Quantum theory of Einstein-Bose particles and nuclear interaction

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1. Introduction

The description of nuclear interaction in terms of a neutron-proton “exchange force” appears to be well justified and generally accepted. However, there has hitherto been no satisfactory suggestion as to the nature of the field of charged particles, the virtual emission and reabsorption of which, according to Heisenberg’s (1932) picture, would give rise to this type of interaction. It may be now considered certain that this field is not identical with the electron-neutrino field of Fermi’s theory, the magnitude of nuclear forces being far too large to be compatible with the small empirical value of the constant of β-decay.

As an alternative and simpler description of the nuclear field Yukawa (1935) put forward the idea that the interaction is transmitted by charged particles obeying Einstein-Bose statistics. He showed that the resulting nuclear potential would be proportional to \( r^{-1} \exp(-2\pi m_0 cr/h) \), \( m_0 \) being the rest mass of the Bose particles. Thus, forces of a correct range would be obtained with a rest mass about 100 times that of the electron.

The apparent discovery of particles ("heavy electrons") with a mass of this order of magnitude in cosmic radiation by Neddermeyer and Anderson (1937) has aroused considerable interest in Yukawa’s suggestion, and various aspects of this possibility have been discussed by a number of authors (Yukawa 1935, 1937; Yukawa and Sakata 1937; Oppenheimer and Serber 1937; Stueckelberg 1937; Fröhlich and Heitler 1938; Kemmer 1938; Bhabha 1938b). In particular Yukawa and Sakata (1937) have studied one possible form of the theory in considerable detail.

It is the purpose of this paper to consider this theory from a more general point of view. There are various ways of generalizing Yukawa’s treatment, the most important concerning the spin of the new particle. From the mechanism suggested it is clear that the spin must be taken to be integral, that is, \( 2n \) times that of the neutron or proton. Yukawa’s equations are one way of describing the case \( n = 0 \). For this value there is a second alter-
native theory, and there are also two independent possibilities for \( n = 1 \). On the other hand, a consistent theory for higher values of \( n \) does not appear possible.

The expression for the interaction of the neutrons and protons with the Bose particles will also be taken in a very general form, a generalization being possible even in the case treated by Yukawa. It will be shown that the similarity of all the cases studied is so great that \emph{a priori} it seems impossible to discriminate in favour of any of them. A decision can, however, be reached if one evaluates the neutron-proton force in all the different cases. Comparing with experimental data, only one possibility proves tenable, and it is of importance to note that this is a case with spin and not the scalar case hitherto usually considered.

In an accompanying paper by Fröhlich, Heitler and Kemmer (1938) this case has been selected for more detailed treatment. As far as comparison with experiment is possible in view of the present rather unsatisfactory state of quantum field theories, the account of known data given by the formalism proposed can be regarded as very good.

2. \textsc{Proca's equations and their quantization}

The wave equations for particles with a spin value twice that of the electron or proton have already been used in other investigations. In particular the theory proposed by Proca (1936) contains much of the formalism used in the following. As an introduction to the systematic development to follow in the next section, we will study and quantize the equations of Proca.† The interaction with heavy particles will not yet be considered here. For the scalar case used by Yukawa the analogous quantization had already been performed earlier by Pauli and Weisskopf (1934). Apart from the difference in the spin eigenvalue the results of the present procedure are found to be identical with theirs.

The wave equations of Proca can be written in the following form:

\[
\frac{\partial \phi_\beta}{\partial x_\alpha} - \frac{\partial \phi_\alpha}{\partial x_\beta} = \kappa \chi_{\alpha\beta},
\]

(1)

\[
\frac{\partial \chi^{\alpha\beta}}{\partial x_\alpha} = \kappa \phi^\beta.
\]

(2)

† It has just come to the writer's notice that the question dealt with in this section has recently been treated independently by Durandin and Erschow (1937). Their results are obviously identical with ours, but it cannot be judged from the short publication available, whether their methods are the same. For methodical reasons we still prefer to include this section.
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Here \( x_0 = c t \), and for any tensor component \( a_0 = - a^0, a_i = a^i (i = 1, 2, 3) \). Greek suffixes are allowed all values from 0 to 3, whereas Latin suffixes run from 1 to 3 only. Unless they occur as summation variables the suffixes \( i, k \) and \( l \) will in future always denote the values 1, 2, 3 in cyclic permutation.

Proca's equations are obviously closely related to Maxwell's equations, the main difference being that the "potentials" \( \phi_\alpha \) recur on the right-hand side of the second set (2). Apart from that a difference consists in the fact that all quantities are here assumed to be complex.

We can immediately deduce the existence of a 4-vector that satisfies an equation of continuity, namely

\[
\mathbf{s}_\alpha = \frac{e c i}{\hbar} (\chi_{\alpha\beta} \phi^\beta - \chi_{\alpha\beta} \phi^{\beta*}).
\]  

(3)

Thus \( \mathbf{s}_\alpha \) can be taken to represent the current density of our particles. Similarly, one can deduce the existence of an energy-momentum tensor. We will here only put down its 00-component, the energy density:

\[
T_{00} = c \kappa (\phi_0^* \phi_0 + \phi_i^* \phi_i + \chi_{0i} \chi_{0i} + \frac{1}{2} \chi_{ik} \chi_{ik}).
\]  

(4)

It is satisfactory that this quantity is essentially positive.

The theory can now be easily stated in Hamiltonian form. Let us express \( T_{00} \) in terms of the six quantities \( \phi_i \) and \( \chi_{0i} \) together with their complex conjugates and their spatial derivatives alone. This can be done by using the equations

\[
\kappa \chi_{ik} = \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k} = (\text{curl} \phi)_i
\]  

(5)

and

\[
\kappa \phi_0 = - \frac{\partial \chi_{0i}}{\partial x_i} = - \text{div} \chi_0
\]  

(6)

and their complex conjugates. We thus obtain

\[
H = \int \! T_{00} \! dV = c \kappa \int \! dV \left[ |\mathbf{\phi}|^2 + |\chi_0|^2 + \frac{1}{\kappa^2} \left| \text{div} \chi_0 \right|^2 + \frac{1}{\kappa^2} |\text{curl} \phi|^2 \right].
\]  

(7)

Equations (5) and (6), which alone of equations (1) and (2) do not involve time derivatives, can be regarded as definitions of \( \chi_{ik} \) and \( \phi_0 \) respectively, and the latter quantities can in this way be completely eliminated as variables of our theory. The possibility of doing this for \( \phi_0 \) is the essential difference from electrodynamics.

It is now seen immediately that we can obtain the remaining equations of (1) and (2) as the canonical equations of our Hamiltonian system. We
must take the $\phi_i$ and $\phi_i^*$ to be the canonical co-ordinates $q$, the $\lambda_{0i}^*$ and $\lambda_{0i}$ being the corresponding momenta $p$.

The equation $\partial H/\partial p = \dot{q}$ here takes the form

$$\frac{\partial \phi_i}{\partial x_0} = \kappa \lambda_{0i} - \frac{1}{\kappa} \frac{\partial}{\partial x_i} \text{div} \lambda,$$

and from the relation $\partial H/\partial q = -\dot{p}$ we obtain

$$-\frac{\partial \lambda_{0i}}{\partial x_0} = \kappa \phi_i + \frac{1}{\kappa} \frac{\partial}{\partial x_k} \text{curl}_i \phi - \frac{1}{\kappa} \frac{\partial}{\partial x_l} \text{curl}_k \phi.$$  

With the use of the definitions (5) and (6) we thus get

$$\frac{\partial \phi_i}{\partial x_0} - \frac{\partial \phi_0}{\partial x_i} = \kappa \lambda_{0i}$$

and

$$\frac{\partial \lambda_{0k}}{\partial x_x} = \kappa \phi^k.$$  

Similarly, the conjugate complex relations are derived. Including (5) and (6) we have the complete set of Proca's equations.

Having found the Hamiltonian form, quantization is simple. We postulate the commutation rules

$$[\phi_i(x), \lambda_{0k}^*(x')] = [\phi_i^*(x), \lambda_{0k}(x')] = -\hbar \gamma^{ik} \delta(x - x'),$$

while all other pairs of quantities are taken to commute. Their invariance can be proved by the usual methods. Assuming (12), the equations of motion (8) and (9) can be derived by means of the relations

$$[H, \phi_i] = \hbar \gamma^{ik} \phi_i$$

and their conjugates.

It should be observed that by making use of (6) we can here avoid the special considerations for $\phi_0$ that are necessary in electrodynamics. This represents a considerable simplification.$^\dagger$

We will now divide each field variable into a "transversal" and a "longitudinal" part:

$$\phi_i = \psi_i + \Pi_i^*$$

with $\text{div} \psi = 0$ and $\text{curl} \Pi^* = 0$.

