MASS MIXING AND CP VIOLATION IN THE B°-B̅° SYSTEM

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Mass mixing and CP violation in the B°-B̅° system are studied in the standard six-quark model. The mass and width differences are obtained by reliable methods, and their radiative corrections included to leading logarithm. Constraints on the Kobayashi-Maskawa parameters are rederived in order to permit a quantitative account of B° phenomenology. Strong evidence against the observability of CP violation in B°-B̅° mixing is presented.

1. Introduction

Ellis, Gaillard, Nanopoulos and Rudaz were first to point out the possibility of large CP-violating effects in neutral B systems [1]. Since the interpretation of the T as bottomonium, the prospect of observing weak mass mixing and CP violation in B°-B̅° decay has motivated several estimates of mixing and charge-asymmetry parameters in the standard model, but conclusions vary widely. The present work shows that despite some rather large unknowns which necessarily enter the analysis, certain quantitative statements can be made which overwhelmingly support the non-observability of CP violation in B°-B̅° mixing in the standard model.

Numerous methodological improvements contribute to the reliability of these findings. In sect. 2, the familiar mixing and CP formulas from kaon physics are extended to accommodate the potentially larger CP violation characteristic of B-mesons. The unjustified use of an approximation for the CP impurity parameter ε valid near the origin is thereby avoided. A new charge-asymmetry quantity is then defined to accurately reflect the observability of CP violation in B°-B̅° mixing. Sect. 3 contains a concise summary of B-meson decay. In sect. 4, the mass and width differences are computed and their radiative corrections are summed to all orders in the leading logarithms. ΔΓ is taken directly from the 2W ± exchange diagrams, eliminating huge errors due to momentum mismatch in analyses based on common decay channels. Sect. 5 contains a conservative reevaluation of the Kobayashi-Maskawa (KM) constraints, to furnish meaningful bounds on CP violation in B°-B̅° mixing. Sect. 6 summarizes B°-B̅° mixing and CP violation with conclusions regarding their experimental detection.
2. Phenomenology of $B^0 - \bar{B}^0$ mixing

In the standard six-quark model of weak and electromagnetic interactions [2] the charged weak current is given by

$$J^\mu = (\bar{u}, \bar{c}, \bar{t}) \gamma^\mu (1 + \gamma^5) U \begin{pmatrix} d \\ s \\ b \end{pmatrix},$$

(2.1)

where $u, c$ and $t$ denote the charge $+\frac{2}{3}$ quark fields and $d, s$ and $b$ denote the quark fields with charge $-\frac{1}{3}$. $U$ is a $3 \times 3$ unitary matrix which can be specified by three Cabibbo-like angles $\theta_1, \theta_2$ and $\theta_3$ and by a phase $\delta$. With the standard choice of quark field phases [3],

$$U = \begin{pmatrix} c_1 & -s_1c_3 & -s_1s_3 \\ s_1c_2 & c_1c_2c_3 - s_2s_3e^{i\delta} & c_1c_2s_3 + s_2c_3e^{i\delta} \\ s_1s_2 & c_1s_2c_3 + c_2s_3e^{i\delta} & c_1s_2s_3 - c_2c_3e^{i\delta} \end{pmatrix},$$

(2.2)

where $s_i = \sin \theta_i$ and $c_i = \cos \theta_i$, for $i = 1, 2, 3$. Without loss of generality, the angles $\theta_1, \theta_2$ and $\theta_3$ may be chosen to lie in the first quadrant, in which case the phase $\delta$ must a priori be allowed to assume all values ($0 < \delta < 2\pi$). This phase gives rise to $CP$ violation because it cannot be removed from the matrix $U$ by adjusting the phases of the quark fields [3]. The six-quark model is consistent with the $CP$ violation observed in the neutral kaon system and with the experimental upper limit on the electric dipole moment of the neutron [4].

While the observation of $CP$ violation has been confined to the neutral kaon system, one could also hope to observe $CP$ violation in $B$-meson decays. The analysis of $CP$ violation in the neutral $B$-meson system is similar to that in the neutral kaon system where we introduce mass and decay matrices $M - \frac{1}{2}i\Gamma$. The phenomenological hamiltonian in the flavor eigenbasis can be written

$$\begin{pmatrix} M - \frac{1}{2}i\Gamma_1 \\ M_{12} - \frac{1}{2}i\Gamma_{12} \end{pmatrix} \begin{pmatrix} B^0 \\ \bar{B}^0 \end{pmatrix}.$$ (2.3)

The mass eigenstates*

$$B_{1,2} = \frac{1}{\sqrt{2(1 + |\epsilon|^2)}} \left[ (1 + \epsilon)B^0 \pm (1 - \epsilon)\bar{B}^0 \right]$$ (2.4)

* We adopt a convention where the $B^0$ meson has quark content $bd$ and the $\bar{B}^0$ meson has quark content $\bar{d}b$. Then $CP B^0 = -\bar{B}^0$ and $CP \bar{B}^0 = -B^0$. If $\epsilon = 0$ the state $B_1$ is odd under $CP$ and the state $B_2$ is even under $CP$. 
correspond to eigenvalues
\[ M_{1,2} = M \pm \text{Re} \sqrt{(M_{12} - \frac{1}{2} i \Gamma_{12}) (M_{12}^* - \frac{1}{2} i \Gamma_{12}^*)}, \quad (M \pm \text{Re} M_{12}), \]
\[ \Gamma_{1,2} = \Gamma \mp \text{Im} \sqrt{(M_{12} - \frac{1}{2} i \Gamma_{12}) (M_{12}^* - \frac{1}{2} i \Gamma_{12}^*)}, \quad (\Gamma \pm \text{Re} \Gamma_{12}). \] (2.5)

The CP impurity parameter \( \epsilon \) is given by
\[ \epsilon = \frac{-\text{Re} M_{12} + \frac{1}{2} i \text{Re} \Gamma_{12} + \sqrt{(M_{12} - \frac{1}{2} i \Gamma_{12}) (M_{12}^* - \frac{1}{2} i \Gamma_{12}^*)}}{i \text{Im} M_{12} + \frac{1}{2} \text{Im} \Gamma_{12}} \]
\[ \left( \frac{\frac{1}{2} \text{Im} \Gamma_{12} + i \text{Im} M_{12}}{\frac{1}{2} i \Delta \Gamma - \Delta M} \right). \] (2.6)

\( \epsilon, \Delta M = M_1 - M_2 \) and \( \Delta \Gamma = \Gamma_1 - \Gamma_2 \) reduce to the familiar expressions in parentheses in the small-\( \epsilon \) "kaon" limit. This limit is not appropriate for the \( B^0 - \bar{B}^0 \) system since the phase of \( M_{12} \), proportional to \( U_{tb} \), is roughly \( e^{2i\delta} \), where \( \delta \) is not known to be small. This does not imply that CP violation need be large, for a careful analysis will show that \( \Gamma_{12} \) also depends largely on \( U_{tb} \), so that \( M_{12} \) and \( \Gamma_{12} \) have nearly the same phase. Inspection of eq. (2.6) reveals that \( \epsilon \) is pure imaginary if \( M_{12} \) and \( \Gamma_{12} \) have the same phase. Certainly, from eq. (2.4), an imaginary \( \epsilon \) can be transformed away through a redefinition of the \( B^0 \) phase, so that CP violation must, in fact, vanish in this limit.

The experimental parameters for weak mixing and CP violation can be expressed in terms of the quantities \( r \) and \( \tilde{r} \) due to Pais and Treiman [5]:
\[ r \equiv \frac{\Gamma(B^0 \to \chi^+ \ell^+ \nu)}{\Gamma(B^0 \to \chi^+ \ell^- \bar{\nu})} = \left| \frac{1 - \epsilon}{1 + \epsilon} \right|^2 \frac{\Delta M^2 + (\frac{1}{2} \Delta \Gamma)^2}{2 \Gamma^2 + \Delta M^2 - (\frac{1}{2} \Delta \Gamma)^2}, \]
\[ \tilde{r} \equiv \frac{\Gamma(\bar{B}^0 \to \chi^- \ell^+ \bar{\nu})}{\Gamma(B^0 \to \chi^- \ell^- \nu)} = \left| \frac{1 + \epsilon}{1 - \epsilon} \right|^2 \frac{\Delta M^2 + (\frac{1}{2} \Delta \Gamma)^2}{2 \Gamma^2 + \Delta M^2 - (\frac{1}{2} \Delta \Gamma)^2}. \] (2.7)

Denoting by \( l^{++} \) the number of events of the type \( (\chi^- \ell^+ \nu)(\chi^- \ell^+ \nu) \) coming from the production, subsequent mixing and decay of \( B^0, \bar{B}^0 \) pairs in \( e^+ e^- \) annihilation, one defines the mixing parameter \( r_2 \) [6]:
\[ r_2 \equiv \frac{l^{++} + l^{--}}{l^{++} + l^{+-} + l^{-+} + l^{--}} = \frac{r + \tilde{r}}{1 + r + \tilde{r} + r \tilde{r}}. \] (2.8)

CP violation appears as a charge asymmetry in the number of these same-sign
dilepton events. It is described by the parameter \( a \) [6]:

\[
a \equiv \frac{l^{++} - l^{--}}{l^{++} + l^{--}} = \frac{r - \bar{r}}{r + \bar{r}} = \frac{-4 \text{Re} \epsilon (1 + |\epsilon|^2)}{\left(1 + |\epsilon|^2\right)^2 + 4(\text{Re} \epsilon)^2}.
\]  

Finally, one can derive the "total lepton charge asymmetry" of all primary leptons coming from decays of the neutral B system:

\[
l^\pm \equiv \frac{l^+ - l^-}{l^++l^-} = \frac{r - \bar{r}}{2 + r + \bar{r}}.
\]  

The total lepton charge asymmetry \( l^\pm \) is a true indicator of the observability of \( CP \) violation, since it reflects both the amount of mixing and of \( CP \) violation. For although the theory may predict a large dilepton asymmetry \( a \), unless this is accompanied by significant mixing, same-sign dileptons are not produced and the asymmetry is unobservable.

