UNITARITY POLYGONS AND CP VIOLATION AREAS AND PHASES IN THE STANDARD ELECTROWEAK MODEL

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A geometrical interpretation of CP violation in terms of areas of unitarity triangles is presented, in the standard electroweak model with three families of quarks. The extension of the results to the case of four families is investigated. A special case is worked out in detail where it is shown how one determines the CP violating invariant phases and the areas of the CP violation triangles as functions of the moduli of the elements of the quark mixing matrix. For the most general four-family case, which is found to be rather involved, the necessary formulae, for determining the invariant phases and the CP violation areas, are given.

1. Introduction

This paper is devoted to a study of the structure of the quark mixing matrix and CP violation in the standard electroweak model [1]. Our emphasize will be on the geometrical aspects of CP violation and on the determination of the invariant phases of the quark mixing matrix and “CP violation areas” in terms of moduli of its elements.

As we shall explain in the next section, when dealing with the quark mixing matrix, what one usually does is to parametrize it in terms of a number of angles. Then one uses data to determine these angles. However, at the present stage of our knowledge, these angles have no deep physical significance. Is there then any need to introduce them at all? In an earlier publication [2] it was found that for the case of three families it is not necessary to introduce the angles. One may simply parametrize the quark mixing matrix (up to a twofold ambiguity) in terms of four independent moduli of its elements. In other words, by measuring the (CP conserving) rates of four appropriately chosen processes one can find out whether CP is violated or not in much the same way as one “tests” parity violation in the standard model by comparing the (parity conserving) rates of say neutrino and antineutrino interactions.

In this paper, we shall take a new look at the problem of the determination of the quark mixing matrix in terms of directly measurable quantities. The plan of the paper is as follows. In the next section we shall give a review of previous results. In section 3 we discuss the six unitarity triangles of the three-family model. The main point is that all these six triangles, in spite of looking very different, have equal areas. Furthermore, this common area is related to CP violation in the sense that all of these triangles collapse into lines if CP is conserved. The invariant phases, of the quark mixing matrix, as well as the area of the triangles are determined by the sides of the triangles (the moduli of the elements of the quark mixing matrix). Section 4 deals with the case of four families. Unitarity, in this case gives quadrangles. Again all these quadrangles collapse into lines if CP is conserved. Furthermore, these quadrangles can be decomposed into subtriangles. It turns out that the formalism becomes quite complicated. Therefore, we first consider a special case, where one of the elements of the quark mixing matrix is assumed to vanish. We then show, explicitly, how one determines the invariant phases and the areas of CP violation subtriangles as functions of (nine independent) moduli, up to an overall twofold ambiguity, just like in the case of three families. This result is nontrivial because, in the special case considered, there are two independent CP phase angles. In section 5 we consider the...
most general four-family model, where in the usual parametrization there are three $CP$ phase angles. We introduce three invariant phases (see below). The question is then to what extend these phases are constrained by the knowledge of the moduli of the quark mixing matrix? Using unitarity, we derive six constraint equations on these three invariant phases.

2. Parametrizations of the quark mixing matrix

As is well known, the quark masses and mixings, in the standard model, originate from the quark mass matrices. There are two types of mass matrices, the up-kind, which we denote by $m$ and the down-kind which we shall call $m'$. These matrices are both $n$ by $n$, if there are $n$ families of quarks. If we had known them we could have computed the quark masses as well as the measurable of the quark mixing matrix. Since we do not know the mass matrices, the quark masses as well as the measurable of the quark mixing matrix are taken from experiment, whenever there are data. However, the standard model does make one very important prediction, namely that the quark mixing matrix must be an $n$ by $n$ unitary matrix which is not the most general unitary matrix but has “only” $(n-1)^2$ parameters. Usually, one parametrizes the quark mixing matrix, à la Murnaghan [3], in terms of $n(n-1)/2$ Euler-type “rotation angles” and $(n-1)(n-2)/2$ “$CP$ phase angles”. For $n=2$ there is only one rotation angle and one obtains the Cabibbo-GIM structure [4]. For $n=3$, which is the smallest $n$ compatible with data, there are three rotation angles ($\theta_1$, $\theta_2$, $\theta_3$) and one $CP$ phase $\delta$. There are many ways \(^{41}\) of introducing these angles, all of them being of course, equivalent to each other and the original among them being the one introduced by Kobayashi and Maskawa [6].

