On the relativistic equation for scattering

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By defining an analytic continuation method, Wick has been able to elucidate the structure of the relativistic equation for bound states. In particular, the equation acquires an 'elliptic' rather than a 'hyperbolic' metric. Taking advantage of a gap in the rest-mass spectrum of one-nucleon states, a similar investigation is here carried through for the relativistic equation for two-nucleon scattering, in the energy region $2\kappa < E < 2\kappa + \mu$, $\kappa$ and $\mu$ being nucleon and meson masses.

1. Introduction

Wick (1954) has recently made an important advance in the study of the Bethe–Salpeter (1951) amplitude for two-nucleon bound states by an analytic continuation of the amplitude to complex values of the relative time. As a result, the integral equation (in the 'ladder' approximation) for the amplitude acquires a positive definite Euclidian metric and the mathematical discussion of its behaviour takes on a much simpler form.

It is the purpose of this paper to consider the possibility of this type of analytic continuation for the case of neutron-proton scattering. Wick's proof depends essentially on the following remark. If all real states are specified as eigenstates of operators $P_i$ and the operator $M = + (P_0^2 - P_i^2)^{1/2}$, where $P_0$ and $P_i$ ($i = 1, 2, 3$) are the displacement operators for energy and momentum, then for one-nucleon states there is a gap in the spectrum of the rest-mass operator $M$ between $M = 0$ (eigenvalue for vacuum) and $M = \kappa$ (eigenvalue for one free nucleon). The investigation below started from the remark that for one-nucleon states, there is also another gap between $M = \kappa$ and $M = \kappa + \mu$, where $\mu$ is the $\pi$ meson mass. Taking advantage of this gap, we consider the scattering of two nucleons in the (centre of mass system) energy range

$$2\kappa < E < 2\kappa + \mu.*$$

We shall find that a continuation of the integral equation is indeed possible. Our results will also show the intimate connexion of the possibility of this type of continuation and the existence of creation thresholds and displaced poles (Dyson 1949).

2. Equation for the Feynman amplitude for scattering

The Feynman amplitude (Matthews & Salam 1953) for proton-neutron scattering is defined as

$$\Psi(12) = \langle 0 | T\bar{\psi}_p(1)\psi_n(2) | E \rangle.$$  \hspace{1cm} (1)

$| E \rangle$ is the one-proton, one-neutron state of total energy $E$ and zero momentum. It

* It is assumed that no bound states of the $\pi$-meson nucleon system exist. If there are such states, with lowest rest mass $\rho$, our considerations hold for the energy range $2\kappa < E < \kappa + \rho$, rather than $2\kappa < E < 2\kappa + \mu$. 

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is identified with the initial free state \(| \psi \rangle \) at \(-\infty\), by the relation (Freese 1953) (true only for scattering states),
\[
| E \rangle = S(0, -\infty) | \mathbf{K}, -\mathbf{K} \rangle. 
\]
(2)

\( \mathbf{K} \) and \(-\mathbf{K} \) are respectively the free initial proton and neutron momenta, and \( E = 2(\mathbf{K}^2 + \kappa^2)^{1/2} \). \( \Psi(12) \) satisfies an inhomogeneous integral equation which can be directly obtained by a method due to Gell-Mann & Low (1951). Using the relation
\[
| 0 \rangle = S^{-1}(\pm \infty, 0) | 0 \rangle_f, 
\]
(3)
which connects the true vacuum state \(| 0 \rangle \) with the free vacuum \(| 0 \rangle_f \),
\[
\Psi(12) = \int \Psi(0 | TS(\infty, -\infty) \Psi_f(1) \psi_N(2) | \mathbf{K}, -\mathbf{K})_f. 
\]
(4)
\( \psi_f(1), \psi_N(2) \) are interaction representation operators. Equation (4) then leads to an integral equation for \( \Psi(12) \). In the lowest (ladder) approximation,
\[
\Psi(12) = \Psi_0(12) - \frac{i}{\hbar} g^2 \int S_{FP}(13) S_{FN}(24) D_F(34) \Psi(34) d^3 d^4 
\]
(5)
for neutral scalar theory.

