3.6) can be solved immediately by using the following theorem:

**Theorem II.** Let $G(X)$ be an arbitrary four-vector function of the world point $X$, and let $C$ be any space-like surface in the space-time world. Further, let $\mathbf{N}(P)$ denote the unit vector which is normal to $C$ at the point $P$ on $C$. The direction of $\mathbf{N}$ is such that its contravariant time component is positive (this means that $\mathbf{N}$ is pointing to the future). Then we have

$$
\frac{\partial}{\partial C_P} \int_{\mathbf{N}} (G(\mathbf{N'}), \mathbf{N}(P'))dF_{P'} = -\text{Div} \ G(P)
(3.7)
$$

where the integral on the left-hand side is the surface integral taken over the surface $C$, $dF_{P'}$, being its surface element at $P'$.

**Proof.** According to Gauss' theorem we have, in general,

$$
-\int_{\text{Vol}} \text{Div} \ G(X)dV = \int_{\text{Surf}} (G(P), \mathbf{N}(P))dF_P
(3.8)
$$

where the integral on the left-hand side is the volume integral taken over a four-dimensional space-time region, and the integral on the right-hand side is the surface integral taken over the surface enclosing this region, $\mathbf{N}(P)$ being the outward normal to this surface at the point $P$. We consider the space-time region to be that part of the world which lies on the past side of the variable surface $C$. Then the surface integral on the right-hand side of (3.8) consists of two parts: that taken over the surface $C$ and that over the surface at infinity:
\[- \int_\mathcal{D} \text{Div} \ G(X) d\sigma = \int_\mathcal{D} (G(P), N(P)) dF_p + \int_\mathcal{N} (G(P), N(P)) dF_p. \quad (3.9)\]

Now, carrying out the differentiation \( \partial / \partial C_p \) on both sides, we obtain the required result (3.8).

Using this theorem the solution of the equation (3.6) is found to be

\[ f_x[C] = + \varepsilon \int_\mathcal{D} (J(P), N(P)) D_1(P - X) dF_p. \quad (3.10)\]

In (3.10) we have put the vanishing integration constant.

We have now to verify that the solution (3.10) satisfies the first relation of (3.5). This can be shown in the following manner:

First we substitute (3.10) into \([H_{1,1}(P), f_x[C]]\):

\[ [H_{1,1}(P), f_x[C]] = + \varepsilon \int_\mathcal{D} [J^t(P), J^t(P')] A_t(P) N_t(P') D_1(P - X) dF_p. \quad (3.11)\]

Since the point \( P \) as well as the point \( P' \) lie on \( C \), and since \( C \) is space-like, one of the two points \( P \) and \( P' \) lie outside the light cone of the other. It follows, as one can verify by using the formula (2.11), that \([J^t(P), J^t(P')]\) has a non-vanishing value only when \( P' \) is an adjacent point of \( P \).

This situation enables us to replace \( J^t(P), N_t(P') \) in the integrand on the righthand side of (3.11) simply by \(-J^t(P')\) if one carries out the integration referring to the coordinate system in which the space-axes are tangent to \( C \) at \( P \); (in this coordinate system the components of a vector are labeled by the barred suffixes) because in this reference system \( N(P') \) has the contravariant components \( (1, 0, 0, 0) \) at \( P' \) adjacent to \( P \). Calculating (3.11) in this system we obtain

\[ [H_{1,1}(P), f_x[C]] = + \varepsilon \int_\mathcal{D} [J^t(P), J^t(P')] A_t(P) D_1(P - X) dF_p, \]

which gives the vanishing result on account of the relation obtained by (2.11):

\[ [J^t(P), J^t(P')] = \{ \phi^s(P) \rho \phi^s(P') - \psi^s(P') \tilde{\phi}(P) \} \delta(x_P - x_{P'}) = 0. \]

In this way we have found the auxiliary condition which is compatible with the fundamental equation (III):
\[
\begin{align*}
\mathcal{E}_x[C] & = 0 \\
\mathcal{E}_x'[C] & = \text{Div } \mathbf{A}(X) + \varepsilon \int_{\mathcal{O}} (J(P), N(P)) D(P-X)dF_P.
\end{align*}
\] (IV)

Since we have conditions of this form, one for each point \(X\) in the world, any one of these must commute with others. That this requirement is also satisfied can be verified as follows:

\[
[\text{Div } \mathbf{A}(X) + f_x[C], \text{Div } \mathbf{A}(X') + f_{x'}[C]] = [\text{Div } \mathbf{A}(X), \text{Div } \mathbf{A}(X')] + [f_x[C], f_{x'}[C]].
\]

The first term of the last expression vanishes because of \(\square D = 0\). That the second term vanishes can be proved by carrying out the integration over \(P'\) in a reference system whose space axes are tangent to \(C\) at \(P\).