Another approach to the parametrization of the quark mixing matrix is to introduce the invariant phases [7,8] of the quark mixing matrix. These phases, unlike the usual phases, are independent of how the phases of the quark fields are chosen. Following Bjorken and Dunietz [9] we may call these phases “the plaquette phases”. In this paper, we shall discuss these phases in some detail (see below).

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\(^{41}\) As an example of such parametrizations see ref. [5].
where the rows $\alpha$, $\beta$ and $\gamma$ are any of the six permutations of $(1, 2, 3)$. The columns must also be different. For example, we may take $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $j = 2$ and $k = 3$. There is also a column “formulation” for $J^2$ given by [2]

\[ 4J^2 = \left( |V_{\alpha j}| |V_{\beta k}| + |V_{\alpha k}| |V_{\beta j}| \right)^2 - |V_{\alpha j}|^2 |V_{\beta k}|^2 \times \left( |V_{\alpha j}|^2 |V_{\beta k}|^2 - (|V_{\alpha j}| |V_{\beta k}| - |V_{\alpha k}| |V_{\beta j}|)^2 \right). \]  

(6)

Again $j \neq k \neq 1$ and $\alpha \neq \beta$. The factors in the above relations are written in such a way that each of them is manifestly non-negative. The expressions above look similar to expressions one obtains when dealing with kinematics of, for example, two-body decays. In fact, what we have is simply

\[ 4J^2 = -\lambda(a^2, b^2, c^2), \]  

(7)

where $\lambda$ is the Källen lambda function [11] given by

\[ \lambda(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx). \]  

(8)

The arguments of the lambda function, in the row formulation, eq. (5), are given by

\[ a = |V_{\alpha j}| |V_{\alpha k}|, \quad b = |V_{\beta j}| |V_{\beta k}|, \quad c = |V_{\gamma j}| |V_{\gamma k}|. \]  

(9)

The lambda function is also sometimes called the triangular function. The reason is that it determines the area of the triangle which has the three sides $a$, $b$ and $c$. Thus $J$ has a simple geometrical interpretation [2] as the area, up to a factor, of the triangle with the sides $a$, $b$ and $c$. The column formulation may also be written and interpreted in a similar way. Thus all the measurable quantities of the quark mixing matrix are expressible in terms of the four independent absolute values of the elements of the mixing matrix. However, in this parametrization of the quark mixing matrix, the sign of $J$ is not fixed. We shall now take a closer look at these triangles which we shall call the “unitarity triangles” or the $CP$ triangles (see below).

3. The unitarity triangles

In order to acquire a better understanding of the structure of the standard model, in this section we shall re-examine the case $n = 3$ in somewhat more detail.

Consider the quark mixing matrix

\[
\begin{pmatrix}
V_{11} & V_{12} & V_{13} \\
V_{21} & V_{22} & V_{23} \\
V_{31} & V_{32} & V_{33}
\end{pmatrix}
\]

(10)

The “row unitarity” and the column “unitarity relations” read respectively

\[ \sum V_{\alpha j} V^{*}_{\beta j} = \delta_{\alpha \beta}, \]  

(11)

and

\[ \sum V_{\alpha j} V^{*}_{\alpha k} = \delta_{jk}. \]  

(12)

Here the repeated indices are summed from 1 to 3. We may consider the quark mixing matrix, eq. (10) as a 3 by 3 lattice with vertices $V_{\alpha j}$ and “links” between them, which we shall now define. We define the $a$-link as being the one which “connects” the first and the second columns through the relation

\[ a_{\alpha} = V_{\alpha 1} V^{*}_{\alpha 2}, \quad \alpha = 1, 2, 3. \]  