As we shall see, the standard model can accommodate large values of \( a \) and \( r_2 \) individually, but the magnitude of \( l^\pm \) is strictly bound. \( l^\pm \) provides a clean check on the standard model, and a comparison against other models with more quark doublets or charged Higgs exchange [7], where \( l^\pm \) could be substantial.

\( a \) has been the favored observable in connection with \( CP \) violation, since the magnitude of \( l^\pm \) is always reduced by the semileptonic decays which do not involve mixing. This advantage, however, is offset by an extremely low probability of observing dilepton events. Small semileptonic branching ratios combined with restricted solid angle and limited detector efficiencies mean that the number of dilepton events observed compared with single primary leptons is \( \frac{1}{10} \). For this reason, \( l^\pm \) may be the best experimental handle on \( CP \) violation for a restricted sample size.

The leptons appearing in eqs. (2.8)–(2.10) are "primary" leptons from the decay of B-mesons. Cascade decays of charmed hadrons produce secondary leptons which must be excluded by a momentum cutoff. A small component of these secondary leptons will mimic \( B^0 - \bar{B}^0 \) mixing, but should not appear as \( CP \) violation. Details of background and detection are considered elsewhere in the literature [8].

### 3. The \( B^0 \) decay rate

It is clear from eqs. (2.3)–(2.7) that the mass and width differences \( M_{12} \) and \( \Gamma_{12} \), as well as the total rate for B decay \( \Gamma \) are required to complete the analysis of \( B^0 - \bar{B}^0 \) mixing. The decay rate is estimated to first approximation by considering the decay as due to that of a free b-quark, with the light quark playing a purely spectator role.

* These fractions are based on solid-angle restrictions and detector efficiencies associated with the CLEO detector.
These "spectator" decays can be hadronic or semileptonic, and of these the semileptonic rate follows immediately by analogy with muon decay:

$$\Gamma_{SL} = \frac{G_F^2 m_b^5}{192 \pi^3} \left[ |U_{cb}|^2 (2 \cdot 0.44 + 0.06) + |U_{ub}|^2 (2 + 0.32) \right]. \quad (3.1)$$

Numerical coefficients reflect phase space allowances for massive particles in the final state, as conveniently tabulated by Cortes, Pham and Tounsi [9]. Eq. (3.1) gets modified by low-energy QCD processes, similar to the radiative corrections to the electron spectrum in muon decay [10]. Because these effects are known to alter the total rate only slightly, they are justifiably ignored for the present purposes [11].

The hadronic decays are similar, but are subject to short-distance QCD renormalizations. Under renormalization group scaling, four-quark operators ($S$) which appear to tree level in the effective hamiltonian at $M_W$ mix with operators of a different color structure ($O$), where

$$S \equiv \bar{q}_1 \gamma^\mu (1 + \gamma^5) q_2 \bar{q}_3 \gamma_\mu (1 + \gamma^5) q_4,$$

$$O \equiv \bar{q}_1 \gamma^\mu (1 + \gamma^5) t_{\alpha \beta} q_2 \bar{q}_3 \gamma_\mu (1 + \gamma^5) t_{\alpha \beta} q_4. \quad (3.2)$$

There is no mixing among operators with different flavor content if quark masses are neglected. The linear combinations which scale multiplicatively are known from early work on strange decays [12]: $O^+ = S + \frac{3}{2} O$, $O^- = S - 3O$. Their Wilson coefficients evolve with the renormalization point such that

$$C^+(m_b) = \left[ \frac{\alpha_s(m_b)}{\alpha_s(M_W)} \right]^{-2/7} C^+(M_W) \equiv f_+ C^+(M_W),$$

$$C^-(m_b) = \left[ \frac{\alpha_s(m_b)}{\alpha_s(M_W)} \right]^{4/7} C^-(M_W) \equiv f_- C^-(M_W). \quad (3.3)$$

In terms of $f_+$ and $f_-$, the spectator rate into hadrons is

$$\Gamma_H = \frac{G_F^2 m_b^5}{192 \pi^3} \left[ 2 f_+^2 + f_-^2 \right] \left[ |U_{cb}|^2 (0.44 |U_{ud}|^2 + 0.42 |U_{us}|^2) 
+ 0.12 |U_{cd}|^2 + 0.10 |U_{cs}|^2) 
+ |U_{ub}|^2 (0.99 |U_{ud}|^2 + 0.97 |U_{us}|^2) 
+ 0.44 |U_{cd}|^2 + 0.42 |U_{cs}|^2) \right]. \quad (3.4)$$
The coefficient \((2f_+^2 + f_-^2)\) replaces a factor of 3 which occurs in the uncorrected parton model from summing over colors. For values of \(\Lambda_{\text{QCD}}\) between 0.1 and 0.5 GeV, \((2f_+^2 + f_-^2)\) ranges from 3.26 to 3.63, so the short-distance effects of QCD are extremely modest. It is important to understand why this is so. The operator \(S\), present at \(M_w\), mixes purely into \(O\) under one-loop renormalization, and not into itself. Therefore, the coefficient of \(O\) gets generated at order \(\alpha_s\), and generates in turn a correction to the coefficient of \(S\) beginning at order \(\alpha_s^2\). The effective hamiltonian in the \(S, O\) basis is

\[
H_{\text{eff}} = \sqrt{\frac{2}{\pi}} G_F \left[ \left( \frac{2}{3} f_+ + \frac{1}{3} f_- \right) S + \left( f_+ - f_- \right) O \right],
\]

where

\[
\left( \frac{2}{3} f_+ + \frac{1}{3} f_- \right) = 1 + O\left( \alpha_s^2 \log^2 M_w \right),
\]

\[
\left( f_+ - f_- \right) = O\left( \alpha_s \log M_w \right).
\]

(3.5)

Since the interference between the spectator decays induced by the operators \(S\) and \(O\) vanishes, corrections to the decay rate begin with order \(\alpha_s^2\). There are no QCD corrections to the spectator decays proportional to \(\alpha_s \log M_w\). As we shall see, this result, first noted by Peccei et al., extends to \(\Delta \Gamma\) as well.

When one is careful to include the t-quark threshold in computing \(H_{\text{eff}}\), new operator structures akin to the “penguin” operators in kaon decay get generated [13]. Although interesting, these operators are not expected to contribute significantly to the total rate. To lowest order in QCD, the penguin diagram contributes to the hamiltonian

\[
H_{\text{eff}}^{\text{penguin}} = \sqrt{\frac{2}{3\pi}} G_F \frac{\alpha_s}{3\pi} \sum_q U_{qb} U_{qd}^* \log m_q^2 \cdot \bar{d} \gamma^\mu (1 + \gamma^5) t^A b
\]

\[
\times \left( \bar{u} \gamma_\mu t^A u + \bar{d} \gamma_\mu t^A d + \cdots \right).
\]

(3.6)

Since the ordinary \((V + A)(V + A)\) operators which occur in \(H_{\text{eff}}\) at the tree level are not suppressed in the matrix elements for B decay (as they are for kaon decay), these penguin terms proportional to \(\alpha_s^2 / 3\pi^2\) in the rate should be negligible [9].