$^\dagger$ In proving the invariance of (12) it is however initially necessary to proceed as in radiation theory, considering $\phi_0$ as a separate variable. This form of the theory has been dealt with in more detail by Bhabha (1938b).
and similarly
\[ \chi_{0i}^* = \pi_i - \Psi_i^* \]  
with \( \text{div} \pi = 0 \) and \( \text{curl} \Psi^* = 0 \).

The Hamiltonian can now be represented by
\[ H = c \kappa \int dV \left[ |\vec{\psi}|^2 + |\pi|^2 + \frac{1}{k^2} |\text{curl} \psi|^2 + |\vec{\Psi}|^2 + \frac{1}{k^2} |\text{div} \Psi|^2 \right], \]  
with the commutation rules
\[ [\pi_i(x), \psi_k(x')] = [\pi_i^*(x), \psi_k^*(x')] = [\Pi_i(x), \Psi_k(x')] = [\Pi_i^*(x), \Psi_k^*(x')] = -\frac{i\hbar}{\kappa} \delta_{ik} \delta(x - x'), \]  
all other combinations commuting. We transform to momentum space, putting
\[
\psi_i = \frac{1}{\sqrt{L^3}} \sum_k q_i(k) \exp(-ikx), \quad \pi_i = \frac{1}{\sqrt{L^3}} \sum_k p_i(k) \exp(+ikx),
\]
\[ \left( k_i = \frac{2\pi n_i}{L}, \, n_i = \text{integer} \right), \]  
\[ \Psi_i = \frac{1}{\sqrt{L^3}} \sum_k Q_i(k) \exp(-ikx), \quad \Pi_i = \frac{1}{\sqrt{L^3}} \sum_k P_i(k) \exp(+ikx), \]  
and denoting the complex conjugates correspondingly. If we now choose the directions of the co-ordinate axes for each Fourier component separately, we can do this in such a way that
\[ q_3 = Q_1 = Q_2 = p_3 = P_1 = P_2 = 0. \]

Further we put
\[ q_3 = q^{(0)}, \quad \frac{1}{\sqrt{2}} (q_1 + iq_2) = q^{(+1)}, \quad \frac{1}{\sqrt{2}} (q_1 - iq_2) = q^{(-1)}, \]  
\[ p_3 = p^{(0)}, \quad \frac{1}{\sqrt{2}} (p_1 - ip_2) = p^{(+1)}, \quad \frac{1}{\sqrt{2}} (p_1 + ip_2) = p^{(-1)}. \]

The energy then becomes
\[ H = c \kappa \sum_{k} \sum_{\epsilon = -1}^{+1} \left[ q^{(\epsilon)*}(k) q^{(\epsilon)}(k) \left( 1 + \frac{k^2}{\kappa^2} \right) + p^{(\epsilon)*}(k) p^{(\epsilon)}(k) \right]. \]
Similarly, the total charge can be shown to be

$$\frac{1}{c} \int s^0 dV = -\frac{e_i}{\hbar} \sum_{k} \sum_{\epsilon = -1}^{+1} [p^{(e)}(k) q^{(e)}(k) - p^{(e)*}(k) q^{(e)*}(k)].$$

We perform yet another transformation of the variables, following the procedure of Pauli and Weisskopf (1934) in the scalar case. We put

$$p^{(e)}(k) = \left( \frac{\hbar}{2\kappa} \right)^{\frac{1}{2}} (k^2 + \kappa^2)^{\frac{1}{4}} [a^{*}_e(k) + b_e(k)],$$
$$q^{(e)}(k) = \left( \frac{\hbar\kappa}{2} \right)^{\frac{1}{2}} \left( k^2 + \kappa^2 \right)^{\frac{1}{4}} [a_e(k) - b^{*}_e(k)].$$

In these variables we find the commutation rules to be

$$[a_e(k), a^{*}_{e'}(k')] = [b_e(k), b^{*}_{e'}(k')] = \delta_{ee'} \delta(k, k'),$$

while all $a(k)$ commute with all $b(k)$, etc. Further, we obtain for the Hamiltonian

$$H = \sum_{k} \sum_{\epsilon = -1}^{+1} \hbar c \sqrt{(k^2 + \kappa^2)} [a^{*}_e(k) a_e(k) + b^{*}_e(k) b_e(k) + 1],$$

and for the total charge

$$\frac{1}{c} \int s^0 dV = e \sum_{k} \sum_{\epsilon = -1}^{+1} [a^{*}_e(k) a_e(k) - b^{*}_e(k) b_e(k)].$$

As a consequence of (22) the operators

$$N^+(k) = a^*(k) a(k) \quad \text{and} \quad N^-(k) = b^*(k) b(k)$$

have all positive integers and 0 as eigenvalues. Thus, taking account of the expressions (23) and (24), we can obviously interpret the $N$ as numbers of particles present in the field, the $N^+$ denoting particles of positive charge, the $N^-$ negatively charged ones. This result is identical with that of the scalar case, except for the additional suffix $\epsilon$ occurring here, which indicates the three possible values that the spin component in the $k$-direction can assume. As the $N$ are not restricted to the values 0 and 1, the particles have Bose statistics.

The energy of a particle of the kind $k$ is $E(k) = \hbar c \sqrt{(k^2 + \kappa^2)} T$, its rest mass therefore $m_0 = \kappa \hbar / c$. It is of importance to note that, in spite of the formal analogy in the final result, the treatment of the "longitudinal" part of the field, which gives the quanta with $\epsilon = 0$, was necessarily separate from that of the "transversal" field which gives the other two spin orientations. The latter part corresponds exactly to the electrodynamical case, whereas there is no analogy to the former there.
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Proca has already given the extension of his formulae for the case of the presence of electromagnetic fields. His equations then are

\[ \left( \frac{\partial}{\partial x_\alpha} - \frac{ie}{\hbar c} A_\alpha \right) \phi_\beta - \left( \frac{\partial}{\partial x_\beta} - \frac{ie}{\hbar c} A_\beta \right) \phi_\alpha = \kappa \chi_{\alpha\beta}, \] (25)

\[ \left( \frac{\partial}{\partial x_\alpha} - \frac{ie}{\hbar c} A_\alpha \right) \chi^{\alpha\beta} = \kappa \phi_\beta, \] (26)

the \( A_\alpha \) being the electromagnetic potentials. Just as in the scalar case the addition of the potentials makes no difference to the procedure of quantization. The commutation rules between the \( \phi_i \) and the \( \chi_{\alpha\beta} \) remain the same as before, and the expression for the charge density remains unaltered. The Hamiltonian acquires two additional terms, which are again very similar to those given by Pauli and Weisskopf in the scalar case:

\[ H_1 = -\frac{ie}{\hbar \kappa} \int dV \{ \text{curl } \Psi^* \cdot [A \times (\Psi + \Psi')] + \text{div } \Psi^* [A \cdot (\Psi - \Psi')] \]
\[ - \text{complex conjugate} \]
\[ - \frac{e^2}{\hbar^2 \kappa c} \int dV \{ |A \times (\psi + \Pi^*)|^2 + |A \cdot (\Psi - \pi^*)|^2 \} \] (27)

and

\[ H_2 = \frac{ie}{\hbar} \int dV A^0 (\Psi - \pi^*) \cdot (\psi^* + \Pi) + \text{complex conjugate}. \] (28)

(The symbol \( \cdot \) means scalar multiplication whereas the vectorial product is denoted by \( \times \).)

This formalism appears sufficiently simple to justify the attempt to connect it with experience. Though more general cases will be studied in the following, the final result will be that the case described in detail above is the one best suited to account for the facts of nuclear interaction.

For the rigorous derivation of the expression describing the interaction of Proca particles with neutrons and protons the considerations of §3 are essential. It may, however, be useful to note that in applications the necessary extension consists solely in adding to the Hamiltonian the term given later as equation (63 b). This expression can be understood using the ideas of the present section alone. It gives rise to emission and absorption of heavy electrons by protons and neutrons.