(13a)

By this definition, there are three $a$-links. The $a$-links are easily visualized by oriented arrows going from the first to the second column. In fig. 1 we have indicated the position of the $a$-links by a symbolic arrow above the lattice. Similarly, the $b$-links will be defined as the ones connecting the second and third columns, viz.,

\[ b_{\alpha} = V_{\alpha 2} V^{*}_{\alpha 3}, \quad \alpha = 1, 2, 3. \]  

(13b)

We define the links among the neighbouring rows in the same fashion via

\[ c_{j} = V_{j1} V_{j2}, \quad j = 1, 2, 3, \]  

(13c)

\[ d_{j} = V_{j2} V_{j3}, \quad j = 1, 2, 3. \]  

(13d)

In addition to the above “short links” there are two “long links” connecting the non-neighbouring rows or columns,

\[ a \quad b \]  

\[ c \quad d \]  

\[ e \quad f \]

Fig. 1. The quark mixing “lattice” for three families. The arrows denote the appropriate oriented links.
\[ e_\alpha \equiv V_{\alpha 1} V_{\alpha 3}^*, \quad \alpha = 1, 2, 3, \]  
\[ f_j \equiv V_{1j} V_{3j}^*, \quad j = 1, 2, 3. \]  

Note that there is no fundamental difference between the short and the long links. By permuting rows and/or columns the short and long links transform into each other. The unitarity relations, eqs. (11) and (12) simply tell us that 
\[ \sum a_j = \sum b_j = \sum c_j = \sum d_j = \sum e_j = \sum f_j = 0, \]  
where, in each sum, \( j \) runs from 1 to 3 and we have used latin indices for all the links. The relations (14) become particularly beautiful if we follow Caspar Wessel [12] and Jean Robert Argand [13] as we shall do next.

Consider first the \( a \)-links. The three \( a_j \) are complex numbers which we may represent as vectors in the complex plane. Eq. (14) then tells us that these three vectors define a triangle, as shown schematically in Fig. 2. Likewise, the \( b \)-links define a "\( b \)-triangle" and so on. From eq. (14) follows that there are altogether six different looking unitarity triangles. In Fig. 2, we have also depicted a schematic picture of the \( f \)-triangle.

The most interesting feature of these triangles is that, in spite of looking very different, they all have equal areas. It is easy to verify this fact (which is a consequence of unitarity) by using eq. (3) and taking different values for \( \alpha, \beta, j \) and \( k \). We have
\[ J = \text{Im}(a_1 a_2^*) = \text{Im}(b_1 b_2^*) = \text{Im}(c_1 c_2^*) \]  
\[ = \text{Im}(d_1 d_2^*) = \text{Im}(e_1 e_2^*) = \text{Im}(f_1 f_2^*). \]  

Thus
\[ J = |a_1 a_2| \sin \phi_{12} = |b_1 b_2| \sin \phi_{12} = \ldots. \]  

Here \( \phi_{12} \) is the angle between the links (vectors) \( a_1 \) and \( a_2 \) and so on. These angles are the invariant or the plaquette phases mentioned before. Thus we see that \( J \) equals twice the area of (any of the) unitarity triangles. Since we have explicit expressions for the square of this area in terms of the moduli [see eqs. (5)–(7)], i.e., we can compute the plaquette phases, up to an overall sign, in terms of the moduli. Note that all the above six triangles collapse to lines if \( CP \) is conserved. Therefore, we may call them the \( CP \) triangles and the area defined by them is "the \( CP \) violation area". Another nice feature of the unitarity triangles is that the unitarity inequalities, may be immediately written down as triangular inequalities, for example, the inequality
\[ \|a_3\| - |a_2| \leq |a_1| \leq |a_3| + |a_2| \]

can be written down immediately (see Fig. 1) but when rewritten in terms of the moduli it does not look so obvious, viz.,
\[ \|V_{31} V_{32}\| - |V_{21} V_{22}| \leq |V_{11} V_{12}| \leq |V_{31} V_{32}| + |V_{21} V_{22}|. \]