There is, however, a second class of tree-level diagram which contributes to B decay, where the b and d quarks exchange a W boson. Since the weak interactions are effectively local, these “exchange” diagrams are proportional to the wave function at the origin, bringing a factor of \(f_B^2\) in the rate. In addition, they are “helicity suppressed”; angular momentum conservation requires that the rate will be proportional to a final-quark mass squared. Considering the greater number of modes available to the spectator decays, these “exchange” graphs, proportional to \(m_B f_B^2 m_c^2\) contribute only at the level of one percent if \(f_B = 0.3\) GeV.
More important may be a related diagram, enhanced by hard gluon emission as in fig. 1. A similar diagram has been proposed to explain the rather significant difference in the $D^0$ and $D^+$ lifetimes [14]. The actual size of this effect is model dependent and highly uncertain. In the non-relativistic quark model, the rate for this process is

$$\Gamma_{\text{gluonic}} = \frac{G_F^2 m_b^5}{324 \pi^2} \alpha_s \left( \frac{f_+ + f_-}{2} \right)^2 \frac{f_B^2}{m_d^3}. \quad (3.7)$$

For an "on-shell" gluon, the light quark propagator is enhanced by a factor of $m_d$ in the denominator. It is clear, however, that the importance of such diagrams must become less as the quark mass gets heavier, and the semileptonic branching ratios suggest its contribution to B decay is already quite small [9]. We shall not attempt to include its contribution to the total rate, but regard this as an uncertainty which could reduce the degree of mixing from the values we quote.

It is important to note that the gluon in this "gluonic" decay is polarized [15]. This can be understood by the following argument. In $B_d$ decay the gluonic diagram is dominated by the time ordering where the valence $\bar{d}$-quark and the virtual left-handed $d$-quark annihilate into the gluon. Because the valence quark's momentum is negligible, the $d$-quark and gluon travel in the same direction. Since there is no orbital angular momentum along this direction, to conserve angular momentum the gluon must be left-handed.

4. The $B^0$-$\bar{B}^0$ transition matrix

\(\Delta \Gamma\) arises from decay modes which are common to the $B^0$ and the $\bar{B}^0$. From such modes, we expect interference effects in the decay of the mass eigenstate fields (respectively the sum and difference of the $B^0$ and $\bar{B}^0$) causing their rates to differ. Fig. 2 shows that the spectator, exchange, and gluonic decays each contain overlap in the $B^0, \bar{B}^0$ final-state particle spectra, and thus may contribute to $\Delta \Gamma$. Of these, the gluonic process (c) cannot contribute since the gluons from the $B^0$ and $\bar{B}^0$ decay have opposite polarizations [15]. The spectator diagrams (a) dominate those which are helicity suppressed (b), but to conclude that $\Delta \Gamma$ behaves like $m_b^5$ is a gross overestimate. The spectator decays from $B^0$ and $\bar{B}^0$ do overlap in their particle
content, but misalign badly in the momentum spectrum. Typically, hard $d(\bar{d})$ quarks coming from $B^0(\bar{B}^0)$ decay have to align with spectator quarks which are at rest.

Such uncertainties are eliminated by computing the $B^0-\bar{B}^0$ transition amplitude directly from the $2W^\pm$ exchange "box" diagrams in fig. 3; $M_{12}$ and $\Gamma_{12}$ are respectively the dispersive and absorptive parts. To lowest order in QCD, one adopts a framework in which the $B^0-\bar{B}^0$ transition is induced by an effective $\Delta B = 2$ interaction Hamiltonian, to which fig. 3 contributes both an hermitian and an antihermitian component. $M_{12}$ and $\Gamma_{12}$ are respectively the hermitian and antihermitian pieces of the matrix element of $H_{\text{eff}}^{\Delta B = 2}$ between conjugate mesons:

$$M_{12} = \frac{1}{2m_B} \langle \bar{B}^0 | H_{\text{eff}}^{\Delta B = 2} | B^0 \rangle. \quad (4.1)$$

Evaluation of figs. 3a and b is greatly simplified by ignoring momentum dependence in the $W$ propagators. In this approximation we have, from fig. 3a,

$$i\mathcal{G} = \left(\frac{\sqrt{2}}{G_F}\right)^2 \sum_{i,j=u} \int \frac{d^4q}{(2\pi)^4} \bar{d}(k')\gamma^\mu(1 + \gamma^5)\gamma^\alpha(1 + \gamma^5)b(p)$$

$$\times \bar{d}(k)\gamma_\nu(1 + \gamma^5)\gamma^\beta\gamma^\mu(1 + \gamma^5)b(p')$$

$$\times q_\alpha(q - p - k_\beta) \left( \frac{\xi_i}{q^2 - m_i^2 + i\epsilon} \frac{\xi_j}{(q - p - k)^2 - m_j^2 + i\epsilon} \right), \quad (4.2)$$

where $\xi_i \equiv U_{ib}U_{a}^{\ast}$. In considering the Dirac structure, we observe that

$$\bar{d}(k')\gamma^\mu(1 + \gamma^5)\gamma^\alpha(1 + \gamma^5)b(p)\bar{d}(k)\gamma_\nu(1 + \gamma^5)\gamma^\beta\gamma^\mu(1 + \gamma^5)b(p')$$

$$= 16 \bar{d}(k')\gamma^\beta(1 + \gamma^5)b(p)\bar{d}(k)\gamma^\alpha(1 + \gamma^5)b(p'). \quad (4.3)$$

Under the assumption $p \gg k$, one obtains

$$\mathcal{G} = \frac{G_F^2}{4\pi^2} \bar{d}(k')\gamma^\beta(1 + \gamma^5)b(p)\bar{d}(k)\gamma^\alpha(1 + \gamma^5)b(p')$$

$$\times \sum_{i,j=u} \xi_i \xi_j \int_0^1 dx \left[ g_\alpha\beta \left[ x(1 - x)m_b^2 - x m_i^2 - (1 - x)m_j^2 \right] + 2 p_\alpha p_\beta \left[ x(1 - x) \right] \right]$$

$$\times \log \left[ -x(1 - x)m_b^2 + x m_i^2 + (1 - x)m_j^2 \right]. \quad (4.4)$$
Fig. 2. Contributions to $\Delta \Gamma$ from spectator, exchange and gluonic channels common to $B^0$ and $\bar{B}^0$.

Fig. 3. Box diagrams contributing to $B^0-\bar{B}^0$ mass mixing.
Summing contributions from both diagrams results in the effective Hamiltonian

\[ H^{A_B=2}_{\text{eff}} = A \bar{d}_a \gamma^\mu (1 + \gamma^5) b_\alpha \bar{d}_\beta \gamma_\alpha (1 + \gamma^5) b_\beta + B \bar{d}_a (1 - \gamma^5) b_\alpha \bar{d}_\beta (1 - \gamma^5) b_\beta + \text{h.c.}, \]  

(4.5a)

where

\[
A = \frac{G_F^2}{8\pi^2} \sum_{i,j=u} \xi_i \xi_j \int_0^1 dx \left[ x(1-x)m_b^2 - xm_i^2 - (1-x)m_j^2 \right] 
\times \log \left[-x(1-x)m_b^2 + xm_i^2 + (1-x)m_j^2\right],
\]

(4.5b)

\[
B = \frac{G_F^2}{8\pi^2} \sum_{i,j=u} \xi_i \xi_j \int_0^1 dx \left[ x(1-x)m_b^2 \right] 
\times \log \left[-x(1-x)m_b^2 + xm_i^2 + (1-x)m_j^2\right].
\]

The Dirac equation has been used to obtain the \((P - S)(P - S)\) interaction from the derivative couplings appearing in eq. (4.4).

In order to evaluate the matrix elements of \(H^{A_B=2}_{\text{eff}}\) we are forced to rely on a valence quark assumption for the meson wave function. In this approximation, the matrix elements are estimated by vacuum insertion, modified by the number of ways in which the operators can act. This procedure yields

\[
\langle B^0 | \bar{d}_a \gamma^\mu (1 + \gamma^5) b_\alpha \bar{d}_\beta \gamma_\alpha (1 + \gamma^5) b_\beta | B^0 \rangle = \frac{8}{3} f_B^2 m_b^2,
\]

(4.6)

\[
\langle B^0 | \bar{d}_a (1 - \gamma^5) b_\alpha \bar{d}_\beta (1 - \gamma^5) b_\beta | B^0 \rangle = \frac{8}{3} f_B^2 m_b^2.
\]

(4.6)

\(\Gamma_{12}\) is now obtained by replacing the logarithms in eq. (4.5b) by \(-i\pi\) and integrating over the domain of negative argument:

\[
\Gamma_{12} = - \frac{G_F^2 f_B^2 m_B m_b^2}{8\pi} \left[ \xi_c \sqrt{1 - 4 \frac{m_c^2}{m_b^2}} \left( 1 - \frac{2}{3} \frac{m_c^2}{m_b^2} \right) 
+ 2\xi_c \xi_u \left( 1 - \frac{4}{3} \frac{m_c^2}{m_b^2} - \frac{1}{3} \frac{m_c^4}{m_b^4} + \frac{2}{3} \frac{m_c^6}{m_b^6} \right) \right] \xi_u^2 \]  

(4.7a)

\[
= - \frac{G_F^2 f_B^2 m_B}{8\pi} \left[ \xi_c^2 m_b^2 + \frac{8}{3} \xi_c \xi_c m_c^2 + O \left( \frac{m_c^4}{m_b^2} \right) \right].
\]

(4.7b)
The leading correction to the dominant contribution (proportional to $\xi_t^2$) must be maintained for reasons to follow.

