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Essays in Theoretical Physics Dedicated to Gregor Wentzel

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The Spatial Extent of Magnetic Monopoles

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Dirac\(^1\) showed that if magnetic charge exists, it must occur in units \(e^1 = e^{1/2}(\hbar = c = 1)\); this follows merely from quantization of angular momentum. Schwinger\(^2\) argued that the unit should be \(e^1 = e^{-1}\), which amounts to requiring integral quantization of the field angular momentum; we shall accept this. So a particle which carries a magnetic charge, a monopole, is coupled very strongly to the electromagnetic field; its coupling to a photon is characterized by the number

\[
e^{12} = e^{-2} = 137.\tag{1}
\]

This is an order of magnitude stronger than the pion-nucleon coupling, \(g^2/4\pi = 15\); it is so strong that many calculations that have been attempted, such as ionization energy loss of a monopole in matter, and monopole pair production cross sections, are very doubtful.

One's first reaction is that strong coupling theory (we mean the well known static model strong coupling theory) ought to be applicable; unfortunately it is not. For instance, consider the photon-monopole elastic amplitude, Compton scattering. The theory of this is identical (up to vacuum polarization effects) to ordinary Compton scattering, if we exchange \(\vec{E}\) and \(\vec{H}\) fields of the radiation field, and substitute \(e^1\) for \(e\). It is thus a rigorous result that the amplitude at zero frequency, the scattering length, is given by the Thompson formula

\[
f(\omega = 0) = -e^{12}/M \equiv -r_m^{-1}\ ("\text{classical magnetic radius}\")\tag{2}
\]

where \(M\) is the mass of the monopole. The difficulty is that this is a large repulsive scattering length, which loosely speaking requires\(^3\) that the range of interaction between photon and monopole be at least \(r_m^{-1} = 137M^{-1}\). [Because this is large compared to the Compton radius \(M^{-1}\), relativistic effects do not help, unlike the case of electrically charged particles, where the classical electron luminosity \(r_e\) is dynamically irrelevant because \(r_e = eM^{-1} = m^{-1} << m^{-1}\).]

In contrast, in meson models in which the strong coupling theory applies, the meson-nucleon scattering is strong and attractive, which induces a bound state (isobar) pole which saturates the amplitude; the corresponding phenomenon in the present case would be the binding of a ghost.

This is well known in the classical theory, where the magnetic dipole amplitude for an electromagnetic wave scattering on a point monopole is

\[
f_N(\omega) = -[r_m^{-1} + 2/3 \, i\omega]^{-1}.\tag{3}
\]

This has a non-causal "ghost" pole at \(\omega = i\,3/2 \, r_m^{-1} = i\,3/2 \, M/137\); it represents Dirac's pre-acceleration.\(^4\) In order for the classical theory to be causal, the monopole's charge must be spread out, with a radius of at least \(r_m\), so that the form factor singularities replace the ghost pole. (Put more dynamically, the condition for the absence of runaway motion, or pre-acceleration, is that the bare mass \(m_0\) be non-negative, which requires that the electromagnetic self mass \(\sim e^{12} \frac{1}{r_m} \geq 0\) must be no larger than the total mass \(M\), which means \(\frac{1}{r_m} \leq M/e^{12} = 1/r_m\).)

We argue\(^5\) that the same must be true in the quantum theory of the monopole: The radius of a monopole must be of the order of \(r_m\).\(^6,7\) We start with a rigorous formula, the dispersion relation for the reciprocal of the (laboratory) forward Compton amplitude, \(f^{-1}(\omega)\), which implies the sum rule

\[
r_m^{-1} = \frac{M}{137} = \frac{2}{\pi} \int_0^\infty d\omega \, R(\omega)\tag{4}
\]
where
\[ R(\omega) = \frac{\sigma_{\text{tot}}}{4\pi |f|^2} \]  
(5)

The l.h.s. of (4) is small (on the scale of \( M \)) and so this puts a limit on \( R \). For instance, \( R \) could not be of order 1 over a frequency range larger than \( 0 < \omega < M/137 \). Thus, on the scale of \( M \), \( R \) must quickly fall, which means that higher partial waves must quickly come in, corresponding to a range of interaction of order \( r_m^{-1} \). One argues this by writing
\[ R = \frac{\sigma_{\text{tot}}}{\sigma_{\text{et}}} \frac{\sigma_{\text{et}}}{4\pi |f|^2} = \frac{\sigma_{\text{et}}/4\pi}{(\partial \sigma_{\text{et}}/\partial q)|_{q=0}} \leq \frac{1}{4(\Delta \theta)^2} \]  
(6)

where \( \Delta \theta \) is the width of the forward peak of \( \partial \sigma_{\text{et}}/\partial q \), \( \Delta \theta \approx ka \), where \( a \) is the range of the force. Thus (4) puts a lower limit on the range \( a \). For instance, if \( \partial \sigma_{\text{et}}/\partial q \) were of the form \( \partial \sigma_{\text{et}}/\partial q \sim (1+a_0^2k^2)^{-2} \), (4) yields \( a \sim 1 \frac{1}{2} r_m^{-1} \).