In summary, we have six unitarity triangles if there are three families. The lengths of the sides of these triangles are determined by the moduli of the links or, in other words, by the moduli of the elements of the quark mixing matrix. The angles in these triangles are nothing but the invariant phases of the quark mixing matrix. Triangles are very special. Knowing the sides suffices to determine the angles. The utility of the triangular construction was first noted in ref. [2] and also, independently by Bjorken [14]. Note that there is a twofold (orientation) ambiguity in such determinations. This corresponds to the fact that we cannot distinguish between the quark mixing matrix \( V \) and its complex conjugate \( V^* \), by just knowing the moduli. In order to determine \( V \) uniquely, in addition to knowing four independent moduli, we need to know the sign of \( J \).

**4. The case of four families**

As there is no known principle, in the standard model, which limits the number of families to three it is of interest to study the generalization of the results of the previous section to the case of larger number of families. In this and the following section we
will investigate the case of four families.

We know that the number of parameters of the quark mixing matrix is \((n-1)^2\), if there are \(n\) families. This number equals the number of independent moduli of the elements of the mixing matrix, because given the moduli of, for example, the \((n-1)^2\) elements which belong neither to the last row nor to the last column the remaining moduli are fixed by unitarity. An interesting question is then to what extend the knowledge of these \((n-1)^2\) moduli constrains the quark mixing matrix? We know from the very beginning that, by knowing just the moduli there will be at least a twofold ambiguity (just as in the case \(n=3\)) because there is no way to distinguish a matrix \(V\) from its complex conjugate \(V^*\).

We start our investigation again by treating the 4 by 4 quark mixing matrix as a lattice and introduce the links. The links between nearest neighbours are depicted in fig. 3 and are given by

\[
\begin{align*}
\alpha_i &= V_{ij}V_{j*}, & \beta_i &= V_{ij}V_{j*}, & \gamma_i &= V_{ij}V_{j*}, \\
\delta_i &= V_{ij}V_{j*}, & \epsilon_i &= V_{ij}V_{j*}, & \zeta_i &= V_{ij}V_{j*},
\end{align*}
\]

(17)

There are also six long links connecting non-neighbours. We do not specify them here but will introduce them when needed (see below). As mentioned before, by permutations among rows and columns any long link may be transformed into a short link.

The unitarity relations are again of the form given in eq. (14), \(\sum a_i = 0\), etc. There are six row and six column unitarity relations of this kind. Let us consider, for example, the \(\alpha\)-links (see fig. 3). We know from eq. (17) that the four vectors \(a_i\) form a quadrangle. Furthermore, given the moduli of the quark matrix, we even know the lengths of the sides of the quadrangle. But, unlike the case of the triangle, our knowledge does not suffice to determine the shape of the quadrangle or its area. It is also evident that, for example, the area of the quadrangle could even be zero and yet \(CP\) be violated. However, all these quadrangles collapse into lines if \(CP\) is conserved. Moreover, it is clear that the relative phases of the links are related to the areas of sub-triangles, which make up the quadrangles, viz.,

\[
\text{Im}(a_1a_2^*) = |a_1a_2| \sin \varphi_{12}.
\]

(18)

It turns out that the treatment of the general four-family case is quite involved. Therefore, we shall first treat a special case where one of the elements of the quark mixing matrix is assumed to be zero. Without loss of generality, we take, for example \(V_{41} = 0\).

4.1. The special case \(V_{41} = 0\)

In this special case there are, in the usual parameterization with rotation angles and phases, five angles and two phases. We will now show that, apart from the overall ambiguity (\(V\) versus \(V^*\)) the phases are fixed by the moduli. We introduce the plaquette phases by

\[
\begin{align*}
\alpha_i &= |a_ia_i| \exp(\text{i} \varphi_{ik}), \\
\beta_i &= |b_ib_i| \exp(\text{i} \varphi_{ik}),
\end{align*}
\]