\section*{§ 4. Gauge Transformation.}

As mentioned in the preceding paragraph, only those \(\mathcal{F}\)'s which satisfy the condition (IV) represent physically significant states of the system, so that only those linear operators which commute with \(\mathcal{E}_p[C]\) represent physically measurable quantities. We shall show in this paragraph that these measurable quantities are so called gauge-invariant quantities which are invariant under the transformation

\[
\begin{align*}
\mathbf{A} & \rightarrow \mathbf{A} + \text{Grad } \lambda \\
\phi & \rightarrow e^{\frac{ia}{\varepsilon}} \phi \\
\phi^* & \rightarrow \phi^* e^{-\frac{ia}{\varepsilon}}
\end{align*}
\] (4.1)

where \(\lambda\) is an arbitrary (real) scalar function of the world point satisfying

\[\square \lambda = 0.\]

We now introduce the quantity defined by

\[
\begin{align*}
\Gamma[C] & = \Gamma_1[C] - \Gamma_2[C] \\
\Gamma_1[C] & = \int_{\mathcal{O}} \langle \lambda(P') (\text{Grad}_{\mathcal{O}} \mathcal{E}_p[C], N(P'))dF_{P'}; \\
\Gamma_2[C] & = \int_{\mathcal{O}} \mathcal{E}_p[C] (\text{Grad } \lambda(P'), N(P'))dF_{P'}. 
\end{align*}
\]
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We shall now show that the quantity \( \Gamma[C] \) has the following important properties:

\[
[\Gamma[C], \mathcal{A}(\dot{P})] = \frac{\hbar}{i} \text{Grad} \lambda(P) \quad (4.4)
\]

\[
[\Gamma[C], \psi_\mu(P)] = \varepsilon_\mu \lambda(P) \psi_\mu(P) \quad (4.5)
\]

and

\[
[\Gamma[C], \psi_\mu^*(P)] = -\varepsilon_\mu \psi_\mu^* \lambda(P). \quad (4.6)
\]

The proof of (4.4) can be performed in the following way:

First we have

\[
[\Gamma_1[C], \mathcal{A}(P)] = \frac{\hbar}{i} \int \lambda(P') (\mathcal{N}(P'), \text{Grad}_{\mu'}) \text{Grad}_{\mu'} D_1(P' - P) dF_{\mu'}. 
\]

We carry out the integration on the right-hand side, referring to the particular reference system in which the space axes are tangent to \( C \) at \( P \). Then, since \( \mathcal{N}(P') \) has contravariant components \((1, 0, 0, 0)\) at \( P' \) adjacent to \( P \), we have here

\[
(\mathcal{N}(P'), \text{Grad}_{\mu'}) \text{Grad}_{\mu'} D_1(P' - P) = \frac{\partial}{\partial P_{\mu'}} \text{Grad}_{\mu'} D_1(P' - P),
\]

so that, on account of the properties of the \( D \)-function stated in §2,

\[
[\Gamma_1[C], \mathcal{A}_\mu(P)] = \begin{cases} 
0 & \text{for } \mu = 0 \\
-\frac{\hbar}{i} \frac{\partial \lambda}{\partial x_\mu} & \text{for } \mu = 1, 2, 3
\end{cases}
\]

Returning to the general reference system, this can be written as

\[
[\Gamma_1[C], \mathcal{A}(P)] = -\frac{\hbar}{i} \left\{ \text{Grad} \lambda(P) + (\mathcal{N}(P), \text{Grad} \lambda(P)) \mathcal{N}(P) \right\}
\]

A similar calculation can be carried out for the second term, \( \mathcal{A} \), with the result:

\[
[\Gamma_1[C], \mathcal{A}(P)] = \frac{\hbar}{i} \int \text{Grad}_{\mu'} D_1(P' - P)(\mathcal{N}(P'), \text{Grad}_{\mu'} \lambda(P')) dF_{\mu'},
\]

\[
= -\frac{\hbar}{i} (\mathcal{N}(P), \text{Grad} \lambda(P')) \mathcal{N}(P). \quad (4.8)
\]