However, in the present case the effect of soft photons complicates the argument a bit. The elastic Compton cross section vanishes because, except for forward scattering, additional soft plons are always emitted by the recoil of the monopole. Hence it makes no sense to discuss the magnitude of \( R \) by writing it as \( R = \frac{\sigma_{\text{tot}}}{\sigma_{\text{et}}} \frac{\sigma_{\text{et}}}{4\pi |f|^2} \), because the two factors are infinite and vanishing, respectively (and the vanishing has nothing to do with the range of the phonon-monopole interaction, but rather with the distance an almost-real monopole can travel). Instead, one should use the concept of quasielastic scattering, defined in the usual way as scattering in which the incident photon loses less than an energy \( \Delta \). Then \( \sigma_{\text{q-et}}/4\pi |f|^2 \) will properly reflect the size of the monopole.

It might be noted that soft photon emission by a monopole is actually not a strong effect, despite the largeness of \( e' \), if the monopole has size \( r_m^{-1} \). The spectrum of soft photons emitted by a non-relativistic monopole in a process in which the velocity change of the monopole is \( \Delta V \) is of the order
\[ \frac{dE_{\text{rad}}}{d\omega} \sim e'^2(\Delta V)^2. \]  
(7)

If this spectrum is cut off at \( \omega \approx r_m^{-1} \), because the monopole has the size \( r_m^{-1} \), the total radiative soft photon energy is of order
\[ E_{\text{rad}} \sim e'^2(\Delta V)^2 r_m^{-1} = (\Delta V)^2 M. \]

Further, if the monopole size is \( r_m \) means that the momentum transfer to the monopole is never larger than \( r_m^{-1} \), then \( \Delta V \leq r_m^{-1}/M = 1/137 \), and so
\[ E_{\text{rad}} \leq (1/137)^2 M. \]

More specifically, in Compton scattering, we have under these 'size' assumptions,
\[ \omega_{\text{Max}} \sim \min \left( \frac{k}{r_m}, r_m^{-1} \right) \]  
(8)
\[ (\Delta V)_{\text{Max}} \sim \min \left( \frac{k}{M}, \frac{r_m^{-1}}{M} \right), \]

where \( k \) is the incident photon momentum, with the result
\[ \left( \frac{E_{\text{rad}}}{k} \right)_{\text{Max}} \sim \min \frac{e'^2k^2}{M^2}, \frac{e'^2}{r_m^{-3}M^2 k} \leq 1/e^{12} \]  
(9)

which means that the fraction of the incident photon energy that is radiated off as soft photons is at most \( \leq 12 \%. \) (It would be large only in the 'intermediate coupling case', \( e'^2 \sim e'^2 \approx 1 \).)

We have argued that a monopole must be as big as its 'classical radius' \( r_m = 137 M^{-1} \). We now sketch how this can come about self-consistently, in a model in which there are only monopoles
and photons. The monopole size $r_m$ would result from the mediation of neutral mesons with masses of the order of $r_m^{-1}$. [For instance, such a vector $C = -1$ meson would produce a one-photon form factor; a scalar $C = +1$ meson would contribute to a two-photon form factor.] But just such mesons result as bound states of a monopole-antimonopole pair. When the monopole and antimonopole are at the same position, their magnetic charge distributions cancel; for small separations $r$, the magnetostatic energy is of order $Mr^2/r_m^2$. Hence the energy of the system is

$$E = 2M_0 + p^2/M + cHR^2/r_m^2, \quad c = 0(1),$$

(10)

where $M_0$ is the classical bare mass of each monopole.

The level spacing of the harmonic oscillator (10) is

$$\omega = c' r_m^{-1}, \quad c' = 0(1).$$

(11)

(One verifies that in low lying states of (10) the monopoles are nonrelativistic and localized in the harmonic region of the potential, $<r^2/r_m^2> \approx \frac{1}{M_0} \left(1 + 137 \right) \ll 1$). Thus if $M_0$ is small, the bound states of the monopole pair system have masses of order $r_m^{-1}$, as advertised.

