Analogies in Theoretical Physics

Giovanni JONA-LASINIO

Dipartimento di Fisica, “Sapienza”, Università di Roma, Piazzale A. Moro 2, Roma 00185, Italy

and

Istituto Nazionale di Fisica Nucleare, Sezione di Roma 1, Roma 00185, Italy

Analogies have had and continue to have an important role in the development of theoretical physics. They may start from similarities of physical concepts followed by similarities in the mathematical formalization or it may be a purely mathematical aspect to suggest the development of analogous physical concepts. More often a subtle nonobvious interplay between these levels is involved. In this paper I will discuss two cases sufficiently intricate to illustrate some ways of how analogies work. The first topic is the introduction of spontaneous symmetry breaking in particle physics. The second one is the use of the renormalization group in the theory of critical phenomena and its statistical interpretation.

§1. Introduction

The transfer of ideas from one domain of science to another is a complicated process which involves that ill-defined concept that we call intuition. This is something very subjective and uses entirely different paths according to the cultural background and inclinations of each scientist. In theoretical physics an intuition may have as a starting point some imperfect or incomplete parallelism of physical concepts, but sometimes it is a mathematical analogy which is at the origin of a new development and physical concepts are shaped along the way. In this talk I will try to call your attention on the role played by analogies in the construction of some physical theories. Several cases can be found in the history of physics in which analogies were viewed as an important methodological tool. It is interesting to recall Boltzmann’s comments on Maxwell’s theory:¹)

“Most surprising and far-reaching analogies revealed themselves between apparently quite disparate natural processes. It seemed that nature had built the most various things on exactly the same pattern; or, in the dry words of the analyst, the same differential equations hold for the most various phenomena. Thus thermal conduction, diffusion and the distribution of charge in electric conductors follow the same laws. The same equations may be regarded as the solution of a problem in hydrodynamics and in potential theory. The theory of fluid vortices and that of friction in gases show the most surprising analogy with electrodynamics an so on. (See also Maxwell, Scientific Papers, Vol. I).

........... In his very first paper on the theory of electricity (On Faraday’s lines of force, see Scientific Papers, Vol. I, p. 157), Maxwell declares that he does not intend to propose a theory of electricity; that is he does not himself believe in the reality of the incompressible fluids and resistances that he is assuming, but merely wishes to give a mechanical example that shows much analogy with electric phenomena, which
he wants to present in a form that makes them most readily understandable. In his second paper (On physical lines of force, Scientific Papers, Vol. I, p. 451) he goes much further still, constructing from fluid vortices and friction rollers moving inside cells with elastic walls an admirable mechanism that serves as a mechanical model for electromagnetism."

I wish to recall also two very important more recent examples in which mathematical analogy played a relevant role. In Dirac formulation of quantum mechanics the Hamiltonian formalism reinterpreted in terms of operator variables was a leading thread. The first quantum theories of elementary particles, the theory of β-decay by Fermi and the theory of nuclear forces by Yukawa, had as their model-guide the already established QED.

I will try to illustrate the role of analogy in contemporary theoretical physics with the help of developments in which I have been involved. I hope that it may contribute to stimulate further reflection on these themes. I will discuss in some detail the introduction and formalization of the idea of spontaneous symmetry breaking (SSB) in particle physics which is a case of cross fertilization from condensed matter. As a second example I will consider the use of the renormalization group (RG) in statistical mechanics where the flow of ideas goes in the opposite direction. The RG is a powerful tool for calculations and, suitably reformulated in terms of probabilistic concepts, clarifies the statistical meaning of universality in critical phenomena. This last development was again stimulated by an analogy, a mathematical one.

§2. Spontaneous symmetry breaking in particle physics

Spontaneous breakdown of symmetry is a concept that is applicable only to systems with infinitely many degrees of freedom. Although it pervaded the physics of condensed matter for a very long time, magnetism is a prominent example, its formalization and the recognition of its importance have been an achievement of the second half of the XXth century. Strangely enough the name was adopted only after its introduction in particle physics: it is due to Baker and Glashow. Very often concepts acquire a proper name only when they attain their full maturity.

What is SSB? In condensed matter physics it means that the lowest energy state of a system can have a lower symmetry than the forces acting among its constituents and on the system as a whole. As an example consider a long elastic bar on top of which we apply a compression force directed along its axis. Clearly there is rotational symmetry around the bar which is maintained as long as the force is not too strong: there is simply a shortening according to Hooke’s law. However when the force reaches a critical value the bar bends and we have an infinite number of equivalent lowest energy states which differ by a rotation.