3. General theory of Einstein-Bose particles

In a recent paper Dirac (1936) has studied some wave equations for particles with a spin momentum \( n \) times that of the electron or proton.
Apart from the usual Dirac equation of the electron the simplest case contained in his scheme, for non-vanishing rest mass, is the following (we use spinor notation and take the case of no electromagnetic field only):

\[
\begin{align*}
\frac{p^{\lambda\kappa} A_{\kappa\lambda}}{\sqrt{2}} &= m_0 B^{\lambda}_{\kappa}, \\
\frac{p_{\kappa\lambda} B^{\lambda}_{\kappa}}{\sqrt{2}} &= m_0 A_{\kappa\lambda}.
\end{align*}
\] (29)

This case corresponds to the spin eigenvalue 1 (twice the electron spin). For Dirac's classification the assumption that \( A_{\kappa\lambda} \) is symmetrical in its two suffixes is essential. Let us however compare the equations with those ensuing from the opposite assumption of antisymmetry. For an arbitrary antisymmetrical spinor we can raise one suffix according to the general rule and then put

\[ A^{\kappa}_{\kappa} = A \delta^{\kappa}_{\kappa}, \] (30)

where \( A \) is a scalar. Equations (29) then become

\[
\begin{align*}
\frac{p^{\kappa} A}{\sqrt{2}} &= m_0 B^{\kappa}_{\kappa}, \\
\frac{p_{\kappa} B^{\kappa}_{\kappa}}{\sqrt{2}} &= -m_0 A_{\kappa}. 
\end{align*}
\] (31)

It is not necessary to go into the details of spinor notation here, as we will now abandon it; it was merely introduced to obtain an initial classification of possible wave theories. The transition from equations (29) or (31) to tensor equations can be performed immediately; it is, however, not unique. In the spinor notation no specification of the transformation of the wave functions for reflexions of the co-ordinate system is implied. As a consequence, when reverting to tensor notation we can always choose either of two dual quantities to correspond to any spinor. Thus, one possibility of writing equations (31) in tensor form is

\[
\begin{align*}
\frac{\partial \phi}{\partial x^\alpha} &= \kappa \chi^\alpha, \\
\frac{\partial \chi^\alpha}{\partial x^\alpha} &= \kappa \phi, 
\end{align*}
\] (32a)

\[
(\kappa = m_0 c/h)
\]

and similarly the first possibility for equations (29) is

\[
\begin{align*}
\frac{\partial \phi^\alpha}{\partial x^\alpha} \frac{\partial \phi^\beta}{\partial x^\beta} &= \kappa \chi_{\alpha\beta}, \\
\frac{\partial \chi_{\alpha\beta}}{\partial x^\alpha} &= \kappa \phi^\beta. 
\end{align*}
\] (32b)
Alternatively, however, (29) can be transcribed as

$$\frac{\partial \phi_{\beta\gamma}}{\partial x_{\alpha}} - \frac{\partial \phi_{\alpha\gamma}}{\partial x_{\beta}} + \frac{\partial \phi_{\alpha\beta}}{\partial x_{\gamma}} = \kappa \chi_{\alpha\beta\gamma},$$  \hspace{1cm} (32c)

$$\frac{\partial \chi_{\alpha\beta\gamma}}{\partial x_{\alpha}} = \kappa \phi_{\beta\gamma},$$  \hspace{1cm} (33c)

$$(\phi_{\alpha\beta} = -\phi_{\beta\alpha})$$

and (31) as

$$\frac{\partial \phi_{\beta\gamma\delta}}{\partial x_{\alpha}} - \frac{\partial \phi_{\alpha\gamma\delta}}{\partial x_{\beta}} + \frac{\partial \phi_{\alpha\beta\delta}}{\partial x_{\gamma}} - \frac{\partial \phi_{\alpha\beta\gamma}}{\partial x_{\delta}} = \kappa \chi_{\alpha\beta\gamma\delta},$$  \hspace{1cm} (32d)

$$\frac{\partial \chi_{\alpha\beta\gamma\delta}}{\partial x_{\alpha}} = \kappa \phi_{\beta\gamma\delta},$$  \hspace{1cm} (33d)

$$(\phi_{\alpha\beta\gamma} = -\phi_{\beta\alpha\gamma} = -\phi_{\gamma\alpha\beta}).$$

Now, equations (32a) and (33a) are obviously equivalent to the scalar relativistic Schrödinger equation (Klein-Gordon equation)

$$\left(\sum_{\alpha} \frac{\partial^2}{\partial x_{\alpha}^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \phi = \kappa^2 \phi,$$  \hspace{1cm} (34)

$\chi_{\alpha}$ being merely a symbol for the 4-gradient of the wave function. Wave equation (34) also holds for all the other field quantities in the above four cases, but in general the statement of the second-order relation (34) alone is not sufficient.

(32b) and (33b) are Proca’s equations. We have, however, further found two cases of obviously highly analogous structure and equal simplicity which we have not considered before. It is true, of course, that their difference from the two first cases is to a large extent a matter of notation, the quantity $\chi_{0123}$ in the last set, for instance, behaving like the scalar $\phi$ in case $a$ in all but reflexions (with $\phi_{123} \leftrightarrow -\frac{1}{c} \phi$, $\phi_{01k} \leftrightarrow -\text{grad}_i \phi$). This difference in reflexion character does not in the slightest affect the quantization of the theory; in case $c$ the method of the preceding section, in case $d$ that of Pauli and Weisskopf is immediately applicable, and we obtain particles of spin 1 or zero just as before. Nevertheless, the separate treatment of the dual cases is necessary as soon as the terms describing the interaction of this field with protons and neutrons are introduced, as these are characteristically different in either of two dual cases. From now onwards we will therefore consider the four cases independently.
We begin by noting that the four sets of equations can each be derived from a Lagrangian, namely,

\[ L_0^a = -\kappa (\dot{\phi}^* \phi + \chi^* \chi), \]  
\[ L_0^b = -\kappa (\dot{\phi}^* \phi + \frac{1}{2} \chi^* \chi), \]  
\[ L_0^c = -\kappa (\dot{\phi}^* \phi + \frac{1}{2} \chi^* \chi), \]  
\[ L_0^d = -\kappa (\dot{\phi}^* \phi + \frac{1}{2} \chi^* \chi). \]

Taking \( L^n \) as a function of the independent variables \( \phi \), and expressing the \( \chi \) as derivatives of the \( \phi \) by means of the equations (32n), the equations (33n) can be deduced as variational equations in the usual manner. We can further easily determine the complete expressions for the energy-momentum tensor and the current density vector:

\[ T^a_{\alpha \beta} = (\chi^* \chi_{\alpha \beta} + \text{complex conjugate}) - \delta_{\alpha \beta} L_0^a, \]  
\[ T^b_{\alpha \beta} = (\dot{\phi}^* \phi_{\alpha \beta} + \chi^* \chi_{\alpha \beta} + \text{complex conjugate}) - \delta_{\alpha \beta} L_0^b, \]  
\[ T^c_{\alpha \beta} = (\dot{\phi}^* \phi_{\alpha \beta} + \frac{1}{2} \chi^* \chi_{\alpha \beta} + \text{complex conjugate}) - \delta_{\alpha \beta} L_0^c, \]  
\[ T^d_{\alpha \beta} = (\dot{\phi}^* \phi_{\alpha \beta} + \frac{1}{2} \chi^* \chi_{\alpha \beta} + \text{complex conjugate}) - \delta_{\alpha \beta} L_0^d, \]  
\[ s^a_\alpha = \frac{\epsilon i}{\hbar} (\chi^* \phi - \chi \phi^*), \]  
\[ s^b_\alpha = \frac{\epsilon i}{\hbar} (\chi^* \phi_{\alpha \beta} - \chi_{\alpha \beta} \phi^*), \]  
\[ s^c_\alpha = \frac{\epsilon i}{2\hbar} (\chi^* \phi_{\alpha \beta} - \chi_{\alpha \beta} \phi^*), \]  
\[ s^d_\alpha = \frac{\epsilon i}{6\hbar} (\chi^* \phi_{\alpha \beta} - \chi_{\alpha \beta} \phi^*). \]

In each case the energy density is essentially positive; this is a special characteristic of these simple cases. If we had proceeded from any more complicated examples of Dirac's spinor equations, we would not in general have obtained this result. There thus appears to be some justification for choosing these four cases of simple and similar type for preferential consideration, and it seems certain that any alternative formalism would have to be a great deal more complicated for physically satisfactory results to be expected. A theory for spin values greater than unity therefore immediately leads to serious difficulties. On the other hand, the strength of the analogy among the four possibilities here stated is so evident that it would not
appear justified to exclude any of them a priori. For reasons of economy it is of course to be hoped that a comparison with experience will prove that one of the cases alone is sufficient to give a description of reality.

When one proceeds to the formulation of the law of interaction of the new particles with protons and neutrons one observes yet again that all four cases appear equally adequate for these purposes. The quantities occurring in our possible fundamental equations are one scalar \( \phi \), two 4-vectors \( (\chi_\alpha \text{ and } \phi_\alpha) \), two antisymmetrical tensors of the second order \( (\chi_{\alpha\beta} \text{ and } \phi_{\alpha\beta}) \), two "pseudovectors", that is, totally antisymmetrical tensors of the third order \( (\chi_{\alpha\beta\gamma} \text{ and } \phi_{\alpha\beta\gamma}) \), and finally one "pseudoscalar" \( (\chi_{0123}) \).

