1. Introduction. The theory of heavy electrons recently developed by several authors† may be considered to give a satisfactory account of the empirically known neutron-proton interaction. However, it now seems well established that there exists a proton-proton interaction of comparable magnitude which is not accounted for equally well. Owing to the fact that the emission of a heavy electron involves the change of a neutron into a proton or vice versa, the first approximation of this theory gives only an exchange force between unlike particles, whereas a force between like particles must be due to double transitions and thus only appears in the second approximation. It is true that the expansion in terms of the number of particles emitted is actually so badly convergent that the second order proton-proton force at distances of about $10^{-13}$ cm. is found to be not essentially smaller than the first order neutron-proton force (see FHK), but nevertheless this does not explain experimental facts, since the calculated second order force is always repulsive.

According to the measurements of Tuve, Heydenburg and Hafstad‡ the proton-proton force is definitely attractive, and also apparently is very nearly equal to the neutron-proton force in corresponding anti-symmetrical states. The latter fact has served as the basis of the hypothesis that the forces between the elementary nuclear particles are essentially independent of the charges of the particular particles concerned. In the following work we denote this “charge-independence hypothesis” by the abbreviation CI-hypothesis. According to this view the observed proton-proton interaction should be exactly equal to the corresponding neutron-proton force, and the neutron-neutron force, which cannot be so directly measured, should also be exactly the same. In the theory of binding energies this hypothesis has proved very useful, particularly because it has made it possible to determine the exchange character of nuclear forces practically uniquely.

‡ Phys. Rev. 50 (1936), 806; 53 (1938), 239.
If the attractive proton-proton force is to be explained by any means, clearly some new field interacting with the heavy particles must be assumed. This is not very satisfactory, since there are few observations which could serve to determine the nature of this field. However, if we admit the CI-hypothesis to be true, a natural explanation of the proton-proton force which contains much less arbitrariness presents itself immediately: if we accept the view that there is symmetry between charged and uncharged heavy particles (proton-neutron)—and probably also between electron and “neutrino”—it is only natural to assume that this symmetry also exists for the “heavy electron”. We are thus led to expect that the latter too has a neutral counterpart which can interact with protons and neutrons. If we assume this to be the case, a proton-proton force arises in the first approximation. It is shown in the present paper that the CI-hypothesis can be accounted for exactly by this assumption†.

The introduction of an uncharged heavy quantum might be criticized for the following reason. Just as there is an “antiparticle” to the heavy electron, so there must exist a neutral antiparticle, but whereas in the former case particle and antiparticle are physically distinguished by the sign of their charge, no difference whatever appears to exist between the two kinds of neutral particles. In the case of the electron-neutrino theory the situation is somewhat different, since the neutrino could be defined as being emitted in β¬-decay, the antineutrino in β⁺-decay. Here, however, no relativistically invariant expression for the interaction is possible which would, for instance, allow the emission of the “particle” only by neutrons, the “antiparticle” only by protons. It is satisfactory, therefore, that it can be proved that the neutral antiparticle can be completely eliminated from the theory, so that only a positive, a negative and one kind of neutral particle need be assumed to exist. In spite of the apparent asymmetry of this procedure, the CI-hypothesis still holds and it is shown that the heavy particle interaction which then results, though only a special form of

† It has been previously shown (Kemmer, Phys. Rev. 52 (1937), 906) that a similar extension of Fermi’s theory of the β-field will also account for the symmetries of the CI-hypothesis. The writer would like to use this opportunity to point out an error of sign in the paper referred to. As direct calculation shows, the sign in equation (14) must be reversed. The alteration also affects a statement of v. Weizsäcker (Zeits. f. Physik, 102 (1936), 572). In the case studied by him the force for large distances is actually attractive in the deuteron ground state, and thus not in contradiction with experience. The neutron-proton force for the general case given by Fierz (Zeits. f. Physik, 104 (1937), 553) in his equation (3.4) has the correct sign only if the wave functions in that equation are considered to be anticommuting operators; there is a change of sign if we use their expectation values. The same is true for equation (4) of the writer’s aforementioned paper. The results of the latter are only affected by this alteration in that the term γν¬σ(σρ)(σρν) cannot be omitted if agreement with experiment is to be established. There is, however, no necessity for this omission, and it is interesting to note that such a term must also be included in the new heavy electron theory.
that derivable from the original theory, is in fact just the form corresponding to experience.