Heisenberg was probably the first to consider SSB as a possibly relevant concept in particle physics but his proposal was not the physically right one. The theory of superconductivity of Bardeen, Cooper and Schrieffer which appeared in 1957 provided the key paradigm for the introduction of SSB in relativistic quantum field theory and particle physics on the basis of an analogy proposed by Nambu. In his Nobel lecture Nambu emphasizes the importance of his previous exposure to
condensed matter physics.

To appreciate the innovative character of this concept in particle physics one should consider the strict dogmas which constituted the foundation of relativistic quantum field theory at the time. One of the dogmas stated that all the symmetries of the theory, implemented by unitary operators, must leave the lowest energy state, the vacuum, invariant. This property does not hold in presence of SSB and degenerate vacua. These vacua cannot be connected by local operations and are orthogonal to each other giving rise to different Hilbert spaces. If we live in one of them SSB will be manifested by its consequences, in particular the particle spectrum.

The BCS theory of superconductivity, immediately after its appearance, was reformulated and developed by several authors including Bogolyubov, Valatin, Anderson, Ricayzen and Nambu. The following facts were emphasized

1. The ground state proposed by BCS is not invariant under gauge transformations.
2. The elementary fermionic excitations (quasi-particles) are not eigenstates of the charge as they appear as a superposition of an electron and a hole.
3. In order to restore charge conservation these excitations must be the source of bosonic excitations described by a long range (zero mass) field. In this way the original gauge invariance of the theory is restored.

The peculiarity of the paper of Nambu,7) was that he used a language akin to quantum field theory, that is the Green’s functions formalism, and the role of gauge invariance was discussed in terms of vertex functions and the associated Ward identities. The search for analogies in particle physics became quite natural. In particular, following the suggestion of 5), the study of chiral symmetry breaking was developed in detail in two papers by Nambu and Jona-Lasinio8), 9) which had a considerable influence on the evolution of elementary particle theories.

Let us illustrate the elements of the analogy.

Electrons near the Fermi surface are described by the following equation:

\[
E\psi_{p,+} = \epsilon_p \psi_{p,+} + \phi \psi_{-p,-}^\dagger
\]
\[
E\psi_{-p,-}^\dagger = -\epsilon_p \psi_{-p,-} + \phi \psi_{p,+}^\dagger,
\]
with eigenvalues

\[
E = \pm \sqrt{\epsilon_p^2 + \phi^2}.
\]

Here, \(\psi_{p,+}\) and \(\psi_{-p,-}^\dagger\) are the wavefunctions for an electron and a hole of momentum \(p\) and spin \(+\); \(\phi\) is the gap.

In the Weyl representation, the Dirac equation reads

\[
E\psi_1 = \sigma \cdot p \psi_1 + m\psi_2
\]
\[
E\psi_2 = -\sigma \cdot p \psi_2 + m\psi_1,
\]
with eigenvalues

\[ E = \pm \sqrt{p^2 + m^2}. \]  

(2.4)

Here, \( \psi_1 \) and \( \psi_2 \) are the eigenstates of the chirality operator \( \gamma_5 \). Particles with mass are superpositions of states of opposite chirality. The similarity is obvious.

The bosonic excitations necessary to restore gauge invariance in a superconductor appear in the approximate expressions for the charge density and the current in a BCS superconductor,

\[ \rho(x, t) \simeq \rho_0 + \frac{1}{\alpha^2} \partial_t f, \]

\[ j(x, t) \simeq j_0 - \nabla f, \]  

(2.5)

where \( \rho_0 = e\psi^\dagger \sigma_3 Z \psi \) and \( j_0 = e\psi^\dagger (p/m) Y \psi \) are the contributions of the quasi-particles, \( Y, Z, \alpha \) are constants and \( f \) satisfies the wave equation

\[ \left( \nabla^2 - \frac{1}{\alpha^2} \partial_t^2 \right) f \simeq -2e\psi^\dagger \sigma_2 \phi \psi. \]  

(2.6)

Here, \( \psi^\dagger = (\psi_1^\dagger, \psi_2^\dagger) \)

In the elementary particle context the axial current \( \bar{\psi} \gamma_5 \gamma_\mu \psi \) is the analog of the electromagnetic current in BCS theory. In the hypothesis of exact conservation, the matrix elements of the axial current between nucleon states of four-momentum \( p \) and \( p' \) have the form

\[ \Gamma^A_{\mu}(p', p) = (i\gamma_5 \gamma_\mu - 2m\gamma_5 q_\mu / q^2) F(q^2), \quad q = p' - p. \]  

(2.7)

Exact conservation is compatible with a finite nucleon mass \( m \) provided there exists a massless pseudoscalar particle.