The disperive calculation is most easily accomplished through expanding the logarithms in powers of $1/m_t^2$. One obtains

$$M_{12} = \frac{G_F^2 f_B^2 m_B}{12 \pi^2} \left[ \xi_t^2 \left( m_t^2 + \frac{1}{3} m_b^2 + \frac{1}{3} m_b^2 \log \frac{m_t^2}{m_b^2} \right) + O \left( m_c^2, \frac{m_b^4}{m_t^2} \right) \right]. \quad (4.8)$$

Before we state the important results of this analysis, a consideration of radiative effects is in order.

The leading contribution to $M_{12}$ (proportional to $m_t^2$) is a local one; the disperive part of the loop integral is dominated by loop momenta of order $m_t^2$ where external momenta are negligible. Radiative corrections to this Wilson coefficient are genuinely short distance—the low-energy effects of QCD are confined to the matrix element. These short-distance effects are known from work by Gilman and Wise on the radiative corrections to the $K_L - K_S$ mass difference [16]. Extending their results to the present analysis shows that the leading contribution to $M_{12}$ gets modified by a factor $\eta$, where

$$\eta = \left( \frac{\alpha_s(m_b)}{\alpha_s(m_t)} \right)^{-6/23} \left[ \frac{3}{2} \left( \frac{\alpha_s(m_t)}{\alpha_s(M_W)} \right)^{-4/7} - \left( \frac{\alpha_s(m_t)}{\alpha_s(M_W)} \right)^{2/7} + \frac{1}{2} \left( \frac{\alpha_s(m_t)}{\alpha_s(M_W)} \right)^{8/7} \right]. \quad (4.9)$$

The square bracket results from integrating out the $t$-quark at an energy scale $\mu = m_t$ to generate the effective four-Fermi operator $O_A$ [eq. (4.5a)] with Wilson coefficient proportional to $\xi_t^2$. The factor in front comes from the renormalization group scaling of this Wilson coefficient as $\mu$ is decreased from $m_t$ to $m_b$. $\eta$ can range from 0.85–0.78 for values of $\Lambda_{QCD}$ between 0.1 and 0.5 GeV, and $m_t$ taken between 20 and 40 GeV. Its main effect is to suppress the level of $B^0 - \bar{B}^0$ mixing (proportional to $\Delta M^2$) when compared with the free quark prediction. Radiative corrections to the subleading terms in $M_{12}$ are of little consequence.

The radiative analysis of $\Gamma_{12}$ is more subtle. $\Gamma_{12}$ is not the matrix element of a local operator. In this respect eqs. (4.1), (4.5) are specious—a quark model contrivance designed to turn a complicated phase-space problem into a simple loop integral. The radiative corrections to $\Gamma_{12}$ cannot be ascribed to a single Wilson coefficient. Each physical intermediate state which contributes to the width difference has to be considered separately.

Figs. 3a, b both contribute an absorptive part to the $B^0 - \bar{B}^0$ transition amplitude. These figures are redrawn in figs. 4a, b in such a way that the absorptive contribution from physical intermediate states is more evident. Cutting figs. 4a, b corresponds respectively to the exchange and spectator contributions to $\Gamma_{12}$. The spectator
contribution dominates the width difference since the exchange diagram is helicity suppressed. Both figures contribute to the subleading terms proportional to $m_c^2$ on which $CP$ violation critically depends. Their radiative corrections are consequently of equal importance and will be considered in turn.

Radiative corrections to fig. 4b are clearly absent to order $\alpha_s \log M_W$. Any gluonic correction which might involve a logarithm of $M_W$ in Landau gauge must connect the initial and final fermion lines; such graphs vanish by the tracelessness of the SU(3) generators. In fact, by carefully adding the contribution to each weak vertex in fig. 4b from both operators present in $H_{\text{eff}}$ [eq. (3.5)], one finds these spectator contributions are proportional to $(2f_+^2 + f_-^2)$—the same enhancement which characterizes the spectator decays. The leading contribution to $\Gamma_{12}$ proportional to $m_b^2$, for which this diagram is solely responsible, receives a modest enhancement which is of relatively small consequence to $B^0-\bar{B}^0$ mixing or $CP$ violation.

The statistics connected with fig. 4a are different, and the exchange contribution is found proportional to $(2f_+ - f_-)^2$. This highly non-trivial effect can amount to a suppression by an order of magnitude or more, for $(2f_+ - f_-)^2$ is equal to 0.18 (0.01) for $\Lambda_{\text{QCD}}$ equal to 0.1 GeV (0.5 GeV). The short-distance effects of QCD can virtually eliminate the exchange contribution to the width difference.

To determine the impact of these effects upon $\Gamma_{12}$, one separates the two contributions to eq. (4.7) due to figs. 4a,b and incorporates their respective QCD enhancements. The exchange contribution of fig. 4a is easily obtained from the related decay process—there is no momentum mismatch associated with this channel:

$$\Gamma_{12}^{4a} = \frac{G_F^2 f_B^2 m_b m_c^2}{3 \cdot 8\pi} \left[ 2\xi_c \sqrt{1 - \frac{m_c^2}{m_b^2}} + 2\xi_c \xi_u \left( 1 - \frac{m_c^2}{m_b^2} \right) \right]^2.$$  (4.10)
where the two terms are respectively the $c\bar{c}$ and the $c\bar{u}, u\bar{c}$ modes. The only new physics in this result is a factor of $\frac{1}{2}$ which enters due to a color mismatch which does not occur in the rate. Eq. (4.7) for $\Gamma_{12}$ is now rearranged to separate the contribution which is helicity suppressed:

$$\Gamma_{12} = -\frac{G_F^2 f_B^2 m_B}{8\pi} \left\{ m_c^2 \left[ \xi_c^2 \sqrt{1 - 4 \frac{m_c^2}{m_b^2}} \left( 1 + 2 \frac{m_c^2}{m_b^2} \right) + 2 \xi_c \xi_u \left( 1 - 3 \frac{m_c^4}{m_b^4} + 2 \frac{m_c^6}{m_b^6} \right) + \xi_u^2 \right] \right\}$$

$$- \frac{8}{3} m_c^2 \left[ \xi_c^2 \sqrt{1 - 4 \frac{m_c^2}{m_b^2}} + \xi_c \xi_u \left( 1 - \frac{m_c^2}{m_b^2} \right)^2 \right]. \quad (4.11)$$

It is interesting to note that the operator $O_B$ of eq. (4.5) which is $(P - S)(P - S)$ contributes to the first piece only. It is the second bracket, due entirely to $O_A$ which is $(V + A)(V + A)$, that we call helicity suppressed. (The first bracket contains terms which are higher order in $m_c$ but none, coincidently, that are proportional to $m_c^2$.) Comparing this with eq. (4.10) it is clear that fig. 4b must also give rise to a contribution which is helicity suppressed, since the second bracket cannot be attributed to fig. 4a alone. By incorporating the radiative corrections associated with these two figures, we obtain a result for $\Gamma_{12}$ which is correct to leading logarithm:

$$\Gamma_{12} = -\frac{G_F^2 f_B^2 m_B}{8\pi} \left\{ m_c^2 \left[ \frac{1}{3} \left( 2 f_+^2 + f_-^2 \right) m_b^2 \xi_c^2 \sqrt{1 - 4 \frac{m_c^2}{m_b^2}} \left( 1 + 2 \frac{m_c^2}{m_b^2} \right) \right. \right.$$  

$$+ 2 \xi_c \xi_u \left( 1 - 3 \frac{m_c^4}{m_b^4} + 2 \frac{m_c^6}{m_b^6} \right) + \xi_u^2 \right\}$$

$$- m_c^2 \left[ \frac{2}{3} (2 f_+^2 + f_-^2) + \frac{2}{3} (2 f_+ - f_-)^2 \right]$$

$$\times \left[ \xi_c^2 \sqrt{1 - 4 \frac{m_c^2}{m_b^2}} + \xi_c \xi_u \left( 1 - \frac{m_c^2}{m_b^2} \right)^2 \right] \right\}, \quad (4.12a)$$

$$\Gamma_{12} = -\frac{G_F^2 f_B^2 m_B}{8\pi} \left\{ \frac{1}{3} \left( 2 f_+^2 + f_-^2 \right) \xi_c^2 m_b^2 \right.$$  

$$+ \frac{8}{9} \left( \frac{1}{2} f_+^2 - f_+ f_- + \frac{1}{2} f_-^2 \right) \xi_c \xi_u m_c^2 + O \left( \frac{m_c^4}{m_b^2} \right) \right\}. \quad (4.12b)$$
In addition to the overall enhancement mentioned earlier, there is a relative suppression of subleading terms from the dependence on \((2f_+ - f_-)^2\). Had we not discovered a helicity-suppressed contribution due to the spectator diagram, this suppression would be extreme with severe repercussions for CP violation.