For the purposes of the following considerations it is very convenient to use the "isotopic spin" formalism. Proton and neutron are then taken to be two "states" of the one heavy particle and we combine the wave functions of the two in a one-column matrix

$$\Phi = \begin{pmatrix} \Phi_P \\ \Phi_N \end{pmatrix}.$$ 

The operator $\tau_3$ is defined as having the value $+1$ in a proton state and $-1$ in a neutron state, so that it can be represented by the matrix

$$\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Further, we introduce the matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix},$$

which describe transitions from proton states to neutron states and vice versa. Formally we combine the three matrices by using a vector symbol

$$\tau = (\tau_1, \tau_2, \tau_3),$$

thus accentuating the analogy with the spin vector. Though this is, of course, purely formal, and a "rotation" in the "space" of this "vector" has no physical meaning, it is extremely convenient to use the analogy. Not only do some results then appear obvious by analogy with known theorems on the true spin, but also many considerations of sign and symmetry which are often confusing are avoided in this manner. It is interesting that the final equations of the present paper appear in a particularly short and convenient form if this notation is used.

2. The equations of the neutral particle; the nuclear potential. The theory of the heavy electron, as expounded in the papers previously referred to, need not be given here in any detail. For our purposes it is sufficient to set down the equations in an abbreviated one-dimensional symbolism. The three-dimensional extension both to the vector case (Proca's equations) and to the scalar case (of Pauli and Weisskopf) is obvious.

The energy of the free heavy electrons is taken to be

$$H_0 = \int dx \left[ \kappa^2 \psi^* \psi + c^2 n^* n + \frac{\partial \psi^*}{\partial x} \frac{\partial \psi}{\partial x} \right], \quad \left( \kappa = \frac{m_e c}{\hbar} \right),$$

and their interaction energy with protons and neutrons schematically as

$$H_1 = \int dx \left[ g^* \Phi^* \frac{\tau_1 + i \tau_2}{2} \Phi \psi + g \Phi^* \frac{\tau_1 - i \tau_2}{2} \Phi \psi^* \right].$$

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We thus ignore the dependence of the interaction on the spin of the proton (neutron) and also omit the interaction terms involving $\partial \psi / \partial x$ or $\pi$. It is easy to see that their inclusion would not alter the following arguments.

We know that the nuclear potential resulting from (2) is

$$ V = \frac{1}{2} |g|^2 (\tau_1^{(1)} \tau_2^{(2)} + \tau_2^{(1)} \tau_1^{(2)}) \psi(r), $$

and that in the three-dimensional case $\psi(r)$ is proportional to $r^{-1} \exp(-\kappa r)$. The superscripts (1) and (2) respectively denote the two interacting heavy particles.

We now introduce similar neutral particles having the wave function $\psi'$ and the unperturbed energy

$$ H_0' = \int dx \left[ \kappa^2 \psi'^\ast \psi' + c^2 \pi' \pi' + \frac{\partial \psi'\ast \partial \psi'}{\partial x} \right]. $$

The interaction energy with the nuclear particles differs, however, from (2). The most general interaction energy with protons is

$$ H_1' = \int dx \left[ g' \Phi \ast \frac{1}{2} \tau_3 \Phi \psi' + g'\ast \Phi \ast \frac{1}{2} \tau_3 \Phi \psi' \ast \right], $$

and with neutrons is

$$ H'_n = \int dx \left[ g'' \Phi \ast \frac{1}{2} \tau_3 \Phi \psi' + g''\ast \Phi \ast \frac{1}{2} \tau_3 \Phi \psi' \ast \right]. $$

The evaluation of the radial (and spin) dependence of the ensuing heavy particle potential is identical with that of the foregoing case, but the dependence on $\pi$ is different. We find

$$ V' = \frac{1}{2} \left[ |g' + g''|^2 + |g' - g''|^2 (\tau_3^{(1)} \tau_3^{(2)}) + (|g'|^2 - |g''|^2) (\tau_3^{(1)} + \tau_3^{(2)}) \right] \psi(r). $$

With the most general form of $V$ and $V'$ the CI-hypothesis is not satisfied; we now adjust the constants in (3) and (7) in such a way as to conform to it.

