TOWARDS A GRAND UNIFIED THEORY OF FLAVOR

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I describe a set of guidelines for the construction of grand unified theories which incorporate multiple flavors. These theories contain the standard SU(5) theory as a subgroup, but their gauge structure also includes non-trivial flavor group structure in the same simple group. I find an example based on an SU(11) gauge symmetry in which six flavors of quarks appear naturally.

1. Beyond SU(5)

Why is electric charge quantized? Why do leptons carry integral charge while quarks carry just the right fractional charge to make the color singlet hadrons integrally charged? Why is it that quark and lepton fields of only one chirality participate in the charged current weak interactions? Why is the color SU(3) coupling constant $\alpha_s$ so much larger (at ordinary momenta) than the electromagnetic coupling $\alpha$? Why is $\sin^2 \theta_w \simeq 0.2$? These questions and others can be answered simply and elegantly if the SU(2) × U(1) gauge theory of weak and electromagnetic interactions is unified with the color SU(3) gauge theory of strong interactions in a grand unified theory with SU(5) gauge symmetry [1,2]. But this remarkable theory leaves two important questions completely unanswered.

(i) The theory necessarily contains very massive particles ($\sim 10^{15}$ GeV) associated with the breaking of the grand unified gauge symmetry down to SU(3) × SU(2) × U(1) [2]. Why then are ordinary particles so much lighter? This is the gauge hierarchy problem.

(ii) In the SU(5) theory, the fermions come in families. Each family consists of a charge $\frac{2}{3}$ quark triplet, a charge $-\frac{1}{3}$ quark triplet, a charge $-1$ lepton and a massless neutrino, and fits into the smallest complex anomaly-free representation of SU(5): a 10 and $\overline{5}$ of left-handed (LH) fermion fields. The theory gives no explanation for the experimental fact that more than one family exists. This is the problem

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of flavor. It is possible to incorporate more than one family in the SU(5) theory by doubling (tripling, etc.) the simplest fermion representation, but there is no good theoretical reason to do so.

I believe that these two unanswered questions indicate that the SU(5) grand unification is not incorrect but incomplete. I expect the SU(5) gauge group to appear as a subgroup of the complete grand unified gauge group of the world. In fact, the particular embeddings of SU(5) into larger groups which I discuss below preserve all the good features of the original SU(5) model.

In this paper, I will attempt to solve the second problem, the problem of flavor. I will show that it is possible to incorporate some flavor symmetry into the unified gauge group along with the SU(5) structure. I will not address the gauge hierarchy problem directly, but my entire approach is based on a view of the nature of the ultimate solution. In sect. 2, I will review this approach and compare it with dynamical symmetry breaking.

In sect. 3, I will formulate a set of principles for the construction of grand unified theories. In sect. 4, I will apply these principles to construct a variety of generalizations of the one-family SU(5) model, some of which necessarily involve more than one family.

2. The grand plateau

The gauge hierarchy problem is the subject of a recent paper [3] by Weinberg in which the issues are very clearly stated. Consider the situation in SU(5). Two irreducible multiplets of Higgs bosons are needed to break the SU(5) symmetry down to SU(3) × U(1), a 24 and a 5. The self-couplings of the 24 can be naturally chosen so that in the absence of 5-24 coupling, the vacuum expectation value (VEV) of the 24 would break the gauge symmetry down to SU(3) × SU(2) × U(1). In the realistic SU(5) theory, this VEV must be large, of the order of $10^{15}$ GeV. Under the SU(3) × SU(2) × U(1) subgroup of SU(5), the 5 transforms like a (3, 1) color triplet and a (1, 2) weak doublet. In the realistic theory, the (3, 1) must develop a large mass $O(10^{15}$ GeV) while the (1, 2) develops a small VEV, ~300 GeV. This can happen if the 5-5 bare mass term and the 5-5-24 quartic Higgs coupling terms conspire to leave the (1, 2) component light.

I have no wish to minimize the problem. It is certainly true that it requires an incredibly accurate tuning of the various components of the (1, 2) mass to obtain the gauge hierarchy. Instead, I want to extract those aspects of the theory which may survive the eventual solution of this problem. The picture that emerges is rather dramatic. There is one explicit mass scale in the theory, the grand unification mass $\sim 10^{15}$ GeV. Many (perhaps most) elementary particles have a mass of order $10^{15}$ GeV, but some remain massless. The particles which are massless at the scale of $10^{15}$ GeV are of three kinds:
(i) gauge particles which are massless because they are associated with an unbroken gauge symmetry \( SU(3) \times SU(2) \times U(1) \);
(ii) fermions which are massless because they carry an unbroken chiral symmetry \( SU(2) \times U(1) \);
(iii) a Higgs \( SU(2) \) doublet which is massless for reasons which are slightly obscure.

Our world is built out of this debris, after a further spontaneous symmetry breaking at 300 GeV.