Now (4.7) and (4.8) together yield just the required relation (4.4).
The relations (4.5) and (4.6) are also easily proved; for the first term of \( \Gamma \) we have

\[
[\Gamma[C], \phi_{\nu}(P)] = \varepsilon \int_{\partial} dF_{\nu}(\lambda(P'), N(P'), \text{Grad}_{\nu'}) \times \int_{\partial} -(W(P-P'')\alpha, N(P''))_{\nu'\nu} \phi_{\nu'}(P')D_{\nu'}(P' - P')dF_{\nu''},
\]

where we have applied the formula

\[
[\lambda(P), \phi_{\nu}(P)] = -W_{\nu\sigma}(P' - P')\alpha_{\sigma\nu} \phi_{\nu}(P').
\]

Then, carrying out the integration with respect to \( F_{\nu''} \) in the coordinate system, the space axes of which are tangent to \( C \) at \( P \), thereby noticing \( W_{\nu\sigma}(P' - P') = \delta_{\nu\sigma}\delta(\mathbf{x} - \mathbf{x}') \) in this system, we find

\[
[\Gamma[C], \phi_{\nu}(P)] = \varepsilon \int_{\partial} \lambda(P') (N(P'), \text{Grad}_{\nu} D_{\nu}(P - P')) \phi_{\nu}(P)dF_{\nu},
\]

\[
= -\varepsilon \lambda(P) \phi_{\nu}(P).
\]

On the other hand, \( \Gamma[C] \) contributes nothing to the commutator \([\Gamma[C], \phi_{\nu}(P)]\). This results from the fact that

\[
[\Gamma[C], \phi_{\nu}(P)] = \varepsilon \int_{\partial} dF_{\nu}(\text{Grad} \lambda(P'), N(P')) \times \int_{\partial} -(W(P-P'')\alpha, N(P''))_{\nu'\nu} \phi_{\nu'}(P')D_{\nu'}(P' - P')dF_{\nu''}
\]

\[
= -\varepsilon \int_{\partial} dF_{\nu}(\text{Grad} \lambda(P'), N(P')) \phi_{\nu}(P')D_{\nu}(P - P').
\]

Here appears in the integrand the function \( D_{\nu} \) and not its derivative so that, when integrated over a space-like surface, the integral identically vanishes. Thus we have obtained (4.5). It goes without saying that the relation (4.6) can be verified in the same way.

The expressions (4.4), (4.5) and (4.6) mean that the quantity \( \Gamma[C] \) can be regarded as the generating function of the gauge transformation (4.1). Namely, every dynamical quantity \( G \) transforms in this transformation according to

\[
G(P) \rightarrow \varepsilon \phi_{\nu}^{\Gamma[C]} G(P) \phi_{\nu}^{\Gamma[C]} \quad (4.9)
\]
From (4.9) we can conclude that the necessary and sufficient condition for that \( G \) is a gauge-invariance is

\[
[I[C], G(P)] = 0, \quad (4.10)
\]

or explicitly written

\[
\int \lambda(P')(\text{Grad}_p, \mathcal{E}_p[C], G(P), \mathcal{N}(P'))dF_p,
- \int [\mathcal{E}_p[C], G(P)](\text{Grad} \lambda(P'), \mathcal{N}(P'))dF_p = 0. \quad (4.11)
\]

Since \( \lambda \) satisfies (4.2) but otherwise may be chosen arbitrary, (4.11) is equivalent to

\[
[\mathcal{E}_p[C], G(P)] = 0. \quad (4.12)
\]

We have thus obtained the required result: every gauge-invariant quantity commutes with \( \mathcal{E} \), and \textit{vice versa}.