In order that the proton-proton force should be equal to the neutron-neutron force, the last term in (7) must vanish, so that we must take

$$ |g'|^2 = |g''|^2. $$

If further the neutron-proton interaction in antisymmetrical states is also to be the same, we must have

$$ |g' - g''|^2 = |g|^2, $$

so that

$$ V + V' = \frac{1}{2} \left[ |g|^2 (\pi^{(1)} \cdot \pi^{(2)}) + |g' + g''|^2 \right] \psi(r). $$

Let us now put

$$ g = \sqrt{2} G e^{i\theta}, $$

and satisfy (8) by putting

$$ g' = G' e^{i\theta'} \quad \text{and} \quad g'' = G' e^{i\theta''}. $$

(9) then becomes

$$ G^2 = 2 G'^2 \sin^2 \frac{1}{2}(\theta' - \theta''), $$

and (10) becomes

$$ V = G^2 [ (\pi^{(1)} \cdot \pi^{(2)}) + \cot^2 \frac{1}{2}(\theta' - \theta'')]. $$
It is obvious that this potential conforms to the CI-hypothesis. To find the most general form of interaction which is still possible, we insert the quantities $G$ and $\theta, \theta', \theta''$, as defined by (11), (12) and (13), into (2), (5) and (6). We find

$$H_1 = \int dx \frac{G}{\sqrt{2}} e^{i\theta} \Phi^*(\tau_1 + i\tau_2) \Phi \psi + \text{the complex conjugate},$$

$$H_1' + H_1'' = \int dx \frac{G}{\sqrt{2}} \frac{1}{\sin \frac{1}{2}(\theta' - \theta'')} \left[ e^{i\theta'} \Phi^*(1 + \tau_3) \Phi \psi' \right. + e^{i\theta''} \Phi^*(1 - \tau_3) \Phi \psi''] + \text{the complex conjugate.}$$

(15)

It is to be noted that the CI-hypothesis cannot be satisfied merely by adding a proton-proton and neutron-neutron potential to the neutron-proton potential due to the charged heavy electrons. The neutron-proton interaction must itself be modified. This question is studied more closely in the final section of this paper.

3. Elimination of the antiparticle. The theory of heavy electrons based on Proca’s equations is strikingly similar to the quantum theory of the electromagnetic field. However, it is significant that the problem of the antiparticle arises here, whereas it does not exist for the photon. The reason for this is that the field quantities in the electromagnetic theory are, by their nature, real, while it is necessary to assume the heavy electron wave function to be complex in order to ensure the existence of a charge density. For the neutral particle this “charge density” has no physical meaning, and it is therefore clear that the way to eliminate the antiparticle is to restrict ourselves to real wave functions for the neutral component. It must then be proved that the CI-hypothesis still holds in such a restricted theory. For the study of reality conditions it is convenient to split all the wave quantities hitherto considered into their real and imaginary parts. If we put

$$\psi = \frac{1}{\sqrt{2}} (\psi_1 + i\psi_2), \quad \pi = \frac{1}{\sqrt{2}} (\pi_1 - i\pi_2),$$

$$\psi' = \frac{1}{\sqrt{2}} (\psi_3 + i\psi_4), \quad \pi' = \frac{1}{\sqrt{2}} (\pi_3 - i\pi_4),$$

(16)

with real $\psi_i$ and $\pi_i$, we obtain a canonical transformation, the $\pi_i$ being the

† Recently Majorana (Nuovo Cimento, 14 (1937), 171) has shown that it is possible to eliminate the antiparticle in the same way in the case of any uncharged particle satisfying the Dirac equation. The simplest, though not the most general, way of proving this fact is by noting that a representation of Dirac’s matrices can be chosen in which all three $\alpha_i$ are real, while $\beta$ is imaginary. Then Dirac’s equation is seen to be a purely real differential equation, and we can confine ourselves to the consideration of its real solutions. Majorana shows that the quantized theory then gives a particle without its antiparticle, as in the case studied above (cf. also Kramers, K. Akad. Amsterdam Proc. 40 (1937), 814).
momenta conjugate to the corresponding $\psi_i$. In these variables the unperturbed Hamiltonian of our system is

$$H_0 + H' = \sum_{i=1}^{4} \frac{1}{2} \int dx \left[ \kappa^2 \psi_i^* \psi_i + c_2 \partial \psi_i \partial \bar{\psi}_i \right].$$