Here
\[
\Psi_0(12) = \int \Psi(0 | T \psi_f(1) \psi_N(2) | \mathbf{K}, -\mathbf{K})_f. 
\]
(6)
It will be convenient also to introduce
\[
\chi(12) = D_f(1) D_N(2) \Psi(12) = g^2 \int \Psi(0 | T \psi_f(1) \phi(1) \phi_N(2) \phi(2) | E), 
\]
(7)
where \( D_f(1), D_N(2) \) are Dirac operators. It is easy to see using (5), or directly by the procedure indicated above, that
\[
\chi(12) = \frac{i}{2} D_F(12) \Psi_0(12) - \frac{i}{\hbar} g^2 \int D_F(12) S_{FP}(13) S_{FN}(24) \chi(34) d^3 d^4. 
\]
(8)
Now for \( t_1 > t_2 \)
\[
\chi(12) = \sum_p \int \Psi(0 | \psi_f(1) \phi(1) | p, \kappa) (p, \kappa | \psi_N(2) \phi(2) | E) 
+ \sum_R \int \Psi(0 | \psi_f(1) \phi(1) | R) (R | \psi_N(2) \phi(2) | E). 
\]
Here \(| p, \kappa \rangle \) are the totality of one-proton states with rest mass \( M = \kappa \), while \(| R \rangle \) are the states with other rest masses, the minimum rest mass concerned being \( M = \kappa + \mu \).

We shall now show that
\[
(0 | \psi_f(1) \phi(1) | p, \kappa) = 0. 
\]
(9)
The proof consists in remarking that the one-proton state of momentum \( p \) and rest mass \( \kappa \) is a steady state. This, besides the vacuum state, is the only Heisenberg state, for which, after a self-energy renormalization, a result similar to (3) above holds.

Thus
\[
(0 | \psi_f(1) \phi(1) | p, \kappa) = \int \Psi(\infty, -\infty) \psi_f(1) \psi(1) | p, \kappa)_f. 
\]
(10)
This, on evaluation, gives \( \int dx_1 \Sigma(x - x_1) \int (0 | \psi(x_1) | p, \kappa)_f \), where \( \Sigma(x - x_1) \) is the total contribution from self-energy graphs. After renormalizing mass, and using
\[ \chi(12) = \sum_R \langle 0 | \psi_R(1) \phi(1) | R \rangle \langle R | \psi_R(2) \phi(2) | E \rangle \quad \text{for} \quad t_1 > t_2, \]

\[ = -\sum_R \langle 0 | \psi_R(2) \phi(2) | R \rangle \langle R | \psi_R(1) \phi(1) | E \rangle \quad \text{for} \quad t_2 > t_1. \]

(11)

Since all states \( | R \rangle \) have rest masses greater than \( \kappa + \mu \), Wick’s ‘stability criteria’ apply to our case as well. Writing

\[ \chi(x) = \chi(12) e^{\frac{i}{2} E(t_1 + t_2)}, \]

(12)

we have, for \( t > 0 \),

\[ \chi(x) = \int dp \int_{w_{\text{min}}}^{\infty} dw f(p, w) \exp(i p x - i w t), \]

(13)

where

\[ w = (M^2 + p^2)^{\frac{1}{2}} - \frac{1}{2} E, \]

\[ w_{\text{min}} = \left[ (\kappa + \mu)^2 + p^2^{\frac{1}{2}} - (\kappa + \mu) > \frac{1}{2} \mu > 0 \right]. \]

(14)

\( \chi(x) \) therefore, for \( t > 0 \), is a superposition of positive frequency terms only. Similarly for \( t < 0 \), \( \chi(x) \) contains only negative frequencies. Following Wick, we set

\[ \phi_1(p, p_0) = \frac{1}{(2\pi)^2} \int d\xi \ e^{-ip\xi} \int_0^{\infty} dt \ e^{ipxt} \chi(x) \]

\[ = \frac{1}{2\pi i} \int_{w_{\text{min}}}^{\infty} dw f(p, w)/(w - p_0 - i\epsilon), \]

(15)

with a corresponding definition for \( \phi_2 \).