Assuming exact conservation of the chiral current, a picture of chiral SSB may consist in a vacuum of a massless Dirac field viewed as a sea of occupied negative energy states, and an attractive force between particles and antiparticles having the effect of producing a finite mass, the counterpart of the gap. The pseudoscalar massless particle, which may be interpreted as a forerunner of the pion, corresponds to the bosonic field associated to the fermionic quasi-particles in a superconductor.

To implement this picture the construction of a relativistic field theoretic model was required. At that time Heisenberg and his collaborators had developed a comprehensive theory of elementary particles based on a nonlinear spinor interaction: the physical principle was that spin \( \frac{1}{2} \) fermions could provide the building blocks of all known elementary particles. Heisenberg was however very ambitious and wanted at the same time to solve in a consistent way the dynamical problem of a nonrenormalizable theory. This made their approach very complicated and not transparent. Nambu considered Heisenberg theory very formal, but the four spinor interaction was attractive due to its simplicity and analogy with the many-body case. I was more enthusiastic. I had been exposed several times to the nonlinear spinor theory, first in a meeting in Venice where a very interesting discussion between Heisenberg
Analogies in Theoretical Physics

and Pauli took place, then in Rome that Heisenberg visited just to explain his theory. At that time I believed in such fundamental theories!

A Heisenberg type Lagrangian was adopted without pretending to solve the nonrenormalizability problem and introducing a relativistic cutoff to cure the divergences. This model is known in the literature with the acronym NJL. The energy scale of interest was of the order of the nucleon mass and one hoped that higher energy effects would not change substantially the picture.

The Lagrangian of the NJL model is

$$\mathcal{L} = -\bar{\psi}\gamma_\mu\partial_\mu\psi + g \left[ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right]. \quad (2.8)$$

It is invariant under ordinary and chiral gauge transformations

$$\begin{align*}
\psi &\rightarrow e^{i\alpha}\psi, & \bar{\psi} &\rightarrow \bar{\psi}e^{-i\alpha} \\
\psi &\rightarrow e^{i\alpha}{\gamma}_5\psi, & \bar{\psi} &\rightarrow \bar{\psi}e^{i\alpha}{\gamma}_5. \quad (2.9)
\end{align*}$$

To investigate the content of the model a simple mean field approximation for the mass was adopted

$$m = 2g[\langle \bar{\psi}\psi \rangle - \gamma_5\langle \bar{\psi}\gamma_5\psi \rangle]$$

$$= -2g[\text{tr}S^{(m)}(0) - \text{tr}\gamma_5S^{(m)}(0)], \quad (2.10)$$

where $S^{(m)}$ is the propagator of the Dirac field of mass $m$, or more explicitly,

$$\frac{2\pi^2}{gA^2} = 1 - \frac{m^2}{A^2} \ln \left(1 + \frac{A^2}{m^2}\right), \quad (2.11)$$

where $A$ is the invariant cutoff. This equation is very similar to the gap equation in BCS theory. If $\frac{2\pi^2}{gA^2} < 1$ there exists a solution $m > 0$.

From this relationship a rich spectrum of bound states follows,

<table>
<thead>
<tr>
<th>nucleon number</th>
<th>mass $\mu$</th>
<th>spin-parity</th>
<th>spectroscopic notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0$^-$</td>
<td>$^1S_0$</td>
</tr>
<tr>
<td>0</td>
<td>$2m$</td>
<td>0$^+$</td>
<td>$^3P_0$</td>
</tr>
<tr>
<td>0</td>
<td>$\mu^2 &gt; \frac{4}{3}m^2$</td>
<td>1$^-$</td>
<td>$^3P_1$</td>
</tr>
<tr>
<td>$\pm 2$</td>
<td>$\mu^2 &gt; 2m^2$</td>
<td>0$^+$</td>
<td>$^1S_0$</td>
</tr>
</tbody>
</table>

The bosonic field in the superconductor and the pseudoscalar particle in the NJL model are special cases of a general theorem formulated by Goldstone in 1961.$^{10}$

Whenever the original Lagrangian has a continuous symmetry group, which does not leave the ground state invariant, massless bosons appear in the spectrum of the theory.