(17)

Our wave function thus has four real "components" which can be treated independently. We can, for instance, perform the Fourier analysis

$$\psi_i = \frac{1}{\sqrt{L}} \sum_k q_i(k) e^{ikx}, \quad \pi_i = \frac{1}{\sqrt{L}} \sum_k p_i(k) e^{-ikx},$$

(18)

but we must add the reality conditions

$$q_i(-k) = q_i^*(k), \quad p_i(-k) = p_i^*(k).$$

(19)

If we then put

$$q_i(k) = \sqrt{\frac{\hbar}{2}} \left[ -c_i(k) + c_i^*(k) \right],$$

$$p_i(k) = \sqrt{\frac{\hbar}{2}} \left[ c_i^*(k) + c_i(-k) \right],$$

(20)

a short calculation shows that the Hamiltonian is

$$H_0 + H' = \sum_{i=1}^{4} \sum_k \hbar c \sqrt{(k^2 + \kappa^2)} c_i^*(k) c_i(k).$$

(21)

It is clear what the quantum interpretation of these formulae must be. As is well known from analogous cases, the $c_i$, when quantized, obey the commutation rules

$$[c_i(k), c_i^*(k')] = \delta_{kk'},$$

(22)

from which it follows that the eigenvalues of each $c_i^*(k) c_i(k)$ consist of all the positive integers and zero. The Hamiltonian is then expressible in terms of the numbers of particles:

$$H = \sum_{i=1}^{4} \sum_k E(k) N_i(k), \quad E(k) = \hbar c \sqrt{(k^2 + \kappa^2)}. $$

(23)

The particles of the kinds 1 and 2 together form the charged component of the field, but in this representation the charge density $s_0/c$ is not diagonal. We have

$$s_0 = -\frac{ie}{\hbar} (\pi \psi - \pi^* \psi^*) = \frac{ec}{\hbar} (\pi_1 \psi_2 - \pi_2 \psi_1),$$

(24)

and

$$\int s_0 dx = \frac{ec}{\hbar} \sum_k [p_1(k) q_2(k) - p_2(k) q_1(k)]$$

$$= \frac{ec}{\hbar} \sum_k i[\psi_1^* c_2(k) - \psi_2^* c_1(k)]. $$

(25)

Thus, for instance, the state in which one particle of given momentum and of the kind 1 is present alone has the expectation value zero for the total charge. For this reason this representation is certainly not convenient for all purposes, and should sometimes be replaced by the one given by Pauli and Weisskopf,
which is given later in equations (38). The present form can, however, be used with advantage, whenever the eigenvalue of the charge is not of direct interest, as for example for the evaluation of the nuclear potential. In any case the representation given here is very convenient for comparisons with the neutral component of the field. The latter is described here by the two wave functions \( \psi_3 \) and \( \psi_4 \) and is thus readily seen to consist of two kinds of physically indistinguishable particles, 3 and 4, as already discussed. It is, however, possible to assume that the initial state of the system is such that no particles of the fourth kind are present. With a general form of interaction with extraneous matter, these particles would still be created at later times, but provided that we assume that all interaction terms occurring are such that \( \psi_4 \) and \( \pi_4 \) are not contained in them\( ^\dagger \), particles of the fourth kind will never be present.

A theory involving only these specialized interactions is thus equivalent to the complete non-existence of the second kind of neutral particle, and the term \( i = 4 \) in the unperturbed energy (17) may just as well be omitted. Mathematically our procedure is equivalent to taking the wave functions of the neutral particle to be real from the beginning, but, by means of the detour made here, we are able to link up with the considerations concerning the CI-hypothesis given in the previous section. The interaction with protons and neutrons given there in equation (15) is the only one that it has hitherto been necessary to ascribe to the neutral particle. If, therefore, we choose a form of (15) in which the coefficients of \( \psi_4 \) are zero, the resulting theory combines the two advantages of ensuring the non-existence of a neutral “antiparticle” and of satisfying the CI-hypothesis.