\[ \phi_1(p, p_0) \] defines an analytic function of \( p_0 \) in the entire complex plane in the region \( -2\pi < \arg(p_0 - w_{\text{min}}) \leq 0 \), while \( \phi_2 \) defines an analytic function in \( -\pi < \arg(p_0 - w_{\text{max}}) < \pi; \) \( w_{\text{max}} = -w_{\text{min}} \). The function \( \phi(p, p_0) = \phi_1 + \phi_2 \), the Fourier transform of \( \chi \), is defined in the complex \( p_0 \) plane with two cuts from \( w_{\text{min}} \) to \( +\infty \) and from \( -\infty \) to \( w_{\text{max}} \). Analytic continuation from the lower to the upper half-plane is ensured through the gap between \( w_{\text{max}} \) and \( w_{\text{min}} \).

3. Transformation of the Integral Equation

Write

\[ \psi_0 = u(E) u_N(-K) e^{iK(x_1 - x_2)} e^{-i\frac{1}{2} E(t_1 + t_2)}. \]

(16)

In momentum space, (8) reduces to

\[ \Phi(p) = \frac{1}{4} \ u(K) d(p - K, p_0) \]

\[ + \frac{i \gamma^2}{(2\pi)^4} \int d(q - q_0) s(p - q, q_0 + \frac{1}{2} E) s_N(q, -q_0 + \frac{1}{2} E) \Phi(q) d q d q_0. \]

(17)

Here

\[ d(k, k_0) = (k^2 - k_0^2 + \mu^2 - i\epsilon)^{-1}, \]

\[ s(k, k_0) = (\gamma k - \gamma_0 k_0 - i\epsilon)^{-1}, \gamma_0^2 = -\gamma_0 = 1. \]

(18)

The analytic continuation of the left-hand side has been considered. On the right-hand side the first term has poles at \( p_0 = \pm [(p - K)^2 + \mu^2]^{\frac{1}{2}} + i\epsilon \). These poles never lie within the gap from \( w_{\text{max}} \) to \( w_{\text{min}} \). In fact, they always lie on the cut lines, and thus need not be considered any further.
The $q_0$ integration in (17) is along the real axis, the integrand having six poles. We want to rotate the contour so that it lies along an imaginary axis.

The kernel that Wick considered had just one factor $d(p - q_0, p_0 - q_0)$. Taking advantage of the fact that $d$ is a function of $p_0$ as well as of $q_0$, Wick was able to show that the poles of $d$ cause no real difficulties in the attempt to rotate the contour in $q_0$ plane. In our case, besides $d$, there are two additional factors $s_p$ and $s_N$, which are functions of $q_0$ alone. It is the poles of these at

$$q_0 = \frac{1}{2} E + (q^2 + \kappa^2)^{\frac{1}{2}} \pm i\epsilon \quad \text{and} \quad q_0 = -\frac{1}{2} E + (q^2 + \kappa^2)^{\frac{1}{2}} \pm i\epsilon$$

which have to be considered (figure 1). It is clear that if the contour is closed so as to lie along $q_0 = \frac{1}{2} \mu + iq_4$, no pole is enclosed in the first quadrant, while the pole at

\[ q_0' = -\frac{1}{2} E + (q^2 + \kappa^2)^{\frac{1}{2}} - i\epsilon \]

is enclosed only for $\text{Re}(q_0') < \frac{1}{2} \mu$. The residue $R$, at this pole, gives

$$R = \frac{-g^2}{16\pi^3} \int d(p - q, p_0 - \frac{1}{2} E - \sqrt{(q^2 + \kappa^2)}) s_p(q, \sqrt{(q^2 + \kappa^2)})$$

$$\times \frac{\gamma q - \gamma_0(\sqrt{(q^2 + \kappa^2) - \frac{1}{2} E})}{(\sqrt{(q^2 + \kappa^2) - E - i\epsilon})} \Phi(q, \sqrt{(q^2 + \kappa^2) - \frac{1}{2} E}) dq,$$  \hspace{1cm} (19)