By including the short-distance effects of QCD, we have placed our understanding of the mass and width differences on a par with the total rate, and are now led to state the important results of this analysis:

1. \(\Gamma_{12}\) is considerably smaller than previous estimates based on the simple analysis of decay channels which place \(\Gamma_{12}\) proportional to \(m_\psi^5\).

2. The leading contributions to \(M_{12}\) and \(\Gamma_{12}\) are both proportional to the same combination of KM angles, and thus share a common phase. As a result, \(\epsilon\) is almost pure imaginary, which places severe restrictions on CP violation.

The last result is a consequence of the near degeneracy of the charm and up quark masses on the scale of the B-meson mass. If two quarks of the same charge have equal masses there is enough freedom to remove the phase \(\delta\) from the matrix \(U\) by redefining the quark fields. Hence in the limit that the charm quark mass equals the up quark mass there is no CP violation and \(\epsilon\) is pure imaginary [15].

CP violation depends critically on the subleading terms proportional to \(m_c^2\), and is thus a measure of SU(4) breaking in \(\Gamma_{12}\). SU(4) is a much better symmetry for \(\Gamma_{12}\) than for the total rate \(\Gamma\). This is because \(\Gamma\) is dominated by three-body (spectator) decays where the effect of \(m_c\) on the rate is important. Although \(\Gamma_{12}\) is also dominated by spectator processes, the momentum mismatch effectively reduces the phase-space integral to a two-body one. For a two-body phase space, the effect of \(m_c\) is less restrictive, and SU(4) is a correspondingly better symmetry [15]*.

It is important to emphasize that these restrictions on CP violation in the standard model do not apply when a fourth quark doublet is added. The argument that CP violation vanishes if \(m_c = m_u\) fails for more than six quarks. In addition, for the six-quark model CP violation vanishes when one of the quark mixing angles is taken to zero, and thus the smallness of the Cabibbo angle acts to suppress CP violation. In a model with eight quarks this is not the case: even if one mixing angle vanishes there is still non-trivial mixing among three generations. Therefore, the small same-sign dilepton and total lepton charge asymmetries we derive for the six-quark

* It is pedagogically interesting that the gluonic decay had the potential of introducing substantial CP violation if a helicity mismatch had not prevented its contributing to \(\Gamma_{12}\). Since the phase space is genuinely three-body (there is no momentum mismatch), SU(4) breaking is potentially large and could give \(\epsilon\) a significant real part. Explicit evaluation of the gluonic decay \(B \rightarrow c\bar{c}g\) finds a phase space suppression factor \(g(2m_c/m_B)\), where

\[
g(x) = \sqrt{1-x^2} \left(1 - \frac{1}{3}x^2 + \frac{1}{3}x^4 \right) + \frac{1}{3}x^6 \log \frac{1 + \sqrt{1-x^2}}{x}.
\]

With \(m_c = 1.5\) GeV and \(m_B = 5.2\) GeV the phase-space suppression factor \(g\) is equal to 0.39, which therefore supports this contention.
5. Constraints on the KM parameters

Quantitative statements concerning $B^0 - \bar{B}^0$ mixing and CP violation depend upon $f_B$, $m_t$ and the KM parameters. Theoretical estimates of $f_B$ range from 0.1 to 0.5 GeV, with little evidence for a growing consensus [17]. The only present bound on $m_t$ appears to be a lower one. Regarding the KM angles, by comparison, much more can be said.

One of the KM parameters, the Cabibbo angle, is well known. Nuclear beta decay permits a precise determination of $U_{ud}$. Even here, however, the precise value of $\theta_c$ depends critically upon radiative corrections computed within the standard model. A consideration of such effects by Shrock and Wang [18] has resulted in $c_1 = 0.9737 \pm 0.0025$. These authors proceed to determine $U_{us}$ from semileptonic strange decays. Here there are considerable uncertainties associated with the hadronic matrix element which must be obtained from approximate SU(3) invariance. In spite of such uncertainties, they find $U_{us} = s_1 c_3 = 0.219 \pm 0.011$, from which $U_{ub}$ follows by unitarity: $U_{ub} = s_2 c_3 = 0.06 \pm 0.06$. The upper bound on $s_2$ of $s_2 \leq 0.5$ which results from this analysis concludes what one can say about the KM angles without additional theoretical input.

Most of this input comes from the well-known CP properties of the neutral kaons. The $K^0 - \bar{K}^0$ system is described by two CP-violating quantities, $\epsilon'$ and $\epsilon$, which respectively measure the “direct” and superweak contributions to $K_L \rightarrow 2\pi$ decays. $\epsilon$, the CP impurity parameter defined in eq. (2.6), is determined from an asymmetry in the rates for $K_L \rightarrow \pi^- \ell^+ \nu$ versus $K_L \rightarrow \pi^+ \ell^- \bar{\nu}$, or from the experimental quantities $\eta_{+-}$ and $\eta_{00}$:

$$
\eta_{+-} = \frac{\langle \pi^+ \pi^- | H | K_L \rangle}{\langle \pi^+ \pi^- | H | K_S \rangle}, \quad \eta_{00} = \frac{\langle \pi^0 \pi^0 | H | K_L \rangle}{\langle \pi^0 \pi^0 | H | K_S \rangle}.
$$

(5.1)

If $\epsilon$ were the only source of CP violation, one would expect the quantities $\eta_{+-}$ and $\eta_{00}$ to be equal, with $\eta_{+-} = \eta_{00} = \epsilon$, and that the amplitudes $A_0 \equiv \langle 2\pi(I=0) | H | K^0 \rangle$ and $A_2 \equiv \langle 2\pi(I=2) | H | K^0 \rangle$ could both be chosen real and positive. If, however, there is a mechanism which contributes directly either to $K_L \rightarrow 2\pi(I=0)$ or to $K_L \rightarrow 2\pi(I=2)$ decays, the quantities $\eta_{+-}$ and $\eta_{00}$ will differ. In general one finds $\eta_{+-} = \epsilon + \epsilon'$ and $\eta_{00} = \epsilon - 2\epsilon'$, where

$$
\epsilon' \equiv \sqrt{\frac{1}{2}} e^{(\pi/2 + \delta_2 - \delta_0)} \text{Im} \left( \frac{A_2}{A_0} \right),
$$

(5.2)

and $\delta_0$ and $\delta_2$ are respectively the strong interaction phase shifts for the $2\pi(I=0)$
and $2\pi(I = 2)$ final states [19]. No such difference has been observed, since the experimental ratio $\eta_{00}/\eta_{+-} = 1.008 \pm 0.041$ is consistent with unity [19]. These data do provide a highly useful upper bound on the magnitude of $\epsilon'$. Experimental values for $\delta_0, \delta_2$ [20] and the phase of $\eta_{+-} = (45.9^\circ \pm 1.6^\circ)$ [21] place $\epsilon$ and $\epsilon'$ nearly parallel (or antiparallel) in the complex plane, from which it follows that $|\epsilon'| < 0.02 |\epsilon|$ [22].

The phase of $\eta_{+-}$ is consistent with the phase of $\epsilon$ defined in eq. (2.6) with $\text{Im} \, \Gamma_{12}$ set equal to zero. Setting $\text{Im} \, \Gamma_{12}$ equal to zero is consistent with the standard convention in which $A_0$ is chosen to be real and positive, and derives from the fact that the $2\pi(I = 0)$ contribution dominates the width difference. The only other modes with non-trivial branching ratios are the $2\pi(I = 2), 3\pi$ and $\pi \ell \nu$ decays. The imaginary contribution to $\Gamma_{12}$ from $2\pi(I = 2)$ is (roughly) proportional to $2 \text{Im} \, A_2 \text{Re} \, A_2$. When compared with the contribution from the $(I = 0)$ channel, this yields $2 \text{Im} \, A_2 \text{Re} \, A_2 \approx 0.1 |\epsilon'|$ in the ratio, which is negligible compared to the contribution from $\text{Im} \, M_{12}$. The $3\pi$ modes are sufficiently small that their contribution to $\Gamma_{12}$ is of limited importance. Current algebra also predicts these $3\pi$ matrix elements are real to the extent that $A_0$ dominates the $2\pi$ amplitude, so their contribution to $\text{Im} \, \Gamma_{12}$ is almost certainly negligible. $\pi \ell \nu$ decays do not contribute at all to $\Gamma_{12}$ in a theory with charged weak bosons. It follows from eq. (2.6) and the approximate relation $\Delta M = \frac{1}{2} \Delta \Gamma$ that

$$
\epsilon = \frac{e^{i\pi/4}}{2\sqrt{2}} \left( \frac{\text{Im} \, M_{12}}{\text{Re} \, M_{12}} \right),
$$

or (ref. [24])

$$
\frac{\text{Im} \, M_{12}}{\text{Re} \, M_{12}} = 2\sqrt{2} |\epsilon| = 6.5 \cdot 10^{-3} \text{ (experiment)}.
$$