(19)

and so on. Some of these phases are schematically depicted in fig. 3. In the case under consideration the phase \(\varphi_{ik}\) is not a measurable quantity because the link \(a_i\) is zero. The \(\alpha\)-links form, again, a triangle and thus we can determine the plaquette phases \(\varphi_{12}\) and \(\varphi_{23}\). Because of the twofold ambiguity \(\sin \varphi_{12}\) can either be positive or negative. We take the positive solution for \(\sin \varphi_{12}\) whereby we have

\[
\begin{align*}
\sin \varphi_{12} &= [-\lambda(A)]^{1/2} \sin |a_1a_2|, \\
\cos \varphi_{12} &= (|a_3|^2 - |a_1|^2 - |a_2|^2) / 2 |a_1a_2|,
\end{align*}
\]

(20a)

and

\[
\begin{align*}
\sin \varphi_{23} &= [-\lambda(A)]^{1/2} \sin |a_2a_3|, \\
\cos \varphi_{23} &= (|a_1|^2 - |a_2|^2 - |a_3|^2) / 2 |a_2a_3|.
\end{align*}
\]

(20b)

Here \(\lambda(A)\) denotes \(\lambda(|a_1|^2, |a_2|^2, |a_3|^2)\) as defined by eq. (8). Note that if we take the negative solution for \(\sin \varphi_{12}\) then \(\sin \varphi_{23}\) would also be negative. The angles are thus "correlated", a concept which we will

\[\text{Fig. 3. The quark mixing lattice for four families. Some of the links and the invariant phases are explicitly indicated.}\]
often refer to in the following. In order to determine the remaining phases we may, for example, introduce the links, \( g_j \), between the first and the third column, viz.,

\[
g_j = V_{1j} V_{3j}^\dagger
\]

and the appropriate phases. We then have

\[
\sin \phi_{t2} = \eta \left[ -\lambda(G) \right]^{1/2} / g_1 g_2,
\]
\[
\cos \phi_{t2} = \left( |g_3|^2 - |g_1|^2 - |g_2|^2 \right) / 2 |g_1 g_2|,
\]

\[
\sin \phi_{t3} = \eta \left[ -\lambda(G) \right]^{1/2} / g_2 g_3,
\]
\[
\cos \phi_{t3} = \left( |g_1|^2 - |g_2|^2 - |g_3|^2 \right) / 2 |g_2 g_3|.
\]

In eqs. (22), \( \lambda(G) = \lambda \left( |g_1|^2, |g_2|^2, |g_3|^2 \right) \). Furthermore, \( \eta \) denotes an, as yet, undetermined sign, i.e., \( \eta = \pm 1 \). In the absence of further ambiguities only one of the signs is allowed and the angles are correlated. In fact, that is exactly what happens. The \( d \)-unitarity (see fig. 3) correlates the “\( g \)-angles” to the “\( a \)-angles”. The point is that the \( d \)-unitarity gives

\[
|d_1 d_2| \cos(\phi_{t2}) + |d_2 d_3| \cos(\phi_{t3}) + |d_1 d_3| \cos(\phi_{t4}) = \left[ |d_1|^2 - |d_2|^2 - |d_3|^2 \right] / 2.
\]

From fig. 3 we see that

\[
\phi_{t2} = \phi_{t2}, \quad \phi_{t4} = \phi_{t2} - \phi_{t2}.
\]

Substituting eqs. (20), (22) and (24) into eq. (23) determines the sign \( \eta \). All the remaining angles are subsequently uniquely determined by unitarity. Evidently, the areas of all the subtriangles are also determined.

In conclusion, if any of the elements of the quark mixing matrix is zero the matrix is (apart from the twofold ambiguity) completely determined by the moduli. In order to resolve the ambiguity the sign of one plaquette phase needs to be measured.

Actually the special case considered here may turn out to be quite close to that in the real world. It is known that the mixing of the second and the third families is much smaller than that of the first and the second families. If this trend would continue, it is quite possible the element farthest away from the diagonal, in the quark mixing matrix, is very small, for example of order \( \lambda^6 \), where \( \lambda \) is the parameter of the Wolfenstein parametrization [15].