Other examples are,
G. Jona-Lasinio

physical system | broken symmetry | massless bosons
ferromagnets | rotational invariance | spin waves
crystals | translational and rotational invariance | phonons

These massless bosons are now known in the literature as Nambu-Goldstone bosons. In nature, however, the axial current is only approximately conserved. The model could make contact with the real world under the hypothesis that the small violation of axial current conservation gives a mass to the massless boson, which is then identified with the $\pi$ meson.

Later, a reinterpretation in terms of quarks of the NJL model provided a successful effective theory of low energy Quantum Chromodynamics, see for example.\(^{11}\) After the NJL model, SSB became a key ingredient in elementary particle physics. Electroweak unification\(^{12}\) is based on a mechanism which has also its roots in the theory of superconductivity. For this mechanism we refer to the papers by Anderson, Brout, Englert and Higgs.\(^{13}-^{15}\) This important part of the story however goes beyond the purpose of the present talk.

2.1. The effective action

The argument showing that SSB actually takes place in the NJL model was based on a self-consistent field approximation and a formulation independent of any kind of approximation was desirable. This was the first motivation of my paper.\(^{16}\)

In field theory we can define a formal analog of the partition function

$$Z(J) = \langle 0 | T e^{i \int dx (\mathcal{L}_I + \sum J_i \Phi_i)} | 0 \rangle,$$  

(2.12)

where $|0\rangle$ is the bare vacuum, the $J_i$ are external sources and the fields $\Phi_i$ transform according to a representation of some symmetry group, e.g., the fundamental representation of the orthogonal group. Its logarithm,

$$G(J) = -i \log Z(J),$$

(2.13)

is the generator of the connected time ordered Green’s functions (in statistical mechanics the analog of $G$ is the free energy in presence of an external field $J$). Define the “classical” fields

$$\frac{\delta G}{\delta J_i} = \langle \Phi_i \rangle = \phi_i.$$  

(2.14)

Assuming that this relationship can be inverted, the effective action is the dual functional $\Gamma[\phi]$ defined by the Legendre transformation

$$\Gamma(\phi) = G(J(\phi)) - \int dx \sum J_i(\phi) \phi_i.$$  

(2.15)

We have the conjugate relation to (2.14)

$$\frac{\delta \Gamma}{\delta \phi_i} = -J_i.$$  

(2.16)
The vacuum of the theory is defined by the variational equation

$$\frac{\delta \Gamma}{\delta \phi_i} = 0.$$  \hspace{1cm} (2.17)

$\Gamma[\phi]$ is the generator of the one particle irreducible vertex functions and can be constructed by simple diagrammatic rules.

A symmetry breaking solution is a solution $\phi_i$ of the variational equation transforming in a nontrivial way under the symmetry group. This is the way in which SSB is nowadays introduced in textbooks.\textsuperscript{17)–20)} The effective action provides the natural setting to analyse the stability of the solutions of the variational problem (2.9). Furthermore the derivation of the Goldstone theorem in this formalism is particularly simple.\textsuperscript{16)}

Now some history. The similarity of the formalisms of quantum field theory and statistical mechanics was part of the common wisdom. It had been emphasized for instance in the book by Bogolyubov and Shirkov.\textsuperscript{21)} A characteristic feature of statistical mechanics, both classical and quantum, is the existence of variational principles determining the stable states of a system. Variational principles in quantum statistical mechanics have been introduced by Lee and Yang\textsuperscript{22)} followed by Balian, Bloch and De Dominicis\textsuperscript{23)} and by De Dominicis and Martin.\textsuperscript{24)} The variables appearing in these principles are typically quantum averages of operators, that is c-numbers. The work of De Dominicis and Martin used functional methods typical of Schwinger’s school with which I had some familiarity. A general tool to derive such variational principles was the functional Legendre transform with respect to space-time dependent potentials. I found then natural to introduce the effective action, a c-number action functional for quantum field theory, to characterize the vacuum in terms of a variational principle. For homogeneous systems the more restricted concept of effective potential, a limiting case of the effective action, had been introduced via perturbation theory in 10) with a more systematic treatment in 25). However the effective potential does not provide a complete description of the dynamics. The effective action differs from a classical action as it is nonlocal in time and involves the whole history of the system. An interpretation of the dynamical equations in terms of an initial value problem is possible only in certain limiting cases.\textsuperscript{26)}