Eq. (5.4) offers a basis for comparison between the standard model and experiment, and provides a useful constraint upon the unknown KM parameters. The derivation of $M_{12}$ is rather similar to our analysis for $B^0 - \bar{B}^0$ mixing. However, the evaluation of fig. 3 is very much simplified by ignoring the external momenta associated with initial and final quark lines. The result of such an analysis is well known [4, 25]:

$$
H_{\text{eff}}^{\Delta S=2} = \frac{G_F^2}{16\pi^2} \left( \xi_t^2 m_t^2 + 2 \xi_t \xi_c m_c^2 \log \frac{m_t^2}{m_c^2} + \xi_c^2 m_c^2 \right)
\times d_{\alpha} \gamma^\mu (1 + \gamma^5) s_{\alpha} d_{\beta} \gamma_\mu (1 + \gamma^5) s_\beta + h.c.,
$$

* For a detailed discussion see, for example, ref. [23].
where $\hat{q}_q \equiv U_{qs} U_{qd}^*$. To the extent this local $\Delta S = 2$ interaction actually dominates the dispersive $K^0 - \bar{K}^0$ transition, we obtain $M_{12}$ from the matrix element of $H_{eff}^{\Delta S=2}$ between conjugate kaons:

$$M_{12}^{box} = \frac{1}{2m_K} \langle K^0 | H_{eff}^{\Delta S=2} | \bar{K}^0 \rangle$$

$$= \frac{G_F^2}{32\pi^2 m_K} \left( \hat{\xi}_t^2 m_t^2 + 2 \hat{\xi}_t \hat{\xi}_c m_c^2 \log \frac{m_t^2}{m_c^2} + \hat{\xi}_c^2 m_c^2 \right)$$

$$\times \langle K^0 | \overline{d}_\alpha \gamma^\mu (1 + \gamma^5) s_\alpha \overline{d}_\beta \gamma^\nu (1 + \gamma^5) s_\beta | \bar{K}^0 \rangle. \quad (5.6)$$

Based on this result, the constraint in eq. (5.4) reads

$$\frac{2\sqrt{2} |\epsilon|}{\text{Re} M_{12}^{box}} = \frac{\text{Im} \left( \hat{\xi}_t^2 m_t^2 + 2 \hat{\xi}_t \hat{\xi}_c m_c^2 \log \frac{m_t^2}{m_c^2} + \hat{\xi}_c^2 m_c^2 \right)}{\text{Re} \left( \hat{\xi}_t^2 m_t^2 + 2 \hat{\xi}_t \hat{\xi}_c m_c^2 \log \frac{m_t^2}{m_c^2} + \hat{\xi}_c^2 m_c^2 \right)}. \quad (5.7)$$

This constraint has been the basis of several investigations of the KM angles [26]. It is a relation among three unknown parameters which allows an unambiguous prediction for one angle in terms of the remaining two. Its most attractive feature is that it is independent of the unknown matrix element which accompanies $M_{12}$. It is not, however, a reliable quantitative constraint, and several modifications must be made if it is to limit the $B^0$ phenomenology in any meaningful way. Its predictions are nonetheless sufficiently distinctive and unambiguous (a feature which does not apply in the least to some of its modifications) that it pays to be somewhat familiar with its solutions. Solutions to eq. (5.7) exist in all but the fourth quadrant ($0 > \delta > -\frac{1}{2} \pi$). These are displayed in figs. 5 and 6 by plotting $s_2$ as a function of $s_3$ and $\sin \delta$. Quad. I [fig. 5a] is observed to obey the approximate relation $2 s_2 s_3 \sin \delta \approx 10^{-3}$, except in its upper regions which are an artifact of eq. (5.7). The solutions for quad. II are similar except that when $s_3$ is taken large, larger values of $s_2 s_3 \sin \delta$ are apparently allowed. This property is greatly exaggerated when $m_t$ is increased from 20 GeV, or when radiative corrections to eq. (5.7) are included. Rather different are the solutions of quad. III, which appear in the very center of fig. 6 and spread radially outward as $s_3$ increases. Here also it is evident that much larger values of $s_2 s_3 \sin \delta$ are consistent with $\epsilon$, which shows that this constraint does not effectively restrict CP violation elsewhere in the theory.

The first question to be raised regarding eq. (5.7) concerns the manner in which $H_{eff}^{\Delta S=2}$ was computed. The box diagram has been simplified by ignoring momentum dependence in the W-boson propagators. Although rapid convergence due to a GIM cancellation makes such an approximation possible, the present experimental situation regarding the t-quark mass raises doubts concerning this lowest order in $m_t^2/M_W^2$ approach. To identify the errors which result from this approximation, we evaluate the box diagram in a general $R_\xi$ gauge [27], with full attention to the gauge...
propagators, and obtain the following gauge-invariant result:

\[
A = \frac{G_F^2}{8\pi^2} \left[ \bar{d}_a \gamma^\mu (1 + \gamma^5) s_a \bar{d}_\beta \gamma^\nu (1 + \gamma^5) s_\beta \right] \sum_{i,j=u}^{c,t} \hat{\xi}_i \hat{\xi}_j \\
\times \left\{ \frac{m_i^2}{M^2 - m_i^2} \left( \frac{-M^4}{(M^2 - m_i^2)^2} \ln \frac{M^2}{m_i^2} + \frac{m_j^4}{(m_j^2 - m_i^2)^2} \right) \\
\times \ln \frac{m_i^2}{m_j^2} + \frac{M^2}{M^2 - m_i^2} + \frac{m_j^2}{m_j^2 - m_i^2} \right\} \\
+ \frac{1}{2} m_i^2 m_j^2 \left[ \frac{m_i^2}{(M^2 - m_i^2)(m_i^2 - m_j^2)} \ln \frac{M^2}{m_i^2} \right. \\
\left. + \frac{m_j^2}{(M^2 - m_j^2)(m_j^2 - m_i^2)} \ln \frac{M^2}{m_j^2} \right] + 2m_i^2 m_j^2 M^2 \\
\times \left[ \frac{m_i^2}{(M^2 - m_i^2)^2(m_i^2 - m_j^2)} - \ln \frac{M^2}{m_i^2} + \frac{m_j^2}{(M^2 - m_j^2)^2(m_j^2 - m_i^2)} \right] \\
\times \ln \frac{M^2}{m_j^2} - \frac{1}{(M^2 - m_i^2)(M^2 - m_j^2)} \right\}. \tag{5.8a}
\]

Expanding this result in powers of \(1/M_W^2\) and dropping those terms which vanish upon summation recovers the lowest order result. In addition, we are encouraged to find that the coefficient which multiplies the next order contribution is small:

\[
A = \frac{G_F^2}{8\pi^2} \left[ \bar{d}_a \gamma^\mu (1 + \gamma^5) s_a \bar{d}_\beta \gamma^\nu (1 + \gamma^5) s_\beta \right] \sum_{i,j=u}^{c,t} \hat{\xi}_i \hat{\xi}_j \\
\times \left\{ \frac{m_i^2 m_j^2}{m_i^2 - m_j^2} \ln \frac{m_i^2}{m_j^2} + \frac{1}{M^2} \left( -\frac{3}{4} m_i^2 m_j^2 \right) + \frac{1}{M^4} \cdots \right\}. \tag{5.8b}
\]
For values of $m_t$ as large as 40 GeV, the numerical effect of the higher order terms on the KM curves defined in eq. (5.7) is negligible.

Much more important is the short-distance renormalization of $H_{\text{eff}}^{\Delta S=2}$. We draw again from the analysis of Gilman and Wise who find important corrections associated with the three contributions to $H_{\text{eff}}^{\Delta S=2}$ (proportional to $\xi_t^2$, $\xi_t \xi_c$ and $\xi_c^2$) [16]. By including these short-distance enhancements found in table 1, we consider the Wilson coefficient in $H_{\text{eff}}^{\Delta S=2}$ to be reliable to within a factor of two or so, particularly the CP-violating part which depends only upon physics above the charm-quark mass. Henceforth we shall assume these radiative corrections have been incorporated into $M_{12}^{\text{box}}$.

* An analysis of KM parameters in the presence of such corrections has been performed in ref. [28]. See also ref. [29].
Fig. 6. Solutions to eq. (5.7) in quad. III emerge as $s_3$ exceeds 0.28 and 0.19 for $m_t = 20$ and 40 GeV, respectively.