the integrating being over the volume of the finite sphere

$$\sqrt{(q^2 + \kappa^2) < \frac{1}{2}(E + \mu)}.$$

The integrand in (19) has the expected behaviour, at $(q^2 + \kappa^2)^{\frac{1}{2}} = \frac{1}{2} E$, corresponding to the coincidence of poles in (17). For $q_0 = 0$, $(q^2 + \kappa^2)^{\frac{1}{2}} = \frac{1}{2} E$, real nucleon emission and rescattering is possible and the singularity of the integrand, with its behaviour as a $\delta$ function plus a principal part integration, correctly describes just this possibility. A detailed discussion of this has been given by Eden (1950) and Hamilton (1952).* The result of this rotation then is

$$\Phi(p, p_0) = \frac{1}{i} u(K) d(p - K, p_0) \frac{g^2}{(2\pi)^4} P \int d'(q - q_4) q_4 \{ -q - q_4 + \frac{1}{2}(\mu - E) \} \Phi(q, i\mu + q_4) dq dq_4 + R.$$  \hspace{1cm} (20)

Here the $q_4$ integration goes from $-\infty$ to $+\infty$;

$$d'(q, q_4) = (q^2 + q_4^2 + \mu^2)^{-1}, \quad s'(q, q_4) = (\gamma q + \gamma_4 q_4 - ik)^{-1},$$

* If we had closed the contour along the imaginary axis itself, the poles in (17) would always coincide on the path of integration. Our choice of the ordinate $q_0 = \frac{1}{2} \mu + iq_4$, makes them coincide at an interior point of the sphere $(q^2 + \kappa^2)^{\frac{1}{2}} < \frac{1}{2}(E + \mu)$.
and \( \gamma_4 = -i\gamma_0 \). One of the denominators in the integrand,
\[
[q^2 + (q_4 - \frac{i}{2}(\mu + E))^2 + \kappa^2]
\]
can vanish for the two-dimensional surface \( q_4 = 0 \), \( q^2 + \kappa^2 = \frac{1}{2}(E + \mu)^2 \), and the principal value \( P \) refers to this possibility. The integration in \( R \) is a three-dimensional integration extending through the region \( q^2 + \kappa^2 < \frac{1}{2}(E + \mu)^2 \).

Finally, an analytic continuation in the \( p_0 \) plane can now be carried through. We choose to continue from real \( p_0 > \frac{1}{2} \mu \) to \( p_0 = ip_4 + \frac{1}{2} \mu \) \((p_4 > 0)\); and for \( p_0 < \frac{1}{2} \mu \) to \( p_0 = \frac{1}{2} \mu + ip_4 \) \((p_4 < 0)\). Defining \( \Psi'(p, p_4) = i\Phi(p, \frac{1}{2} \mu + ip_4) \), we obtain
\[
\Psi'(p, p_4) = u(K) d'(p - K, p_4 - \frac{1}{2}i\mu) - \frac{g^2}{(2\pi)^4} P \int d'(p - q, p_4 - q_4) \times \frac{s}{E}(q, q_4 - \frac{1}{2}(\mu + E)) s_N(-q, -q_4 + \frac{1}{2}(\mu - E)) \Psi'(q, q_4) dq dq_4
+ iK \text{ (with its argument } p_0 \text{ changed to } ip_4 + \frac{1}{2} \mu). \tag{22}
\]
We have thus succeeded in obtaining an elliptic metric. As already stated, in the second term on the right there is a two-dimensional surface (in four-dimensional space) where one of the denominators in the integrand vanishes. The real price, however, has been paid in the appearance of the term \( R \) with its three-dimensional integration over a finite sphere.

Mixed integral equations of the type (22) are in principle exactly soluble (see Appendix). In practice, however, one would set up an iteration between the second and the third terms on the right-hand side. Thus, as a first step, the inhomogeneous term, \( ud' \), is substituted for \( \Psi' \) in \( R \), and the three-dimensional integration performed. The sum of \( ud' \) and \( R \) now serves as a new inhomogeneous term for the four-dimensional integral equation. Physically, this means that we are treating all virtual processes properly, but are limiting ourselves to the case when a formation and rescattering of real nucleons is allowed only once in the interaction zone.