In the variables \( \psi_3 \) and \( \psi_4 \) (15) becomes

\[
H_1' + H_1'' = \int dx \left[ \cos \left( \frac{1}{2} (\theta' - \theta'') \right) \Phi^* \Phi \left[ \left| \frac{\cos \left( \frac{1}{2} (\theta' + \theta'') \right)}{\sin \left( \frac{1}{2} (\theta' - \theta'') \right)} \right| \Phi^* \tau_3 \Phi \left| \frac{\sin \left( \frac{1}{2} (\theta' + \theta'') \right)}{\sin \left( \frac{1}{2} (\theta' - \theta'') \right)} \right| \Phi^* \psi_3 \right] - \frac{\sin \left( \frac{1}{2} (\theta' - \theta'') \right)}{\sin \left( \frac{1}{2} (\theta' - \theta'') \right)} \Phi^* \tau_3 \Phi \left[ \sin \left( \frac{1}{2} (\theta' + \theta'') \right) \psi_3 + \cos \left( \frac{1}{2} (\theta' + \theta'') \right) \psi_4 \right] \right],
\]

so that we now choose

\[
\cos \left( \frac{1}{2} (\theta' + \theta'') \right) = 0, \quad \cos \left( \frac{1}{2} (\theta' - \theta'') \right) = 0.
\]

Then the interaction is reduced to the very simple form

\[
H_1' + H_1'' = \pm \int dx G \Phi^* \tau_3 \Phi \psi_3.
\]

Let us also express the interaction with the charged component, as given by (14), in our new variables. There appears to be no physical reason for retaining the arbitrary phase \( \phi \) included in that equation, so we assume \( \phi = 0 \). We then have

\[
H_1 = \int dx G \left[ \Phi^* \tau_1 \Phi \psi_1 + \Phi^* \tau_2 \Phi \psi_2 \right].
\]

\( ^\dagger \) The corresponding assumption for the Pauli-Weisskopf representation (38) is not compatible with the relativistic invariance of the interaction.
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Thus, if we choose the positive sign in (28), the total interaction can be formally written as

$$H_1 + H_1' + H_1'' = \int dx \, G(\Phi^* \tau \Phi \cdot \Psi) \tag{30}$$

with the use of the symbolic "vector"

$$\Psi = (\psi_1, \psi_2, \psi_3).$$

The specialization (27) necessary to eliminate the neutral antiparticle results in a corresponding specialization of the nuclear potential given by (13). We now have

$$\cot \frac{1}{2}(\theta' - \theta'') = 0, \tag{31}$$

so that

$$V + V' = G^2(\tau^{(1)} \cdot \tau^{(2)}). \tag{32}$$

It is very satisfactory to note that the best data hitherto available in fact indicate that the specialized potential (32) corresponds to reality†.

4. Summary and discussion. Having deduced by specialization a satisfactory theory of heavy quanta, charged and uncharged, we collect here the essential equations in a form suitable for practical purposes, and then discuss the most important application, i.e. the resulting nuclear potential, in more detail.

We retain the notation of three-component quantities as vectors in isotopic spin space. The total Hamiltonian of the system then is

$$H = H_0 + H_1, \tag{33}$$

with

$$H_0 = \frac{1}{2} \int dx \left[ \kappa^2 (\psi \cdot \psi) + c^2 (\pi \cdot \pi) + \left( \frac{\partial \psi}{\partial x} \cdot \frac{\partial \psi}{\partial x} \right) \right] \tag{34}$$

and

$$H_1 = \int dx \, G(\Phi^* \tau \Phi \cdot \Psi). \tag{35}$$

The transition to momentum space is performed by means of

$$\psi_k = \sum_k q_k(k) e^{ikx}, \quad \pi_k = \sum_k p_k(k) e^{-ikx}, \tag{36}$$

with

$$q_k(-k) = q_k^*(k), \quad p_k(-k) = p_k^*(k),$$

and then we can introduce either the variables defined by

$$q_k(k) = \frac{\hbar e^{-i k}}{\sqrt{2} \, (k^2 + \kappa^2)^{1/2}} \left[ -c_k(k) + c_k^*(-k) \right], \tag{37}$$