Many years after my first paper I learnt that the effective action in a perturbative form had appeared in a work by Heisenberg and Euler in the thirties\textsuperscript{27)} where they studied quantum corrections to the Maxwell equations. I also heard from Bryce DeWitt that it was considered by Schwinger in unpublished notes. However the usefulness of the effective action was fully appreciated only after its introduction in connection with SSB. Inspired by the work of De Dominicis and Martin I studied also higher order Legendre transforms, i.e. involving expectation values of composite operators, and the associated variational principles in quantum field theory.\textsuperscript{28)} Their study was pursued later in a work by Cornwall, Jackiw and Tomboulis.\textsuperscript{29)}
§3. Critical phenomena and renormalization group

Statistical mechanics describes macroscopic systems in terms of an underlying microscopic structure whose configurations are the arguments of a probability distribution, an ensemble in the terminology of physicists. When a system approaches a critical point large islands of a new phase appear so that correlations among the microscopic constituents extend over macroscopic distances. One characterizes these situations by introducing a correlation length which measures the extension of such correlations. At the critical point this length becomes infinite and typically correlations decay with a nonintegrable power law as opposed to an exponential decrease away from criticality.

The exponents in these power laws exhibit a remarkable degree of universality because to systems physically different such as a gas and a ferromagnet correspond the same exponents, e.g. the magnetization,

\[ m = A_m |T - T_c|^\beta, \]

and the difference between the liquid and the gas densities,

\[ \rho_L - \rho_G = A_{LG} |T - T_c|^\beta, \]

are characterized by the same power law.

In this case there was a transfer of ideas from quantum field theory to many-body theory and statistical mechanics. In 1966 there was an important school on critical phenomena at Brandeis University where Kadanoff explained his ideas on the origin of critical scaling and the ensuing equations for the structure of the correlation functions. This school was attended by Carlo Di Castro, a young many-body physicist at that time, and when he came back he told me about Kadanoff theory. My reaction was that Kadanoff’s scaling equations for the correlation functions looked like a simplified version of the multiplicative renormalization group (RG) equations satisfied by Green’s functions in quantum field theory and statistical mechanics. I thought that mathematically, scaling and universality should arise from a resummation of singularities similarly to what happens in certain field theoretic infrared problems. This provided the basis for an analogy.

Kadanoff’s qualitative argument to explain why scaling properties should be expected at the critical point was the following: if correlations extend over macroscopic distances it must be irrelevant whether we consider our system constituted by the original microscopic objects or by blocks containing a large number of constituents.\(^{31}\)

In the limit when the correlations extend to infinity the size of the blocks should not matter and this leads to homogeneity properties for the correlation functions and other thermodynamic quantities.

The equation of the multiplicative renormalization group used in quantum field theory in the simplest case has the following form:\(^{21}\)

\[ d(x, y, \alpha) = Z(t, y, \alpha) d(x/t, y/t, \alpha Z^{-1}(t, y, \alpha) Z^2(t, y, \alpha)), \]

where \( d(x, y, \alpha) \) is a dimensionless two-point Green function depending on a momentum squared \( x = p^2 \), a mass parameter \( y = m^2 \) and the intensity of the interaction.
\( \alpha \) (dimensionless). The scaling functions \( Z \) and \( Z_V \) can be expressed in terms of the Green’s functions themselves via normalization conditions at \( p^2 = t \) for \( d(x, y, \alpha) \) and the dressed interaction (vertex function). This is an exact generalized scaling relation which has a counterpart for the correlation functions in statistical mechanics.\(^{30}\) Di Castro and I expected that in the vicinity of the critical point this relationship would reduce to the phenomenological scaling due to the irrelevance of the coupling constant and the other parameters. The first use of this equation in the study of critical phenomena appeared in our 1969 paper\(^{33}\) and we obtained in this way a qualitative explanation and foundation of scaling from first principles. After the introduction of a noninteger space dimension \( d \) and the introduction of \( \epsilon = 4 - d \) as a perturbation parameter,\(^{32}\) the multiplicative RG became the basis for systematic quantitative calculations.\(^{34)-36}\)

About two years after our paper, an article by Wilson\(^{37}\) appeared where a notion of renormalization group apparently different was used. Actually this notion had been used before by Wilson in connection with the fixed source meson theory. This notion looked closer than the multiplicative RG to Kadanoff’s picture described above. However the mathematically most faithful implementation of Kadanoff’s idea came from another direction and again an analogy provided the key idea.