We now assume that the protons and neutrons obey the Dirac equation; then, from their wave functions, we can form just one covariant of each of these five types and no others. Denoting the Dirac wave function of the proton by \( \phi_P \), that of the neutron by \( \phi_N \), we can form the five covariants

\[
\begin{align*}
\text{Scalar:} & \quad \Phi_N^* \beta \Phi_P = w, \\
\text{Vector:} & \quad (-\Phi_N^* \beta \Phi_P; \Phi_N^* \alpha_i \Phi_P) = (\nu_i; \nu_i), \\
\text{Antisymmetrical tensor:} & \quad (\Phi_N^* \gamma_i \Phi_P; \Phi_N^* \beta \sigma_i \Phi_P) = (u_{0i}; u_{ik}), \\
\text{Pseudovector:} & \quad (\Phi_N^* \sigma_i \Phi_P; -\Phi_N^* \gamma_5 \Phi_P) = (t_{0ik}; t_{123}), \\
\text{Pseudoscalar:} & \quad \Phi_N^* \beta \gamma_5 \Phi_P = s_{0123}.
\end{align*}
\]

\( \alpha_i, \beta \) are the Dirac matrices, \( \gamma_i = -i \beta \alpha_i, \sigma_i = -i \alpha_i \alpha_k \) and \( \gamma_5 = -i \alpha_1 \alpha_2 \alpha_3 \).

Further, we can of course form the complex conjugates of the above quantities, for instance,

\[ \Phi_P^* \beta \Phi_N = w^*. \]

We will now put down the most general interaction satisfying the following conditions:

(i) The interaction term to be added to the Lagrangian involves only the wave functions of protons and neutrons without their derivatives.

(ii) The field quantities describing the heavy electron which enter the interaction expression are the \( \phi \) and \( \chi \) only, derivatives of these quantities not occurring explicitly.

In each of the four cases there are then two terms that may be added to the Lagrangian if the condition of relativistic invariance is to be satisfied. The first in each case involves the \( \phi \), namely:

\[
\begin{align*}
L_\phi^a &= -c \kappa (g_a w \phi^* + \text{conjugate complex}), \\
L_\phi^b &= -c \kappa (g_b v_\alpha \phi^* + \text{conjugate complex}), \\
L_\phi^c &= -c \kappa (g_c \frac{1}{2} u_{\alpha\beta} \phi^{\alpha\beta*} + \text{conjugate complex}), \\
L_\phi^d &= -c \kappa (g_d \frac{1}{2} t_{\alpha\beta\gamma} \phi^{\alpha\beta\gamma*} + \text{conjugate complex}).
\end{align*}
\]
The other term will contain the $\chi$:

\[ L_f^a = -\kappa f_a v_\alpha \chi^{\alpha \star} + \text{conjugate complex}, \quad (40a) \]

\[ L_f^b = -\kappa f_b \frac{1}{2} u_{\alpha \beta} \chi^{\alpha \beta \star} + \text{conjugate complex}, \quad (40b) \]

\[ L_f^c = -\kappa f_c \delta_{\alpha \beta \gamma} \chi^{\alpha \beta \gamma \star} + \text{conjugate complex}, \quad (40c) \]

\[ L_f^d = -\kappa f_d \delta_{\alpha \beta \gamma \delta} \chi^{\alpha \beta \gamma \delta \star} + \text{conjugate complex}. \quad (40d) \]

The $f_n$ and $g_n$ are arbitrary constants.†

The general interaction will of course be a linear combination of the two terms. The assumptions (i) and (ii) are justified on grounds of simplicity; it might, however, be here objected that the interactions from (40n) are already less simple than those given by (39n), as the $\chi$ are in fact, by (32n), derivatives of the "wave functions" $\phi$. From the example treated in § 2, we have, however, already seen that the theory is quite symmetrical with respect to the $\phi$ and the $\chi$, and that in fact, for the longitudinal field, it is natural to consider $\phi$ as defined by (33n), $\chi$ then being considered the "wave function". Besides, there is complete analogy between the $\phi$ of cases $a$ and $b$ and the $\chi$ of cases $d$ and $c$ respectively, and vice versa, and there is therefore no legitimate reason for considering the second kind of interaction to be less simple. Our present procedure of treating the $\phi$ as wave functions and the $\chi$ as their derivatives is necessary only for formal reasons.

The total Lagrangian can now be assumed to have the form

\[ \mathcal{L} = \int dV [L_0^p + L_0^n + L_f^a + L_f^b + L_F + L_N], \quad (41) \]

where $L_F$ and $L_N$ are the Lagrangians of the free protons and neutrons respectively. These are of course well known, as we assume the Dirac equation to hold for the heavy particles, and we need not put them down explicitly.

Defining the canonical conjugate $p(q)$ of any co-ordinate $q$ in the usual way,

\[ p(q) = \frac{\partial \mathcal{L}}{\partial \dot{q}}, \]

† The quantities defined as $g$ and $f$ in the paper by Fröhlich and others (1938) are related to $g_b$ and $f_b$ by

\[ g = -\left( \frac{\kappa c}{4\pi} \right)^{\frac{1}{2}} g_b, \quad f = +\left( \frac{\kappa c}{4\pi} \right)^{\frac{1}{2}} f_b. \]
we obtain

\[ p(\phi) = \xi_0^* = \lambda_0^* + f_d^* v_0^*, \]  

(42a)

\[ p(\phi_i) = \xi_{0i}^* = \lambda_{0i}^* + f_b^* u_{0i}^*, \]  

(42b)

\[ p(\phi_{ik}) = \xi_{0ik}^* = \lambda_{0ik}^* + f_c^* f_{0ik}, \]  

(42c)

\[ p(\phi_{123}) = \xi_{0123}^* = \lambda_{0123}^* + f_d^* s_{0123}^*, \]  

(42d)

\[ p(\phi_0) = p(\phi_{0i}) = p(\phi_{0ik}) = 0. \]  

(43)

As there are no time derivatives of the proton or neutron wave functions in \( L_0^p \) or \( L_0^n \), there are no additions to the canonical conjugates of the heavy particle wave functions due to the interaction.

We can now pass to the Hamiltonian function by the usual procedure:

\[ H = \sum p_d q_d - \tilde{L}. \]

We take case \( b \) as an example:

\[
H^b = c \kappa \int dV \left[ \phi_i^* \phi_i + \phi_0^* \phi_0 + \frac{1}{2} \lambda_i^* \lambda_{ik} + \zeta_{0i}^* \zeta_{0i} + (g_b v_i \phi_i^* + f_b \frac{1}{2} u_{ik} \lambda_{ik} - f_b u_{0i} \zeta_{0i}^* + \text{conjugate complex}) + |f_b|^2 u_{0i}^* u_{0i} - \left\{ \phi_0^* \left( \phi_0 + v_0 g - \frac{1}{\kappa} \frac{\partial}{\partial x_i} \zeta_{0i} \right) + \text{conjugate complex} \right\} \right].
\]

(44)

The final term is derived by means of a partial integration. This form corresponds to the final form of the Hamiltonian of electrodynamics. However, here again, as in the simple case studied in § 2, we can consider the equation

\[
\frac{\partial \zeta_{0i}}{\partial x_i} = \kappa(\phi_0 + v_0 g),
\]

(45)

not as a supplementary condition, but as the definition of \( \phi_0 \), and we proceed to eliminate this quantity from the Hamiltonian. \( H \) is then a function of \( \phi_i \) and \( \zeta_{0i} \) only. Whereas such a consideration is not necessary in case \( a \), it reappears in an almost identical form in cases \( c \) and \( d \). There we will use the abbreviations

\[
\kappa \eta_0 = - \frac{\partial \zeta_{0i}}{\partial x_i}, \ \kappa \eta_{0i} = + \frac{\partial \zeta_{0ik}}{\partial x_k}, \ \kappa \eta_{0ik} = - \frac{\partial \zeta_{0ik}}{\partial x_l}.
\]