TABLE 1

<table>
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<th>$\Lambda_{QCD}$</th>
<th>$m_t$</th>
<th>$H(\xi^2_{\ell})<em>{\text{QCD}}/H(\xi^2</em>{\ell})_{\text{free}}$</th>
<th>$H(\xi_{\ell\ell\ell\ell})<em>{\text{QCD}}/H(\xi</em>{\ell\ell\ell\ell})_{\text{free}}$</th>
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The most serious objection to eq. (5.7) lies in setting $\text{Re} M_{12} = \text{Re} M_{12}^{\text{box}}$. There are important contributions to $M_{12}$ which would seem to have nothing to do with the box diagram. It has long been known that there are contributions to the dispersive $K^0$-$\bar{K}^0$ transition from low-mass intermediate states which are of the order of $\Delta M_K$ itself [30]. The sensitivity of such estimates to the precise value of $m_\pi$ and $m_\rho$ would indicate that these contributions are not accounted for with any accuracy whatsoever by $M_{12}^{\text{box}}$. Wolfenstein, in a characteristically fine analysis [31], adds an error term to accommodate the large uncertainties associated with these “soft” contributions, defining $\text{Re} M_{12} = \text{Re} M_{12}^{\text{box}} + \text{Re} M_{12}^{\text{soft}}$. We have found it more profitable to eliminate this uncertainty entirely by replacing the denominator in eq. (5.7) by its well-known experimental value:

$$\text{Re} M_{12}^{\text{box}} = \frac{G_F^2}{16\pi^2 m_K\Delta M_K} \text{Im} \left( \hat{x}_i^2 m_t^2 + 2\hat{x}_i\hat{x}_c m_c^2 \log \frac{m_t^2}{m_c^2} + \hat{x}_c^2 m_c^2 \right)$$

$$\times \langle K^0 | \bar{d}_a \gamma^\mu (1 + \gamma^5) s_a \bar{d}_\beta \gamma_\mu (1 + \gamma^5) s_\beta | \bar{K}^0 \rangle. \quad (5.9)$$

While eq. (5.9) has the virtue of being correct, and does constitute a marked improvement over eq. (5.7), the appearance of the matrix element is a bit unfortunate. For rough purposes, the valence quark approximation employed in sect. 4 assigns the value $\frac{3}{2} f_K^2 m_K^2$ ($f_K = 160$ MeV). Somewhat better may be the bag model estimate of Shrock and Treiman [32] which predicts a value 0.42 times as large. For the purpose of obtaining constraints, intended to rule out certain values of the KM parameters and to limit the $B^0$ phenomenology, we find it prudent to allow for an order of magnitude variation in the value of such unknown matrix elements, with the bag model estimate, where available, occupying a central value, i.e., $\langle \text{m.e.} \rangle = \langle \text{m.e.} \rangle_{\text{bag}} \cdot 10^{\pm 1/2}$.

A closely related constraint derives from the well-known $K_L - K_S$ mass difference. Because of the “soft” contribution to $\Delta M_K$ which we have discussed, a lower bound on the contribution from $\text{Re} M_{12}^{\text{box}}$ would not be meaningful. Even if the contribution from real $\text{Re} M_{12}^{\text{soft}}$ were negligible, the lower bound on $s_2$ discussed by some authors in this context results from undue faith in the bag model matrix element; this bound disappears in the valence quark approximation. One can believe, however, that the contribution from $\text{Re} M_{12}^{\text{box}}$ cannot be made arbitrarily large. One can write this constraint as

$$\text{Re} M_{12}^{\text{box}} \neq \frac{1}{2} \Delta M_K, \quad (5.10)$$

where the bound is understood to be a loose one. For instance, Itzykson et al. [30] find the $\eta$ plus $\pi$ pole terms give a contribution to $\Delta M_K$ which is $(-1.4)$ times the experimental value, which would allow the contribution from $\text{Re} M_{12}^{\text{box}}$ to be that much larger. Despite these uncertainties, the dependence of $\text{Re} M_{12}^{\text{box}}$ on $s_2^2 m_t^2$ does...
constrain the magnitude of $s_2$, especially for $m_t$ large. It may also be that a separate constraint from $K_L \rightarrow \mu\bar{\mu}$ is useful in this context [33].

If the standard model were purely superweak, eq. (5.9) would constitute the constraint due to $\epsilon$ in its final form. Yet the standard model and the experimental limit on $\epsilon'$ both allow enough "direct" $CP$ violation to substantially invalidate this constraint. The conventional choice of quark fields defined in the matrix $U$ and the convention in which $A_0$ are real are consistent only if the $\Delta S = 1$ hamiltonian responsible $K^0$ decay receives no imaginary contribution. We know, however, that the penguin diagrams generate a local strangeness-changing interaction whose coefficient is complex. Since the matrix element of this operator is purely $\Delta I = \frac{1}{2}$ and enhanced relative to the ordinary $(V + A)(V + A)$ interaction, it is believed to contribute significantly to the observed $\Delta I = \frac{1}{2}$ enhancement [13, 34]. Its contribution to $A_0$ will not be purely real [35]:

$$A_0 = e^{i\xi} |A_0|.$$  \hspace{1cm} (5.11)

The consequences of a phase in the amplitude for $K^0 \rightarrow 2\pi(I = 0)$ are really two-fold. Firstly, we note that the constraint from $\epsilon$ defined by eq. (5.3) is valid only in the convention where $A_0$ is real. We can restore this convention easily by adjusting our definition of phases such that $|K^0) \rightarrow e^{-i\xi}|K^0)$, $|\bar{K}^0) \rightarrow e^{i\xi}|\bar{K}^0)$. Under such a transformation, $\langle K^0 | M_1^2 | \bar{K}^0 \rangle \rightarrow e^{2i\xi}\langle K^0 | M_1^2 | \bar{K}^0 \rangle$ or $M_{12} \rightarrow e^{2i\xi}M_{12}$ so that eq. (5.9) gets modified:

$$2\sqrt{2} |\epsilon| = \frac{\text{Im} M_{12}^{\text{box}}}{\frac{1}{2}\Delta M_K} + 2\xi.$$  \hspace{1cm} (5.12)

The second consequence is less direct. If the effective $\Delta S = 1$ hamiltonian is $CP$ violating, so must in general be the contribution to $M_{12}^\text{soft}$ from low-mass intermediate states, $M_{12}^{\text{soft}}$. Since this soft imaginary component will appear in the numerator of eq. (5.12) along with $\text{Im} M_{12}^{\text{box}}$, the explicit dependence upon $M_{12}^{\text{soft}}$ has not been eliminated, despite our efforts to the contrary. One could argue, however, that $M_{12}^{\text{soft}}$ ought to be nearly real in a basis where $A_0$ is real. To the extent that $\pi$ and $2\pi$ intermediate states dominate $M_{12}^{\text{soft}}$ this follows from current algebra. The $\eta$ is less certain, but to the extent that the penguin operator dominates both matrix elements, the $\pi$ and $\eta$ contributions have the same phase. We shall henceforth assume that $\text{Im} M_{12}^{\text{soft}}$ is negligible in the convention where $A_0$ is real, or that $M_{12}^{\text{soft}} = e^{-2i\xi}|M_{12}^{\text{soft}}|$ with the standard choice of quark fields. It follows that the contribution from $\text{Im} M_{12}^{\text{soft}} (\approx -2\xi \text{Re} M_{12}^{\text{soft}})$ which belongs in the numerator of eq. (5.12) exactly cancels the term in $2\xi$ (from the phase redefinition) that is proportional to $\text{Re} M_{12}^{\text{soft}}$.
[36]. As a result, eq. (5.12) should read

$$2\sqrt{2} |\epsilon| = \frac{\text{Im } M_{12}^{\text{box}}}{\frac{1}{2} \Delta M_K} + 2\xi \frac{\text{Re } M_{12}^{\text{box}}}{\frac{1}{2} \Delta M_K}$$

(5.13a)

$$= \frac{G_F^2}{16\pi^2 m_K \Delta M_K} \langle K^0 | \bar{d}_\alpha \gamma^\mu(1 + \gamma^5)s_\alpha \bar{d}_\beta \gamma_\mu(1 + \gamma^5)s_\beta | \bar{K}_0^0 \rangle$$

$$\times \left[ \text{Im} \left( \hat{\xi}^2 m_i^2 + 2\hat{\xi}_i\hat{\xi}_c m_c^2 \log \frac{m_i^2}{m_c^2} + \hat{\xi}_c^2 m_c^2 \right) \right.$$  

$$+ 2\xi \text{Re} \left( \hat{\xi}^2 m_i^2 + 2\hat{\xi}_i\hat{\xi}_c m_c^2 \log \frac{m_i^2}{m_c^2} + \hat{\xi}_c^2 m_c^2 \right) \right].$$

(5.13b)

Eq. (5.13) constitutes the constraint from the kaon CP impurity parameter $\epsilon$ in its final form. Its dependence on $\xi$ is problematic, and requires some discussion. It is no surprise that the magnitude of $\xi$ is bound by the limit on $\epsilon'$, since it is a source of direct CP violation in $K_L \to 2\pi$ decays. If we assume the CP-violating Hamiltonian is purely $\Delta I = \frac{1}{2}$, then we have that

$$|\epsilon'| = \sqrt{\frac{1}{2}} \text{Im} \left( \frac{A_2}{A_0} \right) = \sqrt{\frac{1}{2}} |\xi| \left| \frac{A_2}{A_0} \right|.$$  

(5.14)

From the experimental bound on $\epsilon'/\epsilon$ and the observed ratio for $A_2/A_0$ of $1/20$,

$$|\xi| < 20\sqrt{2} \ (0.02)|\epsilon| = 1.28 \cdot 10^{-3} \text{ (experiment).}$$

(5.15)

This bound is sufficiently weak that the $\xi$ term in eq. (5.13) constitutes a major uncertainty. If $\text{Re } M_{12}^{\text{box}}$ were equal to $\frac{1}{2} \Delta M_K$, this uncertainty would amount to forty percent. But if $\text{Re } M_{12}^{\text{box}}$ is two and a half times larger, as the analysis of Itzykson might suggest, this term could be as large as $2\sqrt{2} |\epsilon|$ and the constraint loses much of its meaning. One can yet do better, a point to which we later return.