3.1. Renormalization group and probability theory

Forming blocks of stochastic variables, as in Kadanoff’s picture, is common practice in probability, the central limit theorem (CLT) being the prototype of such a way of reasoning. CLT asserts the following. Let \( \xi_1, \xi_2, \ldots, \xi_n, \ldots \) be a sequence of independent identically distributed (i.i.d.) random variables with finite variance \( \sigma^2 = \mathbb{E}(\xi_i - (\mathbb{E}(\xi_i))^2) \), where \( \mathbb{E} \) represents expectation with respect to their common distribution. Then

\[
\frac{\sum_i^n (\xi_i - \mathbb{E}(\xi_i))}{\sigma n^{1/2}} \xrightarrow{n \to \infty} N(0, 1),
\]

where the convergence is in law and \( N(0, 1) \) is the normal centered distribution of variance 1.

The crucial point is that when we sum many random variables we have to normalize properly the sum in order to obtain a regular probability distribution. In the case of the CLT the correct normalization is proportional to the square root of the number of variables, and represents the square root of the variance of the sum. When we deal with processes which have correlations the variance can be written

\[
\mathbb{E} \left( \left( \sum_i \xi_i \right)^2 \right) = N \sigma^2 + N \sum_j \mathbb{E}(\xi_0 \xi_j),
\]

where we have assumed translational invariance. The sum in the second term is the susceptibility which diverges at the critical point and dominates over the first term. We must therefore change the normalization. The normalization is directly related to the rescaling of the variables in the RG.

We now describe a RG derivation\(^{38}\) of the CLT and then explain how an analogy and a generalization provide an interpretation of the Kadanoff-Wilson view which
unveils the statistical meaning of universality of critical exponents and at the same

time the connection with Eq. (3.1).

3.2. A renormalization group derivation of the central limit theorem

To visualize things consider the random variables $\xi_i$ as discrete or continuous

spins associated to the points of a one-dimensional lattice $\mathbb{Z}$ and introduce the block

variables $\zeta_1 = 2^{-n/2} \sum_1^{2^n} \xi_i$ and $\zeta_2 = 2^{-n/2} \sum_{2^n+1}^{2^{n+1}} \xi_i$. Then

$$
\zeta_{n+1} = \frac{1}{\sqrt{2}} (\zeta_1 + \zeta_2).
$$

Therefore we can write the recursive relation for the corresponding distributions

$$
p_{n+1}(x) = \sqrt{2} \int dy \ p_n(\sqrt{2}x - y)p_n(y) = (R_n p_n)(x).
$$

The nonlinear transformation $R$ is what we call a renormalization transformation.

Let us find its fixed points, i.e. the solutions of the equation $R p = p$. An easy

calculation shows that the family of Gaussians

$$
p_{G,\sigma}(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}},
$$

are fixed points. To prove the CLT we have to discuss the conditions under which the

iteration of $R$ converges to a fixed point of variance $\sigma^2$. This amounts to determining

the so-called domain of attraction of the normal law.

The standard analytical way to prove the CLT is the Fourier transform. Here we

shall illustrate the mechanism of convergence in the neighborhood of a fixed point

from the point of view of nonlinear analysis. There are three conservation laws

associated with $R$: normalization, centering and variance. In formulas

$$
\int p_{n+1}(x) dx = \int p_n(x) dx,
\int x p_{n+1}(x) dx = \int x p_n(x) dx,
\int x^2 p_{n+1}(x) dx = \int x^2 p_n(x) dx.
$$

Therefore only distributions with variance $\sigma^2$ can converge to a Gaussian $p_{G,\sigma}(x)$.

We fix $\sigma = 1$ and write $p_G$ for $p_{G,1}$.

Let us write the initial distribution as a centered deformation of the Gaussian

with the same variance

$$
p_\eta(x) = p_G(x)(1 + \eta h(x)),
$$

where $\eta$ is a parameter. The function $h(x)$ must satisfy

$$
\int p_G(x) h(x) dx = 0,
$$
\[ \int p_G(x)xh(x)dx = 0, \]
\[ \int p_G(x)x^2h(x)dx = 0. \] (3.9)

Suppose now \( \eta \) small. In linear approximation we have
\[ (\mathcal{R}p_\eta) = p_G(1 + \eta(\mathcal{L}h)) + \mathcal{O}(\eta^2), \] (3.10)
where \( \mathcal{L} \) is the linear operator
\[ (\mathcal{L}h)(x) = 2\pi^{-1/2} \int dy e^{-y^2/2} h(y + x^{1/2}). \] (3.11)
The eigenvalues of \( \mathcal{L} \) are
\[ \lambda_k = 2^{1-k/2}, \] (3.12)
and the eigenfunctions the Hermite polynomials. The three conditions above on \( h(x) \) can be read as the vanishing of its projections on the first three Hermite polynomials.