(46)
The four Hamiltonians can then be expressed as

\[ H^n = H_0^n + H_1^n + H_2^n, \]  
\[ H_0^n = c_k \int dV [\phi^* \phi + \xi^* \zeta_0 + \chi^* \chi_i], \]  
\[ H_0 = c_k \int dV [\phi_i^* \phi_i + \eta^* \eta_0 + \xi_{0i} \zeta_{0i} + \frac{1}{2} \chi_{ik} \chi_{ik}], \]  
\[ H_c = c_k \int dV [\frac{1}{2} \phi_i^* \phi_i + \eta_{0i} \eta_0 + \frac{1}{2} \xi_{0ik} \zeta_{0ik} + \frac{1}{2} \chi_{ik} \chi_{ik}], \]  
\[ H^d = c_k \int dV [\frac{1}{2} \phi_{ik}^* \phi_{ik} + \eta_{0ik} \eta_0 + \frac{1}{2} \xi_{0ik} \zeta_{0ik}], \]  
\[ H_0^a = c_k \int dV [a \phi^* - f_a (v_0 \xi_0^* - v_i \chi_i^*) + \text{conjugate complex}], \]  
\[ H_0^b = c_k \int dV [b (v_i \phi_i^* - v_0 \eta_0^*) - f_b (u_{0i} \xi_{0i}^* - \frac{1}{2} u_{ik} \chi_{ik}) + \text{conjugate complex}], \]  
\[ H_0^c = c_k \int dV [c (u_{ik} \phi_{ik}^* - u_{0i} \eta_0^*) - f_c (\frac{1}{2} \xi_{0ik} \zeta_{0ik} - \frac{1}{2} \xi_{ik} \chi_{ik}) + \text{conjugate complex}], \]  
\[ H_0^d = c_k \int dV [d (\frac{1}{2} \xi_{ik} \phi_{ik}^* - \frac{1}{2} \xi_{0ik} \xi_{0ik}) - f_d \xi_{0ik} \zeta_{0ik} + \text{conjugate complex}], \]  
\[ H_2^a = |f_a|^2 \xi_0^* \xi_{0i}, \]  
\[ H_2^b = |g_b|^2 \xi_0^* \xi_{0i} + |f_b|^2 \xi_0^* \eta_{0i}, \]  
\[ H_2^c = |g_c|^2 \xi_0^* \eta_{0i} + |f_c|^2 \frac{1}{2} \xi_{0ik} \zeta_{0ik}, \]  
\[ H_2^d = |g_d|^2 \frac{1}{2} \xi_{0ik} \xi_{0ik} + |f_d|^2 \frac{1}{2} \xi_{0ik} \xi_{0ik}. \]

The Hamiltonians of the free heavy particles have been omitted. To the Hamiltonians we must adjoin the definitions

\[ \zeta_i - f_a \xi_i = \chi_i = \frac{1}{\kappa} \frac{\partial \phi}{\partial x_i}; \]  
\[ \zeta_{ik} - f_b u_{ik} = \chi_{ik} = \frac{1}{\kappa} \left( \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_i}{\partial x_k} \right); \]  
\[ \zeta_{ikl} - f_c \xi_{ikl} = \chi_{ikl} = \frac{1}{\kappa} \left( \frac{\partial \phi_{kl}}{\partial x_i} + \frac{\partial \phi_i}{\partial x_{kl}} + \frac{\partial \phi_{ik}}{\partial x_k} \right); \]

\[ \phi_{0ik} + g_d u_{0ik} = \eta_{0ik} = -\frac{1}{\kappa} \frac{\partial \zeta_{0ik}}{\partial x_i}. \]
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and then, in the usual way, we obtain the canonical equations

$$\frac{1}{K} \frac{\partial \phi}{\partial x_0} = \xi_0 - f_a v_0; \quad \frac{1}{K} \left( - \frac{\partial \xi_0}{\partial x_0} + \frac{\partial \zeta_i}{\partial x_i} \right) = \phi + g_a w; \quad (52a)$$

$$\frac{1}{K} \left( \frac{\partial \phi_i}{\partial x_0} - \frac{\partial \phi_i}{\partial x_i} \right) = \zeta_{oi} - f_b u_{oi}; \quad \frac{1}{K} \left( - \frac{\partial \zeta_{oi}}{\partial x_0} + \frac{\partial \zeta_{ki}}{\partial x_k} \right) = \phi_i + g_b v_i; \quad (52b)$$

$$\frac{1}{K} \left( \frac{\partial \phi_{ik}}{\partial x_0} - \frac{\partial \phi_{ik}}{\partial x_i} + \frac{\partial \phi_{ik}}{\partial x_k} \right) = \zeta_{0ik} - f_c t_{0ik}; \quad \frac{1}{K} \left( - \frac{\partial \zeta_{0ik}}{\partial x_0} + \frac{\partial \zeta_{ilk}}{\partial x_l} \right) = \phi_{ik} + g_c u_{ik}; \quad (52c)$$

$$\frac{1}{K} \left( \frac{\partial \phi_{ikt}}{\partial x_0} - \frac{\partial \phi_{ikt}}{\partial x_i} + \frac{\partial \phi_{ikt}}{\partial x_k} - \frac{\partial \phi_{ikt}}{\partial x_l} \right) = \zeta_{0ikt} - f_d t_{0ikt}; \quad \frac{1}{K} \left( - \frac{\partial \zeta_{0ikt}}{\partial x_0} + \frac{\partial \zeta_{ilk}}{\partial x_l} \right) = \phi_{ikt} + g_d t_{ikt}. \quad (52d)$$

The relativistic invariance of the scheme is evident.

In cases c and d the present notation is awkward. For any antisymmetrical tensor let us by definition put

$$\bar{a}_0 = a_{ik}, \quad \bar{a}_{ik} = -a_{ik}, \quad (53)$$

for any pseudovector

$$\bar{b}_0 = b_{123}, \quad \bar{b}_i = b_{0ik}, \quad (54)$$

and for a pseudoscalar

$$\bar{c} = c_{0123}. \quad (55)$$

The expressions (48) to (50) for the Hamiltonians then become

$$H_0^c = cK \int dV[\overline{\xi_1} \overline{\xi_2} + \overline{\chi_0} \overline{\chi_0} + \overline{\phi_0} \overline{\phi_0} + \overline{\eta_{0ik}} \overline{\eta_{0ik}}], \quad (56c)$$

$$H_0^d = cK \int dV[\overline{\xi_1} \overline{\xi_1} + \overline{\chi_0} \overline{\chi_0} + \overline{\eta_{0ik}} \overline{\eta_{0ik}}], \quad (56d)$$

$$H_1^c = -cK \int dV[f_c (\overline{\xi_1} \overline{\xi_1} - \overline{\phi_0} \overline{\phi_0} - \overline{\eta_{0ik}} \overline{\eta_{0ik}}) + \text{conjugate complex}], \quad (57c)$$

$$H_1^d = -cK \int dV[f_d \overline{\xi_1} \overline{\xi_1} - g_d (\overline{\phi_0} \overline{\phi_0} - \overline{\eta_{0ik}} \overline{\eta_{0ik}}) + \text{conjugate complex}], \quad (57d)$$

$$H_2^c = |f_c|^2 \overline{\xi_1} \overline{\xi_1} + |g_c|^2 \overline{\eta_{0ik}} \overline{\eta_{0ik}}, \quad (58c)$$

$$H_2^d = |f_d|^2 \overline{\xi_1} \overline{\xi_1} + |g_d|^2 \overline{\eta_{0ik}} \overline{\eta_{0ik}}. \quad (58d)$$

The form of (48a, b)–(50a, b) is very similar to (56c, d)–(58c, d) respectively, there being a correspondence between the following sets of quantities

$$\phi \leftrightarrow \tilde{\xi}, \quad \eta \leftrightarrow \tilde{\chi}, \quad \zeta \leftrightarrow \tilde{\phi}, \quad \chi \leftrightarrow \tilde{\eta}, \quad g_c \leftrightarrow -f_0, \quad g_d \leftrightarrow -f_a, \quad f_c \leftrightarrow -g_b, \quad f_d \leftrightarrow -g_a. \quad (59)$$
However, this correspondence is not quite complete. Instead of the terms (58c) and (58d) actually found one would, from analogy with (50b) and (50a), have expected

\[ H'_2^c = |f_c|^2 \tilde{t}_0^* t_0 + |g_c|^2 \tilde{w}_{0i}^* w_{0i}, \]  

(59c)

and

\[ H'_2^d = |g_d|^2 \tilde{t}_0^* t_0. \]  

(59d)

This is the form of the terms quadratic in the \( f_n \) or \( g_n \) which we would have obtained if we had neglected to treat the dual cases separately but had put down the results for them from analogy. The reason for the difference is however very simple. In our original Lagrangian we assumed the \( \phi \) to be the independent variables, or in our new notation for cases \( c \) and \( d \) the \( \tilde{\phi}' \). These however correspond to the \( \chi \) of cases \( a \) and \( b \). In these first two cases we would indeed have had final terms of the form

\[ H'_2^a = |g_a|^2 w^* w + |f_a|^2 v_i^* v_i, \]  

(60a)

and

\[ H'_2^b = |g_b|^2 v_i^* v_i + |f_b|^2 \frac{1}{2} u_{ik}^* u_{ik}, \]  

(60b)

instead of (50a) and (50b), if we had adopted the following procedure:

As Lagrangian we take

\[ \tilde{L} = \int dV[-L_0^a - L_0^n - L_0^c + L_P + L_N]. \]  

(61)

We then express the \( \phi \) as derivatives of the \( \chi \) by means of equations (33n), and take the \( \chi \) as the quantities that are varied independently. This method is equivalent to the one actually chosen for the corresponding dual cases. There thus exists an ambiguity in the formulation of the theory which would appear surprising, and there seems to be no reason for preferring either of the two forms. Fortunately the ambiguity is not serious. The undetermined terms \( H_2^a \) or \( H_2^b \), it is important to note, do not involve the wave functions of the heavy electron but only those of proton and neutron. They can be interpreted as a direct interaction between these particles with the \( \delta \)-function as potential. Precisely these interactions have been studied in a previous paper (Kemmer 1937), and it was seen that they do not give any finite binding energy between the particles concerned. Similar considerations can also be employed to show that this type of interaction gives no scattering cross section whatever.† The non-vanishing contributions due to such terms are solely of the self-energy type. For

† This result was also proved very generally in unpublished calculations by Professor Pauli, who gave the writer his kind permission to quote the results.
such questions the theory is, however, in any case inadequate, so that the additional ambiguity introduced here is of no consequence at all. It might even be hoped that one could utilize the indeterminateness here found to reduce the divergence of self energy calculations, but the final section of this paper will show that such an attempt cannot succeed.