$$p_k(k) = \frac{\hbar}{\sqrt{2} \, (k^2 + \kappa^2)^{1/2}} \left[ c_k^*(k) + c_k(-k) \right],$$

as in the preceding section, or the more complicated variables defined by

† The probable form of the nuclear potential was derived from consideration of the binding energy of heavy nuclei by Volz (Zeits. f. Physik, 105 (1937), 537) and still further determined and modified by Kemmer (Nature, 140 (1937), 192). The modified form has since been taken as the basis of calculations by Heisenberg (Naturwiss. 25 (1937), 749) and Flügge (Zeits. f. Physik, 108 (1938), 545). It is interesting that the original constants of Volz do not correspond to the simpler form (32) so that their use would necessitate the acceptance of an antiparticle.
\[ \frac{q_1 + i q_2}{\sqrt{2}} = \sqrt{\frac{\hbar c}{2 (k^2 + \kappa^2)^{\frac{1}{2}}}} \left[ -a(k) + b^*(-k) \right] \]

\[ q_3 = \sqrt{\frac{\hbar c}{2 (k^2 + \kappa^2)^{\frac{1}{2}}}} \left[ -c(k) + c^*(-k) \right] \]

\[ \frac{q_1 - i q_2}{\sqrt{2}} = \sqrt{\frac{\hbar c}{2 (k^2 + \kappa^2)^{\frac{1}{2}}}} \left[ -b(k) + a^*(-k) \right] \]

\[ p_1 - i p_2 = \sqrt{\frac{\hbar}{2c}} (k^2 + \kappa^2)^{\frac{1}{2}} \left[ a^*(k) + b(-k) \right] \]

\[ p_3 = \sqrt{\frac{\hbar}{2c}} (k^2 + \kappa^2)^{\frac{1}{2}} \left[ c^*(k) + c(-k) \right] \]

\[ \frac{p_1 + i p_2}{\sqrt{2}} = \sqrt{\frac{\hbar}{2c}} (k^2 + \kappa^2)^{\frac{1}{2}} \left[ b^*(k) + a(-k) \right] \]

(38)

which correspond to those used by Pauli and Weisskopf† except for the fact that we put \( b(k) \) for their \( b_{-k} \).

By (37), the zero order energy and total charge respectively take the form

\[ H_0 = \sum_{i=1}^{3} \sum_k \hbar c (k^2 + \kappa^2)^{\frac{1}{2}} c_i^*(k) c_i(k) \]

(39)

and

\[ \frac{1}{c} \epsilon_0 = \frac{\epsilon}{\hbar} \sum_k i [c_i^*(k) c_2(k) - c_i^*(k) c_1(k)] \]

(40)

or alternatively, by (38),

\[ H_0 = \sum_k \hbar c (k^2 + \kappa^2)^{\frac{1}{2}} [a^*(k) a(k) + b^*(k) b(k) + c^*(k) c(k)] \]

(41)

and

\[ \frac{1}{c} \epsilon_0 = \frac{\epsilon}{\hbar} \sum_k [1 \cdot a^*(k) a(k) + (-1) \cdot b^*(k) b(k) + 0 \cdot c^*(k) c(k)] \]

(42)

For the discussion of the heavy particle potential we use the former representation. We have already seen that in the first approximation this potential is simply

\[ V_1 = G^2 (\tau^{(1)}, \tau^{(2)}) v(r) \]

(43)

and the CI-hypothesis is satisfied. From the “invariant” form of the energy expressions (34) and (35), the completely symmetrical treatment of the three “components” \( \psi_i \) of the wave function, expressed by (36) and (37), together with the relations

\[ (\tau_1^{(n)})^2 = 1, \ldots, \tau_1^{(n)} \tau_2^{(n)} = -i \tau_2^{(n)} \ldots, \]

we can now further deduce that in any higher approximations of the perturbation

theory the heavy particle interaction is almost as simple; it is in fact proportional
to
\[ A\psi'(r) + B(\psi^{(1)} \cdot \psi^{(2)})\psi'(r). \]  

(44)

The CI-hypothesis is thus exactly satisfied in any approximation (if specifically
electromagnetic effects can be neglected). In the second approximation, for
instance, the potential is
\[ V_2 = G^4(\psi^{(1)} \cdot \psi^{(2)})^2 \psi_2(r) = G^4[3 - 2(\psi^{(1)} \cdot \psi^{(2)})] \psi_2(r), \]  

(45)

the form of \( \psi_2(r) \) being known from the calculations of Yukawa and Sakata†
and from FHK.