The mechanism of convergence of the deformed distribution to the normal law is now clear in linear approximation: if we develop \( h \) in Hermite polynomials only terms with \( k > 2 \) will appear so that upon iteration of the RG transformation they will contract to zero exponentially as the corresponding eigenvalues are \( < 1 \).

The Gaussian belongs to a very important class of distributions called \textit{stable distributions}. They are characterized by the fixed point equation
\[ p(ax + b) = \frac{a_1 a_2}{a} \int dy p(a_1 (x - y) + b_1) p(a_2 y + b_2), \] (3.13)
where \( a, a_1, a_2 \) are positive numbers. In the next subsection we shall briefly describe how to introduce an analogous concept in the case of strongly dependent variables as those appearing at the critical point in phase transitions. The concept of \textit{stable} or \textit{self-similar random field} will correspond to that of stable distribution.

3.3. \textit{Strongly dependent variables: interacting spins at the critical point}

The notion of self-similar random field of discrete argument was introduced informally in 39) to provide a proper mathematical setting for the notion of RG \textit{a la} Kadanoff-Wilson. In rigorous form it was described in 40) followed by 41). In the present exposition we follow 42) and 43).

Let \( \mathbf{Z}^d \) be a lattice in \( d \)-dimensional space and \( j \) a generic point of \( \mathbf{Z}^d \), \( j = (j_1, j_2, \ldots, j_d) \) with integer coordinates \( j_i \). We associate to each site a centered random variable \( \xi_j \) and define a new random field, \textit{block-spin}
\[ \xi_j^n = (\mathcal{R}_{\alpha, n}\xi)_j = n^{-d\alpha/2} \sum_{s \in V_{j}^n} \xi_s, \] (3.14)
where
\[ V_{j}^n = \{s : j_k n - n/2 < s_k \leq j_k n + n/2\}, \] (3.15)
and $1 \leq \alpha < 2$. The transformation (3.14) induces a transformation on probability measures according to

$$ (R_{\alpha,n}^* P)(A) = P'(A) = P(R_{\alpha,n}^{-1} A), \quad (3.16) $$

where $A$ is a measurable set and $R_{\alpha,n}^*$ has the semigroup property

$$ R_{\alpha,n_1}^* R_{\alpha,n_2}^* = R_{\alpha,n_1 n_2}^*. \quad (3.17) $$

A measure $P$ will be called self-similar if

$$ R_{\alpha,n}^* P = P, \quad (3.18) $$

and the corresponding field will be called a self-similar random field. We briefly discuss the choice of the parameter $\alpha$. It is natural to take $1 \leq \alpha < 2$. In fact $\alpha = 2$ corresponds to the law of large numbers so that the block variable (3.14) will tend for large $n$ to zero in probability. The case $\alpha > 1$ means that we are considering random systems which fluctuate more than a collection of independent variables and $\alpha = 1$ corresponds to the CLT. Mathematically the lower bound is not natural but it becomes so when we restrict ourselves to the consideration of ferromagnetic-like systems.

A theory of self-similar random fields of generality comparable to the case of stable distributions so far does not exist and presumably is very difficult. However Gaussian fields are completely specified by their correlation function and self-similar Gaussian fields can be constructed explicitly.\(^{43}\)

The search of non-Gaussian self-similar fields is considerably more difficult. A reasonable question is whether such fields exist in the neighborhood of a Gaussian one. The approach to this problem developed by physicists, the so-called $\epsilon$ expansion, in our context can be interpreted as follows.

Consider a deformation $P_G(1 + h)$ of a Gaussian self-similar field $P_G$ and the action of $R_{\alpha,n}^*$ on this distribution. It is easily seen that

$$ R_{\alpha,n}^* P_G h = E(h|\{\xi^n\}) R_{\alpha,n}^* P_G = E(h|\{\xi^n\}) P_G(\{\xi^n\}). \quad (3.19) $$