For the present it is certainly allowable to ignore the terms $H_{a}^{n}$ and $H_{2}^{n}$ completely.

For quantization let us now revert to vector notation as in § 2. We can then put

$$H_{0}^{a} = H_{0}^{d} = c\kappa \int dV \left[ |\psi|^{2} + |\pi|^{2} + \frac{1}{\kappa^{2}} |\nabla \psi|^{2} \right], \quad (62 \, a, \, d)$$

$$H_{0}^{b} = H_{0}^{e} = c\kappa \int dV \left[ |\psi|^{2} + |\pi|^{2} + \frac{1}{\kappa^{2}} |\nabla \psi|^{2} \right]$$

$$+ |\Psi|^{2} + |\Pi|^{2} + \frac{1}{\kappa^{2}} |\nabla \Psi|^{2} \right], \quad (62 \, b, \, c)$$

$$H_{1}^{a} = c\kappa \int dV \left( \Phi_{N}^{*} \left[ g_{a} \beta \psi^{*} - f_{d} \left( \pi - \frac{1}{\kappa} (\alpha \nabla \psi^{*}) \right) \right] \Phi_{p} \right.$$  

$$+ \text{conjugate complex} \right), \quad (63 \, a)$$

$$H_{1}^{b} = c\kappa \int dV \left( \Phi_{N}^{*} \left[ g_{b} \left( \pi - \frac{1}{\kappa} \alpha \nabla \psi^{*} \right) - f_{b} \left( \frac{\gamma \nabla \psi^{*} \beta}{\kappa^{2}} + \frac{1}{\kappa} \beta \psi^{*} + \frac{1}{\kappa} \gamma \psi^{*} \right) \right] \Phi_{p} + \text{conjugate complex} \right), \quad (63 \, b)$$

$$H_{1}^{c} = -c\kappa \int dV \left( \Phi_{N}^{*} \left[ f_{c} \left( \pi - \frac{1}{\kappa} \nabla \psi^{*} \right) - g_{c} \left( \beta \psi^{*} \right) + f_{c} \left( \beta \psi^{*} \right) \right] \Phi_{p} + \text{conjugate complex} \right). \quad (63 \, c)$$

$$H_{1}^{d} = -c\kappa \int dV \left( \Phi_{N}^{*} \left[ f_{d} \beta \psi^{*} - g_{d} \left( \frac{1}{\kappa} \gamma \psi^{*} \right) \right] \Phi_{p} \right.$$  

$$+ \text{conjugate complex} \right). \quad (63 \, d)$$

Here the $\psi$, $\Psi$ and $\pi$, $\Pi$ are understood to be defined as canonical conjugates. The whole of the further development can now be taken over
without change from the case of no interaction.† In particular the
equations from (12) to (24) in § 2, or the corresponding equations of Pauli
and Weisskopf, remain valid. It is of special importance that the total
charge will still be, in cases a and d:

$$\frac{1}{c} \int s_0 dV = \frac{e}{h} \int (\pi \psi - \pi^* \psi^*) dV,$$

and in cases b and c:

$$\frac{1}{c} \int s_0 dV = \frac{e}{h} \int (\pi \psi + \Pi \Psi - \pi^* \psi^* - \Pi^* \Psi^*) dV. \tag{65}$$

The terms (63n) describe possibility of the emission and absorption of
heavy electrons by protons and neutrons, and therefore the heavy electron
charge alone will of course not be conserved. It can however be shown
that if one adds the usual Dirac charge $\int \Phi_P^* \Phi_P dV$ of the protons to the
quantity defined in (64) or (65), their sum is indeed a constant. As, on
the other hand, $\int s_0 dV$ is the same operator as in the case of no in-
teraction, the interpretation of the field theory in terms of numbers of charges
present will still be rigorously possible.

4. Perturbation theory; the neutron-proton interaction

As is well known, the applications of formalisms of the type sketched
above are in practice confined to perturbation theory. We will first deal
with the most important problem of this kind, namely the derivation of the
neutron-proton interaction from our field theories. This will be given by
the second order perturbation formula

$$(p'_P p'_N | V(r) | p_N p_P) = \frac{-\sum (p'_P p'_N | H_{1n}^1 | k p_N p'_N) (k p_N p'_N | H_{1n}^1 | p_N p_P)}{E'_N - E_P + e^+}
- \frac{-\sum (p'_P p'_N | H_{1n}^1 | k p'_P p_P) (k p'_P p_P | H_{1n}^1 | p_N p_P)}{E'_P - E_N + e^-}. \tag{66}$$

† The equations can immediately be generalized to the case when an elec-
 tromagnetic field is present. One has merely to perform the usual substitutions

$$\frac{\partial \psi}{\partial x} \rightarrow \left( \frac{\partial}{\partial x_i} - \frac{ie}{\hbar c} A_i \right) \psi, \text{ etc.}$$

The resulting addition to (62b, c) has already been given as equation (27). It
should, however, not be overlooked that these are also additions to the $H_{1n}^1$ of
equations (63n).
\( E_N \) and \( E_P \) are the energies corresponding to the momenta \( p_N \) and \( p_P \) of neutron and proton respectively; \( \hbar k \) denotes the momentum of the heavy electron, \( e \) its energy, and the sum in (66) has to be taken over all values of this momentum. The first part of (66) takes into account the transitions of the proton into a neutron state under emission of a positive heavy electron, which is subsequently reabsorbed by the neutron, the latter thereby becoming a proton. In the latter case the neutron is the emitting, the proton the absorbing particle and the heavy electron is of negative charge. The two terms give equal contributions. If one only considers heavy particles at distances \( r \gg \hbar/Mc \) (\( M \) being the proton mass), the recoil of the heavy particle in the heavy electron emissions may be neglected and the energy difference in the denominators of (66) can be put equal to the heavy electron energy \( e \). With this assumption we find the following expressions for the effective neutron-proton potential \( V(r) \):†

\[
V^a(r) = \frac{cK}{4\pi} \left[ -g_\alpha^2 \beta N \beta P + f_\alpha^2 \frac{1}{\kappa^2} (\alpha_N \text{grad}) (\alpha_P \text{grad}) \right] J(r),
\]

\[
V^b(r) = \frac{cK}{4\pi} \left[ g_\beta^2 \left( 1 - (\alpha_N \alpha_P) + \frac{1}{\kappa^2} (\alpha_N \text{grad}) (\alpha_P \text{grad}) \right) + f_\beta^2 \left( \beta N \beta P (\sigma_N \sigma_P) - \frac{1}{\kappa^2} \beta N \beta P (\sigma_N \text{grad}) (\sigma_P \text{grad}) \right) - \frac{1}{\kappa^2} (\gamma_N \text{grad}) (\gamma_P \text{grad}) \right] J(r),
\]

\[
V^c(r) = \frac{cK}{4\pi} \left[ f_\gamma^2 \left( \gamma_N \gamma_P - (\sigma_N \sigma_P) + \frac{1}{\kappa^2} (\sigma_N \text{grad}) (\sigma_P \text{grad}) \right) + g_\gamma^2 \left( (\gamma_N \gamma_P) - \frac{1}{\kappa^2} (\gamma_N \text{grad}) (\gamma_P \text{grad}) \right) - \frac{1}{\kappa^2} \beta N \beta P (\sigma_N \text{grad}) (\sigma_P \text{grad}) \right] J(r),
\]

\[
V^d(r) = \frac{cK}{4\pi} \left[ -f_\delta^2 \beta N \beta P \gamma_N \gamma_P + g_\delta^2 \frac{1}{\kappa^2} (\sigma_N \text{grad}) (\sigma_P \text{grad}) \right] J(r),
\]

\[
J(r) = \frac{2}{\pi r} \int_0^\infty \frac{k \sin kr dk}{k^2 + \kappa^2}.
\]

† For the purposes of this and the following sections the possibility of taking the \( g_n \) and \( f_n \) to be complex quantities is inessential. For brevity we will treat them as real.
In obtaining equations (67n) certain integrals of the type

\[ \frac{1}{r} \int kr \sin kr \, dr \]

were neglected. For \( r \neq 0 \) they are indeed zero, but to be exact they should be represented by

\[ -\frac{1}{r} \delta'(r). \]

However, to these the same considerations apply as were already used in the preceding section; their sole contributions are to the self-energies of the heavy particles. Therefore their omission is obviously justified.