5. The general four-family model

We now give the results, we have obtained so far, for the general case where none of the elements of the quark mixing matrix is zero. As the reader may have experienced herself (himself) the straightforward application of unitarity leads to very complicated (non-linear looking) equations and it is not easy to know what to do with them. Instead we shall combine row and column unitarity relations and will obtain a “linear” system of equations which is much simpler to deal with. Our approach is as follows. We know that in the usual parametrization of the quark mixing matrix there are only three independent plaquette phases. Our lattice, fig. 3 has nine phases. Thus we can compute six of the phases as functions of the three “independent” phases and the moduli. Since the magnitude of \( \exp(i\varphi) \) equals unity, for each of the six phases \( \varphi \) which we compute we get six relations among the three independent plaquette phases and the moduli. The first step is to find these six relations and the second step is to examine to what extent these relations determine/correlate the phases. We have done the first step as follows.

First we introduce \( \phi_{t2}, \phi_{t3} \) and \( \phi_{t4} \) as the three independent phases. By permuting rows and/or columns, our choice may be translated to other choices, such as taking the phases along the antidiagonal direction, etc. Next we introduce the notation

\[
z_{\alpha}^{\pm} = \exp(i\phi_{\alpha}^{\pm}), \quad z_{\alpha}^{\pm} = \exp(-i\phi_{\alpha}^{\pm}), \quad \alpha = a, b, c.
\]

Using unitarity, we find the following linear system of equations for the six phases which we wish to determine

\[
\begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{pmatrix}
\begin{pmatrix}
X_{11} & X_{12} & X_{13} \\
X_{21} & X_{22} & X_{23} \\
X_{31} & X_{32} & X_{33}
\end{pmatrix}
\begin{pmatrix}
X_{14} & X_{15} & X_{16} \\
X_{24} & X_{25} & X_{26} \\
X_{34} & X_{35} & X_{36}
\end{pmatrix}
= \begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{pmatrix}.
\]

(26)

Here we have only written the non-zero elements of the matrix \( x \). These elements are given by
$x_{11} = |a_1| z_{12}^a + |a_2|$, $x_{12} = |a_4|$, $x_{22} = |f_1|$, $x_{23} = |f_5| + |f_4| z_{54}$, $x_{33} = |b_4|$, $x_{34} = |b_1| z_{23}^b$, $x_{44} = |d_1| z_{12}^d + |d_2|$, $x_{45} = |d_4|$, $x_{55} = |c_1|$, $x_{56} = |c_1| + |c_4| z_{54}^c$, $x_{66} = |e_4|$, $x_{61} = |e_1| z_{23}^e$. (27)

The quantities on the right-hand side of the eq. (26) are given by

$y_1 = -|a_3|$, $y_2 = -|f_2|$, $y_3 = -(|b_2| z_{23}^b + |b_3|)$, $y_4 = -|d_3|$, $y_5 = -|c_2|$, $y_6 = -(|e_2| z_{23}^e + |e_3|)$. (28)

One can check that the determinant of the matrix $x$ is, in general, non-zero. Remember that we are taking the case where none of the elements of the mixing matrix is zero because the cases with zeros are much simpler (see the previous section). Of course, by letting one of the elements of the mixing matrix to approach zero the equations become linearly dependent.\[5\] Inverting the matrix $x$ (which is easy to do) supplies the six complex numbers $z_{23}^a, z_{34}^b, z_{45}^c, z_{56}^d, z_{61}^e$ as functions of the moduli and the three input phases $\phi_{12}^a, \phi_{23}^b$ and $\phi_{34}^c$. These are exactly the equations we were looking for. The requirement that the six computed $z$'s be of unit norm give us six relations for the three phases $\phi_{12}^a, \phi_{23}^b$ and $\phi_{34}^c$. Unfortunately, these constraint relations are too complicated to be given here. We intend to examine these equations, in the near future, and look for their solutions. The problem is certainly soluble, on the computer, where one may search for numerical solutions.

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### References