A more specific model results from assuming that the form factor of the monopole is dominated by a vector meson of mass $\mu$; the monopole-antimonopole interaction is then $-\left(\frac{1}{4}\frac{e^2}{q^2}\frac{1}{1 + q^2/\mu^2}\right)^2$, which in coordinate space is

$$-\frac{e^2}{r} \left[1 - \left(1 + \frac{\mu^2}{r^2}\right) e^{-\mu r}\right].$$

(12)

The energy of the pair system for small $r$ is then

$$H = \left(2M - \frac{e^2\mu}{2}\right) + p^2/M + \frac{1}{2} \frac{e^2\mu}{6} \mu^2r^2$$

(13)

which is of the form (10), with $M = M_0$ identified as $e^2\mu/4$; this is indeed just the classical self energy of the assumed charge distribution. According to (13), the low lying states have energies of order $\mu$ if $\mu \approx 4M/\alpha r^2$.

Of course these mesons (monopole-pair bound states) are unstable to decay into photons. But if their decay is not so rapid as to wash them out as states altogether, one would expect their decay to have little effect on their role of producing the structure of the monopole.

We conclude by considering monopoles in the real world. If, as we have argued, a monopole of mass $M$ has a size $r_m$ which is produced by the mediation of neutral mesons of mass $r_m^{-1} = M/137$, these neutral mesons should be observable through their couplings to photons. No such scalar or vector mesons are known, below the $\sigma$ and $\rho_0$, mass $\approx 750$ MeV. This lower limit on $r_m^{-1}$ puts a lower limit on the monopole mass of the order $137 \times 0.75 \approx 100$ BeV, or if one takes Eqs. (12) and (13) seriously,

$$M > \frac{e^2\mu}{4} \quad m_\mu = 25 \text{ GeV}.$$  

(14)

If one chooses to identify the $\sigma$ and $\rho_0$ with the monopole "structure mesons", one gets $M \approx 25$ BeV as an estimate of the monopole mass. The identification of hadrons with bound states of monopole pairs, or more generally of any assortment of monopoles whose total magnetic charge adds up to zero, is attractive but has at least one difficulty: It is hard to disguise the fact that such states are composed of large magnetic charges $e^2\mu$; their magnetic polarizabilities are of order $m^2$ (m = typical hadron mass) rather than the usual $e^2m^3$.

References

3. C. Goebel and G. Shaw, Phys. Letters 27B, 291 (1968); see footnote **.
5. The argument of this paragraph was given in Ref. 3.

6. This is the same estimate of monopole size made by L. I. Schiff, Phys. Rev. 160, 1257 (1967) [also Phys. Rev. Letters 17, 714 (1966)], based on equating its mass to its classical magnetostatic self energy. He did not explain why a bare mass, traditionally arbitrary (even infinite) in field theory, would not upset this argument. He did argue that a monopole might obey classical, not quantum, dynamics, and the sum rule, Eq. (4), does show that the classic causality argument on the monopole size applies. But Schiff's deduction of classical behavior was based on an argument that an extended monopole must continuously acquire angular momentum as a point charge approaches; this is surely faulty, because the argument would imply the same about any extended charged system, such as an \( H^+ \) ion, as a point magnetic charge approaches.

7. If the usual electromagnetic potentials are used, monopoles must have strings attached\(^1\) (string = singular line of the vector potential). It seems rather awkward to treat extended monopoles this way, although in principle it should be possible [cf. G. Wentzel, Prog. Theoret. Phys. (Kyoto) Suppl. Nos. 37 and 38, 163 (1966), and Schiff, Ref. 6]. Perhaps it would be simpler to interchange \( E \) and \( H \), and attach strings to the electric charges (if any) of the system.

8. It amounts to the identification of quarks with monopoles. I thank G. Zweig for a discussion on this matter.

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A New Representation for the Plasma Conductivity Tensor in a Magnetic Field

by

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1. Introduction

Conventional expressions for the conductivity tensor \( \sigma \), or the dielectric tensor

\[
\sigma = 1 + 4\pi i \omega / \omega,
\]

of a collisionless plasma in a magnetic field involve infinite sums of Bessel function terms, each associated with an harmonic of the cyclotron frequency, \( \omega \). For many purposes, it is convenient to express these sums in closed form; Berk and Rosenbluth\(^1\) have shown how the series may be summed in the particular case of electrostatic waves, and Majumdar\(^2\) has examined the special case of propagation perpendicular to the magnetic field, assuming the unperturbed velocity distribution function is Maxwellian. We give here general expressions for the elements of \( \sigma \) in which no Bessel function expansions are used, thus eliminating the necessity for subsequent resuming. The closed form expressions given here are valid for any particle distribution functions, since the integrations over \( v_z \) and \( v_\perp \) are not carried out explicitly. The compact nature of our closed form representation facilitates the extension to more complicated problems (inhomogeneous plasmas, non-linear interaction of waves). It also is well suited to the derivation of asymptotic expressions when the frequency is large (compared to cyclotron frequency) or the wavelength is small (compared to cyclotron radius). The conventional expressions, involving sums over harmonics, can be recovered at any point by making use of a standard identity for Bessel functions.