The value of the integral \( J(r) \) in equations (68) can be immediately given, we have

\[ J(r) = -\frac{2}{\pi r} \frac{d}{dr} \int_0^\infty \frac{\cos kr}{k^2 + \kappa^2} dk = \frac{e^{-\kappa r}}{r}. \]

Having neglected the recoil of the protons and neutrons, it is now not very consequent to use the relativistic terms of equations (67). It is more sensible to neglect the contributions proportional to the heavy particle velocities, that is the terms with \( \alpha \) or \( \gamma \), the same is true for the terms with \( \gamma_5 \). Further we should put \( \beta = 1 \).

Thus we obtain the final formulae:

\[ V^a(r) = -\frac{cK}{4\pi} g^a \frac{e^{-kr}}{r}, \quad (69a) \]

\[ V^b(r) = +\frac{cK}{4\pi} \left[ g^b \sec^2 \left( \frac{\sigma_N \sigma_P}{2} \right) - \left( \sigma_N \text{ grad} \right) \left( \sigma_P \text{ grad} \right) \right] \frac{e^{-kr}}{r}, \quad (69b) \]

\[ V^c(r) = -\frac{cK}{4\pi} \left[ g^c \left( \sigma_N \text{ grad} \right) \left( \sigma_P \text{ grad} \right) + f^c \left( \sigma_N \sigma_P \right) \right. \]

\[ \left. - \left( \sigma_N \text{ grad} \right) \left( \sigma_P \text{ grad} \right) \right] \frac{e^{-kr}}{r}, \quad (69c) \]

\[ V^d(r) = +\frac{cK}{4\pi} g^d \left( \sigma_N \text{ grad} \right) \left( \sigma_P \text{ grad} \right) \frac{e^{-kr}}{r}. \quad (69d) \]

Taking the most general form of our theory that is possible, namely a linear combination of all four cases, we see at once that we get an expression

\[ V(r) = [A + B \left( \sigma_N \sigma_P \right) + C \left( \sigma_N \text{ grad} \right) \left( \sigma_P \text{ grad} \right)] \frac{e^{-kr}}{r}, \quad (70) \]

with completely arbitrary coefficients \( A, B \) and \( C \). This result has been stated before (Kemmer 1938). In this way a form of interaction with \( C = 0 \)
may of course also be chosen, but though the latter assumption has been common to most calculations on nuclear binding, there appears to be no reason why it should be maintained. It is certainly more satisfactory to abandon it, if in this way one could succeed in describing all empirical facts with one of the four possibilities a to d alone. This is actually possible using case b. The detailed discussion of this fact is the subject of the paper of Fröhlich and others. Using the same methods as there described, we will here only briefly state the results for all four cases, which justify the preference for case b.

Case a: The potential in the \(^3\)S ground state of the deuteron is proportional to \(g^3\) and thus repulsive. The potential in the \(^1\)S state is of equal magnitude, but attractive. Interaction \(f_3^2\) gives no non-relativistic contribution.

Case b: The potential in the \(^3\)S state is \(\sim -(g^3 + \frac{2}{3}f^2)\) and thus always attractive. The potential in the \(^1\)S state is \(\sim g^3 - 2f^2\) and so can be chosen to be about half as deep as that of the triplet state, as experiment requires.

Case c: The potential in the \(^3\)S state, \(\sim \frac{5}{2}f^2 + \frac{3}{2}g^3\), is repulsive, there is a repulsion 3 times as strong in the \(^1\)S state.

Case d: The potential in the \(^3\)S state is attractive \((- \frac{1}{3}g^3)\), but there is a larger attraction \((- g^3\) in the singlet state. The interaction \(f_3^2\) gives no non-relativistic contribution.

It is obvious that no other case but b agrees with experience. To achieve agreement with experiment the inclusion of the interaction \(f_b\) is of course essential.† We have stressed the symmetry of our theory with respect to the \(\phi\) and the \(\chi\) sufficiently to make it clear that it cannot be excluded by arguments of simplicity.

Some remarks on perturbation theory in general are of interest in this connexion. If we consider the interaction expression \((63b)\), we note that there are terms proportional to \(g_b\bar{\psi}^*\) and other terms \(\sim \frac{1}{\kappa}g_b \frac{\partial}{\partial x}\Psi^*\) combined in the same invariant, the former arising from the transversal part, the latter from the longitudinal. The transversal part is similar to the electrodynamical case. Therefore the evaluation of succeeding approximations of an effect by perturbation theory will for this term be in powers of an absolute constant, corresponding to \((e^2/hc)^{\frac{1}{4}}\) in radiation theory. In our choice of units this constant is \(g_b(\kappa/4\pi\hbar)^{\frac{1}{4}}\). The contributions due to the additional longitudinal term, on the other hand, will be a power series in \(g_b(1/4\pi\hbar\kappa)^{\frac{1}{4}}\), that is, we have an expansion in powers of a fundamental

† The case \(f_b = 0\) has been considered by Bhabha (1938 a).
length. Using the present field theory the two contributions will always occur together. This is the case even for \( f_b = 0 \), and the same is true for any interaction in case c. The state of affairs is different in cases a and d, as there the interactions \( g_d \) and \( f_d \) give terms of the first type only.

Now Heisenberg (1936) has pointed out that, in a theory involving a fundamental length in the sense stated above, the contribution of high order approximations will increase when the energies in the effects studied increase, and thus multiple processes such as cosmic ray showers should, according to such a theory, occur when sufficiently high energies are available. This is not the case for formalisms involving expansions in powers of a dimensionless constant only. Unpublished calculations by Pauli have also shown that in such a theory the well-known divergency difficulties become mathematically far more serious than in radiation theory. It seems therefore of importance to note that the data on neutron-proton interaction prove definitely that a theory of the simpler type is not sufficient, and that according to the views developed here Heisenberg showers should be expected to occur.

It must be further remarked that quite generally the expansions as performed in perturbation theory can naturally only be valid if the expansion parameters are small. As is shown in the paper of Fröhlich and others by the evaluation of the 4th order proton-proton force, this is here by no means the case. However, it is well known that the question of avoiding these expansions is the main unsolved difficulty of quantum field theory, and therefore the use of any better method does not at present appear practicable. It seems fairly certain, that even with better methods the result that a general linear combination of our possibilities can be made to fit known data would be maintained, but of course the statement that \( b \) alone is sufficient, is more open to criticism. The study of the cases other than \( b \) may therefore still prove to be of more than systematic value.

5. Self-energies

Hitherto we have avoided the consideration of effects which, calculated by our theory, give divergent results. It is well known that all forms of quantum field theory formulated up to now give such divergencies, as soon as the calculations involve the process of virtual emission and reabsorption of a quantum by one and the same particle. The self-energy of a proton or a neutron due to the interaction with our heavy electrons is an effect of this type. We do not intend to discuss here whether one should attach
any physical significance to these calculations in spite of their formal
divergence, but it is in any case of importance to be informed as to the
exact type of these divergencies in simple cases. We will therefore give a
brief summary of the results of such calculations.

For these questions it is not in general sensible to maintain the assumption
of the last section that the recoil (energy change) of the heavy particles in
the processes considered can be neglected; therefore the explicit assumption
is now necessary that protons and neutrons obey Dirac's equation including
the "hole theory" interpretation of negative energy states. The methods
of evaluating self energies in "hole theory" may be found in the paper by
Weisskopf (1934a, b), where the case of electromagnetic interaction is
considered.