Now, we must discuss how to introduce the gauge transformation into the frame of our theory. Such a discussion is necessary because the ordinary scheme of the gauge transformation in which the field quantities are transformed according to (4.1) can not be directly taken over into our theory. In fact, it can be seen at once that, if one adopts (4.1) and replaces the original field quantities by the quantities transformed according to (4.1) it destroys the invariance of various fundamental relations in our theory. The commutation relations, for instance, between new variables (which we distinguish by a prime) become then

\[
\begin{align*}
[A^\mu(P), A^\nu(P')] &= \frac{4}{\pi} g^{\mu\nu} D_\lambda(P - P') \\
[\phi_{a}'(P), \phi_{b}'(P')] &= \epsilon \frac{i \lambda}{\pi} (\lambda(P) - \lambda(P')) W_{ab}(P - P') \\
&\text{etc.}
\end{align*}
\]

while the field equations for the free fields take the form

\[
\begin{align*}
\Box A' &= 0 \\
\{ (a, \text{Grad} - ie/\hbar \text{ Grad} \lambda) + i\epsilon \beta \} \phi' &= 0 \\
\phi'^{\mu} \{ (a, \text{Grad} + ie/\hbar \lambda) - i\epsilon \beta \} &= 0,
\end{align*}
\]
and finally the generalized Schrödinger equation is

\[
\left\{ H' + \varepsilon \left( (\phi^* a \phi'), \text{Grad} \lambda \right) + \frac{\hbar}{i} \frac{\partial}{\partial C_r} \right\} \Psi' = 0
\]

with

\[
H' = -\varepsilon (\phi^* a \phi', \mathcal{A}')
\]

Thus our theory would not be invariant under gauge transformations.

But, as we shall show in the following, our theory is still invariant with respect to the change of the gauge; in other words, we can introduce the law of transformations for various quantities (in this case not only for the field quantities but also for the \(\Psi\)-vector \(\Psi[C]\)) in such a way that the fundamental relations maintain their forms when we go over from one gauge to another.

This can be done by decomposing the generating function \(I[C]\) of the gauge transformation into two parts, one of which transforms the dynamical quantities, the other the \(\Psi\)-vector in such a way that the above-mentioned requirements are fulfilled. This possibility of separation in a relativistically invariant manner (so was the construction of our fundamental relations!) may be regarded as an indirect proof of the compatibility of the Lorentz and the gauge transformations.

Since \(\left[ \mathcal{E}, H + \frac{\hbar}{i} \frac{\partial}{\partial C_r} \right] = 0\), it is obvious from (4.3) that

\[
\left[ I[C], H + \frac{\hbar}{i} \frac{\partial}{\partial C_r} \right] = 0,
\]

whence we get

\[
\frac{i}{\hbar} \Gamma (H + \frac{\hbar}{i} \frac{\partial}{\partial C_r}) e^{\frac{i}{\hbar} \Gamma \Psi} = 0. \tag{4.13}
\]

We introduce now:

\[
I'[C] = \gamma[C] + \gamma'[C] \tag{4.14}
\]

with

\[
\gamma[C] = -\int_c \lambda(P') (\text{Grad} \rho, \text{Div} \mathcal{A}(P'), \mathcal{N}(P')) dF_{P'},
\]

\[
+ \int_c \text{Div} \mathcal{A}(P) (\text{Grad} \lambda(P), \mathcal{N}(P')) dF_{P'} \tag{4.15}
\]
\[ \gamma'[C] = -\epsilon \int_B \lambda(P') dF_{\nu} \{ N(P') \text{ Grad}_{\nu} \{ (J(P'), N(P')) \} D(P' - P) dF_{\nu} \} \\
+ \epsilon \int_B (\text{Grad} \lambda(P'), N(P')) dF_{\nu} \{ (J(P'), N(P')) \} D(P' - P) dF_{\nu} \]
\[ = -\epsilon \int_B \lambda(P') (N(P') J(P')) dF_{\nu} \quad (4.16) \]

and prescribe the transformation rules for the dynamical quantities and the \( \phi \)-vector by means of

\[
\begin{align*}
G(P) & \rightarrow e^{i \frac{\delta}{\delta C_P} \gamma'[C]} G(P) e^{-i \frac{\delta}{\delta C_P} \gamma'[C]} = G'(P) \\
\Psi & \rightarrow e^{-i \frac{\delta}{\delta C_P} \gamma'[C]} \Psi = \Psi'
\end{align*}
\]

for any dynamical quantity; \( (V) \)

for the \( \Psi \)-vector.

The relation \( (V) \) gives rise to the following transformation for \( A, \phi, \phi^* \):

\[
\begin{align*}
A & \rightarrow A + \text{Grad} \lambda \\
\phi & \rightarrow \phi \\
\phi^* & \rightarrow \phi^*
\end{align*}
\]

(unchanged) \( (4.17) \)

as can be easily verified. This guarantees the invariance of the commutation relations and of the free-field equations.