To calculate the scattering cross-section, we require \( \Phi(p, p_0) \) for \( p = K' \) and \( p_0 = 0 \), where \( K', -K' \) are the final nucleon momenta. In other words we need \( \Psi'(K', \frac{1}{2}i\mu) \). On account of the analytic properties of \( \Psi' \) however, it would suffice to solve (22) for a real argument \( p_4 \) and to continue to the complex value \( p_4 = \frac{1}{2}i\mu \). If (22) is solved numerically, this can be achieved as follows. If in the \( q_0 \) plane we rotate the contour on to \( q_0 = -\frac{1}{2} \mu + ip_4 \) (rather than \( q_0 = \frac{1}{2} \mu + iq_4 \) we obtain an equation analogous to (22) for \( \Phi(p, -\frac{1}{2} \mu + ip_4) \), which need not be written down. Setting
\[
\Psi''(p, p_4) = i\Phi(p, -\frac{1}{2} \mu + ip_4), \tag{23}
\]
this equation for \( \Psi'' \) also can be solved for a real argument \( p_4 \). Since \( \Phi(K', p_0) \) is an analytic function of \( p_0 \) in the strip enclosed by the ordinates \(-\frac{1}{2} \mu + ip_4 \) and \( \frac{1}{2} \mu + ip_4 \)
\[
\Phi(K', 0) = \frac{1}{2\pi i} \int_C \frac{\Phi(K', p_0)}{p_0} dp_0, \tag{24}
\]
where \( C \) is the closed contour along these ordinates. Thus
\[
\Phi(K', 0) = \frac{1}{2\pi i} \int \frac{\Phi(K', \frac{1}{2} \mu + ip_4)}{p_4 - \frac{i}{2} \mu} - \frac{\Phi(K', -\frac{1}{2} \mu + ip_4)}{p_4 + \frac{i}{2} \mu} dp_4
= \frac{1}{2\pi} \int \left[ \Psi''(K', p_4) - \Psi''(K', -p_4) \right] dp_4. \tag{25}
\]
The procedure described in this paper works equally well for the meson-nucleon case, in the energy range $\kappa + \mu < E < \kappa + 2\mu$. The resulting equation does not suffer from the renormalization difficulties of the three-dimensional Tamm–Dancoff methods. In the end, it may be pointed out, that unlike the case considered in the preceding paragraph (calculation of the scattering cross-section), there are a number of problems where the amplitude $\chi(x)$ is needed for all values of the relative time. One such case for example, is the relativistic treatment of the deuteron photo-disintegration problem.

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APPENDIX

Integral equation (22) is a mixed equation of the type

$$F(x_1, x_2) = f(x_1, x_2) + \lambda \int K_1(x_1, x_2; y_1, y_2) F(y_1, y_2) dy_1 dy_2$$

$$+ \lambda \int K_2(x_1, x_2; y_1, g(y_1)) F(y_1, g(y_1)) dy_1,$$

or symbolically

$$F = f + \lambda K_1 F + \lambda \mathcal{X}_2 \mathcal{F}.$$  

The script type of $\mathcal{X}_2$ and $\mathcal{F}$ emphasizes the partial nature of the integral operation. Such equations can be solved exactly by methods used by Salam (1953). Assume that the Fredholm determinant $d_1(\lambda)$ for the kernel $\lambda K_1$ exists. Then

$$F = \frac{1}{1 - \lambda K_1} \left[ f + \lambda \mathcal{X}_2 \mathcal{F} \right] = f' + \mathcal{X}' \mathcal{F}.$$  

Setting $x_2 = g(x_1)$ in $F, f'$ and $\mathcal{X}'$, we immediately get an integral equation for $\mathcal{F}$, which is solved by conventional methods.

The kernel in the integral equation (22) is singular, in the sense that all its traces diverge logarithmically. A modification of $\frac{1}{q^2 + \mu^2}$ to $\frac{1}{q^2 + \mu^2} - \frac{1}{q^2 + M^2}$ removes this difficulty. Here $M$ is taken to be arbitrarily large. This type of Feynman cut-off seems justified for scattering problems, where no problems of the existence of the solution are involved (Goldstein 1953).

References

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