There is an independent constraint arising from the bound on $\epsilon'$ which is fortunately less illusive. The effective $\Delta S = 1$ Hamiltonian which is responsible for $K^0$ decay has been thoroughly studied [37]. The CP-violating contribution to $K^0 \to 2\pi(I = 0)$ decay is dominated by a single operator of the penguin type:

$$Q_6 \equiv \bar{s}_\alpha \gamma^\mu(1 + \gamma^5)\bar{u}_\alpha \bar{u}_\beta \gamma_\mu(1 - \gamma^5)u_\beta + \bar{d}_\beta \gamma_\mu(1 - \gamma^5)d_\alpha + \bar{s}_\beta \gamma_\mu(1 - \gamma^5)s_\alpha.$$

(5.16)

The imaginary part of the Wilson coefficient which appears here and elsewhere in
$H_{eff}^{S=1}$ is proportional to a single combination of KM parameters:

$$\text{Im} C_6 \equiv \frac{\hat{C}_6}{\sqrt{2}} G_F s_1 s_2 c_2 s_3 \sin \delta. \quad (5.17)$$

By constraining this combination of angles one effectively limits CP violation for the whole of low-energy physics. Taking the real part of $A_0$ from experiment we have that

$$\xi = \pm \frac{\hat{C}_6}{\sqrt{2}} G_F s_1 s_2 c_2 s_3 \sin \delta \frac{\langle 2\pi (I = 0) | Q_6 | K^0 \rangle}{A_0 = 3.32 \cdot 10^{-4} \text{ MeV}}. \quad (5.18)$$

The sign ambiguity does not reflect an uncertainty in the calculation for $\text{Im} A_0$, but rather an inability to reliably determine the sign of $\text{Re} A_0$ in the standard quark convention. It can be resolved by assuming that the penguin contributes to a $\Delta I = \frac{1}{2}$ enhancement and not a suppression. Then the contribution to $A_0$ from $\text{Re} C_6$ must be taken positive by definition, and the sign of $\xi$ follows from the relative phase of $\text{Im} C_6$. A detailed renormalization group analysis for $H_{eff}^{S=1}$ by Gilman and Wise [37] has found a magnitude for $\hat{C}_6$ of 0.06 to 0.11 depending upon $\Lambda_{QCD}$ and the value of $m_t$. Its sign is such that $\xi$ is opposite the sign of $\sin \delta$.

The matrix element of $Q_6$ can be related to a one-pion matrix element by soft pion reduction, which can then be estimated in the bag model. A recent compendium of bag model results is available from the work of Donoghue, Golowich and Holstein [38]. One obtains

$$\langle 2\pi (I = 0) | Q_6 | K^0 \rangle \approx -\sqrt{\frac{3}{2}} \langle \pi^0 \pi^0 | Q_6 | K^0 \rangle$$

$$\approx \frac{-i \sqrt{3}}{2 f_\pi} \langle \pi^0 | Q_6 | K^0 \rangle$$

$$\approx \frac{\sqrt{3}}{2 f_\pi} \frac{9}{16} (7.8 \cdot 10^{10} \text{ MeV}^4)_{\text{bag}}, \quad (5.19)$$

where $f_\pi = 0.93 m_\pi$. To within the accuracy afforded by these estimates, the magnitude of $s_2 c_2 s_3 \sin \delta$ is constrained by eq. (5.18). Even with a conservative account of the errors associated with this matrix element and the Wilson coefficient, this constraint due to $\varepsilon'$ provides a better bound on $s_2 s_3 \sin \delta$ than the constraint from $\varepsilon$. It is consequently more effective in constraining CP violation elsewhere in the model.

We conclude this section with several comments. The constraints represented by eqs. (5.10), (5.13) and (5.18) comprise the extent of what one can definitively conclude on the basis of present day phenomenology. In contrast to eq. (5.7), they
do not afford a picturesque graphical representation of allowed versus disallowed regions. Their solutions are in crude qualitative agreement with figs. 5 and 6, the most notable exception being that the regions corresponding to large values of $s_2 s_3 \sin \delta$ are now eliminated by the $\epsilon'$ condition. As an independent constraint, eq. (5.13) from $\epsilon$ suffers from a dependence on $\xi$. It can be considerably improved by substituting for $\xi$ its theoretical expression from eq. (5.18), since this is much smaller than the experimental bound for most of the KM regions. An observation of $\epsilon'$ in the forthcoming experiments [22] would determine $\xi$ precisely and remove this uncertainty. Such an observation would also do much to eliminate the quadrant ambiguity. Our analysis finds the sign of $\epsilon'/\epsilon$ equates with the sign of $\sin \delta$, which makes $\epsilon'/\epsilon < 0$ a unique characteristic of the quad. III regions.

6. Conclusions

Numerical values for mixing and $CP$ violation are obtained by scanning the domain of KM angles. It is both surprising and reassuring that none of the bounds we observe derive from constraints on the unknown KM angles, which are not found to be restrictive. Bounds are obtained by exhaustive numerical search on the entire space of unknown angles, and reflect the nature of the model itself rather than a particular choice of angles.

Mixing depends on the fourth power of both $f_B$ and $m_t$; hence, its magnitude is highly uncertain. For values of $f_B$ and $m_t$ as small as 0.3 GeV and 20 GeV, respectively, $B^0$-$\bar{B}^0$ mixing can be complete ($r_2 = 0.5$), or it can vanish depending upon KM parameters. For smaller values of $f_B$, mixing decreases rapidly as $f_B^4$, independent of KM angles. If $m_t = 40$ GeV, mixing can be complete for $f_B$ as small as 0.15 GeV but necessarily decreases when $f_B$ is further reduced. The maxima occur for $\cos \delta < 0$ where a cancellation in $U_{cb}$ acts to suppress the decay rate. Mixing is characteristically smaller in quad. I ($0 < \delta < \frac{1}{2} \pi$), but there are no theoretical arguments which favor this quadrant.

The dilepton charge asymmetry $a$ is independent of $f_B$ and therefore less uncertain. It varies considerably throughout the space of KM angles and can approach 20 percent for $m_t = 20$ GeV despite those arguments which conspire to reduce it. It diminishes when $m_t$ is increased, since $CP$ violation vanishes when $M_{12} \gg \Gamma_{12}$ [see eq. (2.6)]. Further consideration of $a$ by itself is purely academic, for unless regions of KM parameters exist where $a$ and $r_2$ are simultaneously appreciable, $a$ remains an unobservable parameter.

The total lepton charge asymmetry $l^\pm$ is the best indicator of the observability of $CP$ violation, since it reflects the amount of both mixing and $CP$ violation. For though the standard model appears to accommodate large values of $a$ and $r_2$ individually, it is unfortunately the case that the magnitude of $l^\pm$ is strictly bound. $CP$ violation depends on significant values of $\sin \delta$, but when $\sin \delta$ is non-negligible, the cancellation in $|U_{cb}| (\cos \delta < 0)$ required for large mixing is spoiled; the overlap
between $CP$ violation and mixing is therefore small. An exhaustive search over the domain of KM parameters confirms that the total charge asymmetry of primary leptons from neutral B decay is bound below 0.01. This result does not depend upon theoretical bounds constraining $\theta_2$, $\theta_3$ and $\delta$. It is a property of the standard model and our knowledge of $\theta_1$.

To reiterate: independent of theoretical constraints on the unknown KM angles, the total lepton charge asymmetry in the standard model is less than one percent. Since the charged B decays act to further reduce this number by a half, detecting such an effect would be statistically implausible. Systematic errors alone could prohibit such a program. The restrictions this limit imposes on detecting $CP$ violation from an asymmetry in same-sign dileptons are equally severe.

This result is not entirely negative. It is a very clean and unequivocal statement about the standard model. If a total charge asymmetry is seen in excess of $\frac{1}{2}$ percent at the T"e, the KM model as the primary source of $CP$ violation in B decay is ruled out. Additional quark doublets or $CP$-violating Higgs exchange becomes necessary. If no such asymmetry is observed it is a further indication to support the standard model.

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