A relativistic formulation of the SU$_6$ group based on the little group of Wigner was sketched previously [1] and spelled out recently [2] for an arbitrary SU$_6$ multiplet. In view of the divergent statements [3] about the relativistic invariance of SU$_6$, applications of the general theory to the special case of the lowest (6-dimensional) representation of SU$_6$ corresponding to quarks [4] * will be given in this note. For this particular multiplet there are many simplifications and all the operators of the group can be written explicitly in equivalent forms which are already familiar to physicists. The following results which are essentially contained or implied in ref. 2, will be derived directly in this case:

1. The operators $X_i$ of the little group, when applied on quark particle states, are equivalent to the Hermitian mean spin operators of Foldy and Wouthuysen [5].

2. In the representation of Foldy and Wouthuysen for particle quarks the generators of SU$_6$ are $\frac{1}{2}\gamma_i$, $F_\lambda$, $\frac{1}{2}\sigma_q F_\lambda$, where the $\sigma_q$ are the usual Pauli matrices operating on the 3 quarks simultaneously and the $F_\lambda$ are the SU$_3$ matrices.

3. The free quark Hamiltonian is invariant under SU$_6$.


5. If $u(p,n)$ is the c-number Dirac spinor associated with a quark particle state with momentum $p$ and polarization direction $n$, then the SU$_2$ transportations of the little group induce the following Pauli transformation [7] ** in momentum space

$$u(p,n) \rightarrow u(p,n') = au(p,n) + b\gamma_5 u(p,n),$$

$$a^2 + b^2 = 1.$$  \hspace{1cm} (1)

We start with the expansion of the quark operator $\psi(x)$,

$$\psi(x) = \sum_p \Lambda(p) \omega(p)$$

where

$$\Lambda(p) = \left( \frac{E}{m} \gamma_4 \right)^{\frac{1}{2}} = \frac{m+p_0+\gamma_5 \sigma \cdot p}{\sqrt{2m(p_0+p_0)}}, \quad (p_0 = \sqrt{m^2+p^2}),$$

and

$$\omega(p) = e^{i\gamma_4 \beta \lambda X_\lambda} = \begin{pmatrix} a_1 & \bar{a}_2 & \bar{b}_2 \\ -b_2 & a_2 & \bar{a}_1 \end{pmatrix}$$

in the standard Dirac representation with $\gamma_4$ diagonal. $\alpha_i, \beta_i$ are the usual annihilation operators for particles and antiparticles, each with spin up or down. $\omega^{(+)}$ and $\omega^{(-)}$, the positive and negative frequency parts of $\omega$, are projected by $\frac{1}{2}(1+\gamma_4)$.

The non local operators $X_i$ of the little group were shown to be [2]

$$X_i(-i\hbar \nabla) = \frac{1}{2} \Lambda(-i\hbar \nabla) \alpha_i \Lambda^{-1}(-i\hbar \nabla),$$  \hspace{1cm} (5)

where $\Lambda$ is the boost operator that reduces to eq. (3) for a Dirac particle $\dagger$. Operating with $X_i$ on $\psi^{(+)}$, the positive frequency part of $\psi$, we obtain

$$X_i \psi^{(+)}(x) = \frac{1}{2} \sum_p X_i(p) \Lambda(p)(1+\gamma_4) \omega(p).$$

On the other hand, using eq. (3), we have

$$\Lambda(p) \frac{1+\gamma_4}{2} = \int \frac{d^3p_0}{m} U(p) \frac{1+\gamma_4}{2},$$

$\dagger$ In the $s = \frac{1}{2}$ case the operators $X_i$ are identical with the spin operators introduced by Chakrabarti [8], who has also shown their connection with the relativistic classical mechanics of a system of particles. Our operator $\omega(p)$ of eq. 2 is essentially the momentum space field operator in the Chakrabarti representation.
where

\[ U(p) = \left( \frac{m+ip\gamma^\lambda\gamma^\mu p}{p_0} \right)^{\frac{1}{2}} = \frac{m+p_0+ip\gamma^\lambda\gamma^\mu p}{\sqrt{2p_0(p_0+m)}} \]

is the unitary Foldy-Wouthuysen operator \[5\]. Eq. (6) now takes the form

\[ X_i \psi^{(+)}(x) = \sum_p X_i^{(+)}(p) \Lambda(p) \psi^{(+)}(p), \]

where

\[ X_i^{(+)}(p) = \frac{1}{2} U(p) \sigma U^*(p) = \frac{1}{2} \sigma - 1 \gamma_4 \gamma_5 \frac{\sigma \times p}{2p_0} - \frac{p \times (\sigma \times p)}{2p_0(p_0+m)} \]

are the Hermitian and non local mean spin operators. They coincide with the little group operators \( X_i \) on the manifold of the positive frequency states and are known to commute with the Hamiltonian. Thus we have proved statement 1.