We now combine the results found here with those concerning the spin-
dependence found in the three-dimensional theory in FHK. The complete form
of the first approximation for the potential can be stated immediately, but it is
sufficient to give the expression for spherically symmetrical states‡. Then
\[ V = (\psi^{(1)} \cdot \psi^{(2)}) [G^2 + \frac{2}{3} F^2(\sigma^{(1)} \cdot \sigma^{(2)})] \frac{e^{-qr}}{r}. \]  

(46)

The constants \( g \) and \( f \) occurring here differ from the \( g \) and \( f \) of FHK for
two reasons; first because of the difference of a factor \( \sqrt{2} \) in their definition (see
equation (11)), secondly because the non-exchange forces added as the result
of the introduction of the neutral necessitate a change in the relative magnitudes
of \( F \) and \( G \), if the empirical positions of the \( ^3S \) and \( ^1S \) states of the deuteron are
still to be given correctly.

For the triplet state we obtain the potential
\[ V^3 = -3(G^2 + \frac{2}{3} F^2) \frac{e^{-qr}}{r}; \]  

(47)

for the singlet state
\[ V^1 = (G^2 - 2F^2) \frac{e^{-qr}}{r}. \]  

(48)

Now the latter is known to be half the former, so that
\[ 5G^2 = 2F^2, \quad \text{i.e. } F = \sqrt{\frac{5}{2}} G. \]  

(49)

‡ It is to be noted that the above value of the potential corresponds to the choice of
the three-dimensional interaction as
\[ H_1 = \frac{F^2}{\hbar^2} \sum_{i,j} \Phi^{*} \sigma_i \sigma_j \psi \cdot \Phi \psi_i + \frac{G^2}{\hbar^2} \sum_i \Phi^{*} \cdot \Phi \psi_i, \]
where the suffix \( i \) refers to "isotopic spin space" and \( j \) refers to true space, and \( \psi_i \) is so
normalized that the undisturbed energy is
\[ \frac{1}{8\pi} \int dV \left[ \sum_{i,j} \left\{ \kappa^2 \psi_i^* + \kappa^2 \psi_i + \frac{1}{c^2} \left( \frac{\partial \psi_i^*}{\partial t} \right)^2 + \frac{1}{c^2} \left( \frac{\partial \psi_i}{\partial t} \right)^2 + \left( \nabla_i \psi_i \right)^2 + \left( \nabla_i \psi_i \right)^2 \right\} + \sum (\nabla_i \psi_i)^2 \right]. \]
Finally the absolute magnitudes of $F$ and $G$ can be determined, just as was done in FHK. We must have, roughly,†

$$3(G^2 + \frac{1}{2}F^2) \kappa = \frac{8G^2}{\hbar c} m_0 c^2 = 15.8 \times 10^6 \text{e.V.} \quad (50)$$

and therefore

$$\frac{G^2}{\hbar c} \sim \frac{1}{35} \quad \text{and} \quad \frac{F^2}{\hbar c} \sim \frac{1}{14}.$$ 

Thus both $F$ and $G$ must be taken to be smaller here than was necessary in FHK. This, however, scarcely means an improvement of the convergence of the theory, since, as equation (45) demonstrates, it is to be expected that the numerical factors occurring in the expansions will now be correspondingly increased owing to the larger number of intermediate states over which summations generally have to be performed.

† This value was determined by Miss Littleton of Bristol who kindly solved the deuteron wave equation numerically for the particular potential here used, assuming $m_0 = 137m_e$. Dr Bhabha has since informed me that the same integration has been performed mechanically at Cambridge for arbitrary $m_0$. The point determined by Miss Littleton lies exactly on the curve thereby found. Since these calculations still neglect the fact that the true theoretical force is not strictly central, no great weight can be attached to the numerical values.