Noticing that

\[
\frac{i}{\epsilon} \frac{\partial}{\partial C_P} e^{\frac{\delta}{\delta C_P} \gamma} = \frac{i}{\epsilon} \frac{\partial}{\partial C_P} e^{-\frac{\delta}{\delta C_P} \gamma}
\]

which follows from

\[
\frac{\partial \gamma'[C]}{\partial C_P} = \frac{\partial}{\partial C_P} \int_B \text{Div} \{ \text{Grad} \lambda \text{ Div} A - \lambda \text{ Grad} \text{ Div} A \} dV
\]

\[ = \Box \lambda \text{ Div} A - \lambda \Box \text{ Div} A = 0, \]

one obtains from \( (4.13) \) and \( (4.14) \), using \( e^{i \frac{\delta}{\delta C_P} (\gamma + \gamma')} = e^{i \frac{\delta}{\delta C_P} \gamma} e^{i \frac{\delta}{\delta C_P} \gamma'} \) which results from the fact that \( \gamma \) commutes with \( \gamma' \),

\[
e^{i \frac{\delta}{\delta C_P} \gamma'} \left( e^{i \frac{\delta}{\delta C_P} \gamma} H e^{i \frac{\delta}{\delta C_P} \gamma} + \frac{i}{\epsilon} \frac{\partial}{\partial C_P} \right) e^{-i \frac{\delta}{\delta C_P} \gamma'} \Psi = 0
\]

or, multiplying from the left by \( e^{i \gamma'} \) and using \( (V) \),

\[
(H' + \frac{i}{\epsilon} \frac{\partial}{\partial C_P}) \Psi = 0.
\]
Thus the covariance of our generalized Schrödinger equation is also proved. It is obvious that, on account of (4.10) and (4.14), the expectation value of any physically measurable quantity maintains its value in the transformation (V). Thus the invariance of our theory under the transformation of the gauge has been proved.

It may be of some interest to check our assertion by means of a direct calculation. What we must verify is that the equation

\[
\left( \frac{i}{\hbar} \gamma^\prime \hat{A} e^{-\frac{i}{\hbar} \gamma^\prime \Phi} + \frac{i}{\hbar} \frac{\partial}{\partial C_p} \right) \times e^{-\frac{i}{\hbar} \gamma^\prime \Phi} = 0
\]

does really hold. Since \( \gamma^\prime \) commutes with \( \hat{H} \) (see § 3), the first term gives

\[
\frac{i}{\hbar} \gamma^\prime \hat{A} e^{-\frac{i}{\hbar} \gamma^\prime \Phi} = e^{-\frac{i}{\hbar} \gamma^\prime \Phi} \left( \hat{H} + \frac{i}{\hbar} \gamma, \hat{H} \right) \Psi = e^{-\frac{i}{\hbar} \gamma^\prime \Phi} \{ \hat{H} - \epsilon(\psi^a \phi, \text{Grad} \lambda) \} \Psi.
\]

As for the second, we obtain the following results:

\[
\frac{\hbar}{i} \frac{\partial}{\partial C_p} e^{-\frac{i}{\hbar} \gamma^\prime \Phi} = e^{-\frac{i}{\hbar} \gamma^\prime \Phi} \left( \frac{\partial}{\partial C_p} - \frac{\partial \gamma'[C]}{\partial C_p} \right) \Psi.
\]

From (4.16)

\[
\frac{\partial \gamma'[C]}{\partial C_p} = -\epsilon \text{Div}(\mathcal{J}(P) \lambda(P)) = -\epsilon(\mathcal{J}(P) \text{ Grad} \lambda(P)) = -\epsilon(\phi^a \phi, \text{Grad} \lambda).
\]

whence

\[
\frac{\hbar}{i} \frac{\partial}{\partial C_p} e^{-\frac{i}{\hbar} \gamma^\prime \Phi} = e^{-\frac{i}{\hbar} \gamma^\prime \Phi} \left( \epsilon(\phi^a \phi, \text{Grad} \lambda) + \frac{\hbar}{i} \frac{\partial}{\partial C_p} \right) \Psi.
\]

Summing up the results and multiplying from the left by \( e^{\frac{i}{\hbar} \gamma^\prime \Phi} \) we arrive at

\[
\left( \hat{H} + \frac{\hbar}{i} \frac{\partial}{\partial C_p} \right) \Psi = 0.
\]

which is the original equation. (to be continued.)

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