We can now go over to the Foldy-Wouthuysen representation through

\[ \psi^{FW} = U \psi. \]

In this representation the Dirac Hamiltonian \( H^D \) and the little group operators have the form

\[ H^{FW} = U^* H^D U = \gamma_4 \gamma_5 \frac{\sigma \times p}{2p_0} - \frac{p \times (\sigma \times p)}{2p_0(p_0+m)} \]

(12)

\[ (\sigma^{(+)}\sigma^{(+)})^{FW} = U^* \sigma^{(+)} U = \frac{1}{2} \sigma. \]

(13)

This proves the statements 2 and 3. Hence, to formulate \( SU_6 \) we can use \( \frac{1}{2} \sigma \) as spin operators only in the Foldy-Wouthuysen representation. \( X_i \) should be used in the Dirac representation. The confusion between the two representations may have led some authors to claim the non invariance of the kinetic part of the free Hamiltonian under \( SU_6 \).

Statement 4 follows from the fact that the Newton-Wigner \[6\] position operator \( R \) for particle states is identical with the Foldy-Wouthuysen mean position operator. Thus \( R = x \) in the Foldy-Wouthuysen representation, and \( R \) and \( R \times p \) commute with the \( SU_6 \) generators. In the Dirac representation \( R \) and \( R \times p \) are non local operators.

To prove statement 5 we rewrite eq. (2) in the form

\[ \psi(x) = \sum_p [u(p) u^*_2(p) e^{i p_y x_y} + v(p) v^*_2(p) e^{i p_y x_y}], \]

(14)

where \( u, u^*, v \) and \( v^* \) are the 4 columns of the Lorentz matrix \( \Lambda(p) \) in the representation with \( \gamma_4 \) diagonal. Hence \( u, u^* \) and \( v, v^* \) are respectively the spin up and down wave functions in momentum space for particles and antiparticles respectively. Because \( \Lambda(p) \) is determined by its first column, one finds

\[ u^* = \gamma_5 u^C, \quad v = u^C, \quad v^* = \gamma_5 u. \]

(15)

Under a transformation of the little group with parameters \( \theta \), we have

\[ \psi^{(+)}(\theta) = e^{i \sigma \cdot \theta} \Lambda(p, \theta) \psi^{(+)}(p), \]

(16)

where \( \sigma \) is the polarization direction chosen as the \( Oz \) axis in the rest frame. The new polarization direction \( \sigma' \) can be defined by

\[ \Lambda(p, \sigma') = \Lambda(p, \sigma) R(\theta), \]

(17)

where \( R(\theta) \) is the unitary rotation matrix

\[ R(\theta) = e^{i \frac{1}{2} \sigma \cdot \theta} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}. \]

(18)

Rewriting eq. (17) in terms of the first columns of each side we find eq. (1).

We also note that the transformations (1) or (16) can be applied to each quark separately, corresponding to the subgroup \( G_6 = SU_2 \times SU_3 \times SU_2 \) of \( SU_6 \) with generators \( \Pi_1 X_{\mu 1}, \Pi_2 X_{\mu 2} \) and \( \Pi_3 X_{\mu 3} \), where \( \Pi_1, \Pi_2 \) and \( \Pi_3 \) are the projection operators for the three quarks \( q_1, q_2 \) and \( q_3 \).

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2. F. Görsy and L. A. Radicati, Lorentz covariant definition of the \( SU_6 \) group, to be published. In this paper it was shown that the little group operators \( X_i \) can be defined by means of a self dual tensor \( W_K \) constructed from the Bargman-Wigner operators \( W_K \) that commute with the energy-momentum operators.