Here we will have two types of self energies to evaluate, namely those due
to the interactions $H_n^1$ and those arising from the additions $H^n_2$ or $H^n_3$,
which, of course, can now no more be neglected. The former terms give
a self energy in second order perturbation, namely

$$W_1^n = - \sum_{\substack{p \in P_0 \mid H_1^p \mid kp \mid H_3^p \mid p_0 \mid E - E_0 + \epsilon \mid \sum_{\substack{p \in P_0 \mid H_1^p \mid kp \mid H_3^p \mid p_0 \mid E \mid E + + E_0 + \epsilon}}, (71)$$

the notation being the same as in the previous section. $p_0$ and $E_0$ are the
momentum and energy of the proton (or neutron) in its initial state, $p$
and $E$ the corresponding quantities in the intermediate state, and $\hbar k$
and $\epsilon$ the same for the heavy electron. The general expression for $W_1^n$ is found to
be not very simple. We will confine ourselves to giving the result under two
alternative special assumptions only:

1. We take the proton to be at rest: $p_0 = 0$.

Then with the designation $M$ for the mass of the proton and $K = Mc/\hbar$,
we have

$$W_1^q = \int dk [g^2 k^2 A(k) + f_0^2 (k^2 + \kappa^2) A(k) + k^2 B(k)], (72a)$$

$$W_1^p = \int dk (g^2_0 + f_0^2) [k^2 A(k) + (k^2 + 3k^2) B(k)], (72b)$$

$$W_1^p = \int dk (g^2_0 + f_0^2) [(k^2 + 3k^2) A(k) + k^2 B(k)], (72c)$$

$$W_1^d = \int dk [g^2_0 [k^2 A(k) + (k^2 + \kappa^2) B(k)] + f^2_0 \kappa^2 B(k)], (72d)$$
with

\[ A(k) = -\frac{\mathbf{c}}{4\pi^2\kappa} \frac{k^2}{\sqrt{(k^2 + K^2)(k^2 + \kappa^2)}} \left[ \frac{\sqrt{(k^2 + K^2) + K}}{\sqrt{(k^2 + \kappa^2) + \sqrt{(k^2 + K^2) + K}}} - \frac{\sqrt{(k^2 + K^2) - K}}{\sqrt{(k^2 + \kappa^2) + \sqrt{(k^2 + K^2) + K}}} \right] \]  

\[ B(k) = +\frac{\mathbf{c}}{4\pi^2\kappa} \frac{k^2}{\sqrt{(k^2 + K^2)}} \left[ \frac{1}{\sqrt{(k^2 + \kappa^2) + \sqrt{(k^2 + K^2) + K}}} - \frac{1}{\sqrt{(k^2 + \kappa^2) + \sqrt{(k^2 + K^2) + K}}} \right] \]  

(73)

(2) We admit an arbitrary value for \( p_0 \), but give only the first term of the expansion in descending powers of \( k \):

\[ W_1^a = f_a^2 C_1, \quad W_1^b = (g_b^2 + 2f_b^2) C_1, \]

\[ W_1^c = (2g_c^2 + f_c^2) C_1, \quad W_1^d = g_d^2 C_1, \]

\[ C_1 = -\frac{1}{2\pi^2\hbar\kappa} \frac{M^2 + \frac{1}{3} p_0^2}{E_0} \int dk \ k + \ldots \]  

(74)

Thus the divergence of the result is quadratic in every case, but cases \( a \) and \( d \) do not include such highly divergent terms if we put \( f_a = 0 \), or respectively \( g_d = 0 \). In these special cases the divergence can be shown to be logarithmical only. It should further be noted that the representation (74) is only valid for \( k \gg K \). Thus if one "cuts off" the integrals at \( k_0 \) and, as is done by Fröhlich and others, takes \( k_0 \sim \kappa \), expression (74) cannot be used. Taking (73) instead we then find

\[ W_1^a = f_a^2 D, \quad W_1^b = (g_b^2 + f_b^2) D, \]

\[ W_1^c = (2g_c^2 + f_c^2) D, \quad W_1^d = g_d^2 D, \]

\[ D = -\frac{\mathbf{c}}{2\pi^2\kappa} \int dk k^2 + \ldots \]  

(75)

these being the highest terms of an expansion in powers of \( K/k_0 \).

We now determine the contributions of the \( H_2^a \) and \( H_2^c \). These are simpler to calculate, as one needs only first-order perturbation theory:

\[ W_2^a = \sum_{E > 0} (p_0 p \mid H_2^a \mid p_0 p) - \sum_{E < 0} (p_0 p \mid H_2^a \mid p_0 p), \]

(76)
and after a short calculation

\[
\begin{align*}
W_2^a &= f_a^2 C_2, \\
W_2^b &= (g_b^2 - 3f_b^2) C_2, \\
W_2^c &= -3(g_c^2 - f_c^2) C_2, \\
W_2^d &= (3g_d^2 - f_d^2) C_2, \\
W_2^e &= (3g_b^2 - f_b^2) C_2, \\
W_2^f &= (3g_c^2 - f_c^2) C_2, \\
W_2^g &= -3(g_d^2 - f_d^2) C_2, \\
W_2^h &= -g_d^2 C_2,
\end{align*}
\]

(77)

Here, too, the degree of divergence is quadratic. The dependence on \( p_0 \) is, however, different in these terms, and it is therefore evident that they will in no case compensate the other divergencies, even if we make use of the arbitrariness in the choice of the \( H_2^a \).

The fact that the divergence of self-energy is here greater than in radiation theory is noteworthy. Case \( b \) might especially have been expected to yield results identical with those for electromagnetic interaction, at least for \( f_b = 0 \). However, the contribution of the longitudinal quanta is sufficient to alter the result essentially. This is obviously in direct connexion with the facts concerning Heisenberg showers discussed in the foregoing section, as is particularly well shown by the fact that just those interactions, which do not give showers (\( g_a \) and \( f_d \)) also give no quadratically divergent contributions.

A further quantity that must strictly speaking be classed among the divergent effects is the magnetic moment of the proton or neutron caused by the possibility of emitting heavy electrons. This has been studied for case \( b \) in the paper by Fröhlich and others, but as the derivation there is by an abridged method, we would here like to indicate the more rigorous method of calculating the effect. In § 2 we gave the expressions for the interaction of the heavy electrons with the electromagnetic field. It was given there for case 2 only, but it is very similar in all four cases. Denoting the operator there named \( H_1 \) by \( H_R(A) \), we can obtain a third order perturbation self energy dependent on the electromagnetic potentials (recoil being neglected):

\[
W(A) = -\sum \frac{\langle p_0 | H_1^a | k'p \rangle \langle k'p | H_R(A) | kp \rangle \langle kp | H_1^a | p_0 \rangle}{k'k} - \sum \frac{\langle p_0 | H_1^a | k^+p \rangle \langle k^+p | H_1^a | k^+k^-p_0 \rangle \langle k^+k^-p_0 | H_R(A) | p_0 \rangle}{k^+(k^+-k^-)} - \sum \frac{\langle p_0 | H_R(A) | k^+k^-p_0 \rangle \langle k^+k^-p_0 | H_1^a | k^+p \rangle \langle k^+p | H_1^a | p_0 \rangle}{(k^++k^-)k^+}. \tag{78}
\]
The first term describes the virtual emission of a positive heavy electron by the proton, its subsequent interaction with the electromagnetic field and its final reabsorption by the proton. This term is identical with the expression used in the other paper. The second term describes another effect of the same order, namely the creation of a pair of heavy electrons by the field and their consecutive absorption by the proton. Term 3 describes the process inverse to this one. For the evaluation of the magnetic moment we need only take a static magnetic field, which can be described by a time-independent vector potential $A$ satisfying the condition

$$\text{div } A = 0.$$  (79)

The magnetic moment is then defined as

$$\mu = \left( \frac{\partial W(A)}{\partial H} \right)_{H=0},$$  (80)

$H = \text{curl } A$ being the magnetic field strength; to obtain the term proportional to $H$ in (78) we must expand in powers of $k - k'$ or $k^+ - k^-$, i.e. the Fourier component of the magnetic field, and restrict ourselves to the linear term of this expansion.

Using relation (79) it is easy to prove that

(1) In the expansion of the denominators one need only use the zero-order term.

(2) Only the terms in the heavy electron interactions proportional to $\sigma_P$ or $\beta_P \sigma_P$ can give a contribution to the moment (for a proton at rest).

From (1) it follows that the second and third terms of (78) give just the contribution of the first term again, so that the method of calculation used in the other paper is reasonable; from (2) we deduce that the only interactions giving rise to a moment are $f_b$, $g_c$, $f_c$ and $g_d$.

However doubtful the details of these calculations may be considered in view of the divergence of the result, the fact that an interaction has to be preferred which indicates the existence of an additional moment must be considered as satisfactory. The degree of divergence is here in all cases linear (see Fröhlich and others 1938).

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SUMMARY

It is shown that there are four inequivalent but equally simple possibilities of formulating a field theory of Einstein-Bose particles, in which a positive expression for the energy density exists. Any of these formalisms might tentatively be accepted as a description of the "heavy electron". Considerations of relativistic invariance show that two independent expressions for the interaction of these particles with protons and neutrons can be chosen in each of the four cases. Taking account of the interaction terms the general Hamiltonian form of the theories is stated and the quantization is performed. The resulting proton-neutron potential is determined and it is found that its sign and spin-dependence agrees with reality in only one of the four cases, namely in the case based on the equations of Proca (1936). The (divergent) self-energies of the proton or neutron resulting from the interactions studied are evaluated.

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