The conditional expectation on the right-hand side of (3.19) will be called the linearization of the RG at $P_G$ and we want to study its stability as a linear operator. For this purpose we have to find the eigenvectors and eigenvalues of $E(h|\{\xi^n\})$. These have been calculated by Sinai. The eigenvectors are appropriate infinite dimensional generalizations of Hermite polynomials $H_k$ which are described in full detail in 43). They satisfy the eigenvalue equation

$$ E(H_k|\{\xi^n\}) = n^{[k(\alpha/2-1)+1]}d H_k(\{\xi^n\}). \quad (3.20) $$

We see immediately that $H_2$ is always unstable. It is a relevant direction in the physicist terminology. The direction $H_4$ becomes unstable when $\alpha$ crosses from below the value $3/2$. Bifurcation theory tells us that generically we must expect an exchange of stability between two fixed points and we should look for the new one in the direction which has just become unstable. By introducing the parameter $\epsilon = \alpha - 3/2$, one can construct a non-Gaussian fixed point using $\epsilon$ as a perturbation parameter. The formal construction is explained in Sinai’s book\(^{43}\) where one can find also a discussion of the questions, mostly still unsolved, arising in this connection.
3.4. Universality

The previous analysis shows that in the probabilistic interpretation of the RG universality of critical phenomena acquires a clear statistical interpretation. In analogy with the case of the CLT there will be different Gibbs distributions that under the RG will converge to the same limit ensemble. A physical universality class will correspond to a subset of the domain of attraction of the limit ensemble. In general we expect to be a subset because not all distributions in the domain of attraction will admit a natural physical interpretation.

3.5. Multiplicative structure

In this section we show that there is a natural multiplicative structure, in mathematics called a cocycle, associated with transformations on probability distributions induced by the block transformation. This multiplicative structure is related to the properties of conditional expectations. Supposing we wish to evaluate the conditional expectation

\[ E(h|\{\xi_j^n\}), \]

where the collection of block variables \( \xi_j^n \) indexed by \( j \) is given a fixed value. Here \( h \) is a function of the spins \( \xi_i \). It is an elementary property of conditional expectations that

\[ E(E(h|\{\xi_j^n\})|\{\xi_j^{nm}\}) = E(h|\{\xi_j^{nm}\}). \]

Let \( P \) be the probability distribution of the \( \xi_i \) and \( R_{\alpha,n}^* P \) the distribution obtained by applying the RG transformation, that is the distribution of the block variables \( \xi_j^n \). By specifying in (3.22) the distribution with respect to which expectations are taken we can rewrite it as

\[ E_{R_{\alpha,n}^* P}(E_P(h|\{\xi_j^n\})|\{\xi_j^{nm}\}) = E_P(h|\{\xi_j^{nm}\}). \]

This is the basic equation of this section and we want to work out its consequences. In analogy with the theory of dynamical systems we interpret the conditional expectation as a linear transformation from the linear space tangent to \( P \) to the linear space tangent to \( R_{\alpha,n}^* P \) and we assume that in each of these spaces there is a basis of vectors \( H_k^P \), \( H_{R_{\alpha,n}^* P} \) connected by the following generalized eigenvalue equation

\[ E_P(H_k^P|\{\xi_j^n\}) = \lambda_k(n, P) H_{R_{\alpha,n}^* P}^P(\{\xi_j^n\}). \]

Equation (3.23) implies that the \( \lambda \) must satisfy the relationship

\[ \lambda_k(m, R_{\alpha,n}^* P)\lambda_k(n, P) = \lambda_k(mn, P). \]

If \( P \) is self-similar (3.25) implies that the \( \lambda \) are powers of \( n \). An example is provided by (3.20). In the theory of the critical point the corresponding eigenvectors are called scaling fields. When \( P \) is not self-similar the \( \lambda \) can be expressed in terms of suitable correlation functions.

The multiplicative renormalization group of quantum field theory and statistical mechanics, Eq. (3.1), is structurally similar to (3.25). It corresponds to a simple transformation of the probability distribution leaving its form unchanged while the values of its parameters are rescaled together with the random variables.
I will conclude with a remark of a more general character. New ideas are not always immediately understood and I would like to point out an aspect relevant in their evolution: this is the language in which an idea is proposed. The spontaneous breakdown of a symmetry was rapidly absorbed by particle physicists: the NJL model was formulated in the standard language of the particle physics community, quantum field theory.

The situation was somewhat different with the renormalization group. The RG for the first time provided a microscopic theory of critical phenomena of wide applicability but required on the part of the interested community to adapt to a new way of looking at the problems and to a new language originated in particle physics. This was a collective effort and happened in a remarkably short time.

An important consequence in both cases was that condensed matter and particle physicists became closer in their way of thinking.

Coming back to analogies, they have a role in my recent work on nonequilibrium statistical mechanics. This is another story for which I refer to my paper *From fluctuations in hydrodynamics to nonequilibrium thermodynamics* in this same issue of the Progress of Theoretical Physics Supplement.

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**References**

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