If this picture is correct, physics between 300 GeV and \( 10^{14} \) GeV is boring. There is a grand plateau in momentum scale on which the world is well-described by an \( SU(3) \times SU(2) \times U(1) \) gauge theory. Below 300 GeV, there may yet be exotic objects in \( SU(3) \times SU(2) \) representations more complicated than the \((3, 2), (3, 1), (1, 2)\) and \((1, 1)\) of the quarks and leptons. Nothing I have said so far forbids the existence of quixes, queights, etc., but if this view is correct, there will be no new interactions below \( 10^{15} \) GeV. This is the simplest and most conservative implementation of grand unification.

Of course it may be that nature is not so conservative. There has been some speculation concerning the existence of gauge groups with couplings much stronger than those of color \( SU(3) \) [4]. In particular, it has been argued that with several such groups with appropriately chosen fermion representations, the spontaneous symmetry breakdown required to break \( SU(2) \times U(1) \) and give mass to the ordinary quarks can be completely dynamical, involving no fundamental Higgs boson field [5].

In this paper, I will discuss only the conservative theories with an explicit Higgs meson. The dynamical symmetry breaking idea is interesting, but the gauge structure required to implement it is usually so complicated that it is not clear how to grand unify it. The conservative theories with explicit Higgs mesons are so much simpler that I think it is good tactics to investigate them thoroughly.

3. The laws of grand unification

Many of the attractive predictions of the \( SU(5) \) model are automatic consequences of the grand unification. Charge quantization, for example, follows trivially from the fact that the electromagnetic charge is a generator of a spontaneously broken simple gauge group. But some of the nice features of \( SU(5) \) depend on properties of the specific representation under which the fermions transform.

To see how this works, take all the fermion fields to be left-handed (eigenstates of \( \gamma_5 \) with \( \gamma_5 = 1 \): this can always be done by charge-conjugating the right-handed fields). The fermion representation of the grand unified group is said to be real if the generators, \( T_a \), acting on all the LH fermion fields satisfy

\[
-T_a^* = U T_a U^\dagger,
\]  
(3.1)
where $U$ is unitary: in other words if the representation is equivalent to its complex conjugate representation. If not, the representation is complex. I can extend the definition of reality/complexity of a representation to refer only to a specific subgroup. The fermion representation is real with respect to a subgroup if the generators of the subgroup satisfy eq. (3.1). If not, the representation is complex with respect to the subgroup. Clearly, if a representation is real, it is real with respect to any subgroup.

Now I can formulate simple conditions which a good grand unified theory should satisfy.

The first law of grand unification: the representation of the LH fermions must be real with respect to the color SU(3) subgroup.

The second law of grand unification: the representation of the LH fermions should be complex with respect to the SU(3) × SU(2) × U(1) subgroup.

The first law is a clear necessity. If it is not satisfied, the color SU(3) theory of strong interactions does not make sense. The point is that if the color SU(3) representation is complex, then the color gauge symmetry itself is chiral. The spontaneous breakdown of the chiral symmetries associated with confinement will break the color gauge symmetry as well. This is probably obvious, but I will belabor the point because the SU(5) theory itself provides a nice example.

Imagine an SU(5) theory with a 10 and a $\bar{5}$ of LH fermions but no Higgs mesons at all. This is a perfectly renormalizable anomaly free theory [6–8]. If instead of a 10 and $\bar{5}$, the LH fermions were a $\bar{5}$ and $\bar{5}$, the theory (presumably) would confine. The $\bar{5}$ and $\bar{5}$ would combine to form a four-component fermion with a dynamically generated mass. With a 10 and a $\bar{5}$, this kind of trivial dynamical mass generation is not consistent with the gauge symmetry. What I suspect will happen is that the gauge symmetry will break spontaneously down to SU(4). Under the SU(4) subgroup, the $10 + \bar{5}$ transform like $6 + 4 + 4 + 1$. The 4 and $\bar{4}$ will combine to form a four-component massive fermion. The 6 is real with respect to the SU(4) subgroup. It will combine with itself with an SU(4) invariant dynamical Majorana mass. The singlet will remain as an unconfined massless fermion.

If the color SU(3) symmetry is chiral (for example the LH fermions might be a 6 and seven $\bar{3}$s to cancel the anomalies [8]) it will probably break spontaneously down to SU(2), with respect to which all representations are real. I want to avoid this kind of spontaneous breakdown of color SU(3) so I demand that the first law be satisfied.

The second law is related to the V – A nature of the weak interactions. The quarks and leptons that have been observed transform (apparently) according to a complex representation of the SU(3) × SU(2) × U(1) subgroup of the gauge group of the world. It is only the LH quarks and leptons which are doublets under weak SU(2), not the LH antiquarks and antileptons. A single family of quarks, leptons and their antiparticles transforms like $(3, 2) + (1, 2) + 2(\bar{3}, 1) + (1, 1)$ under SU(3) × SU(2): obviously a complex representation. I believe that this should be taken as a hint that the full representation of the world is complex. Only then can
we preserve the explanation of the chiral nature of the weak interactions exhibited in the SU(5) model.

There is a deeper reason to require the fermion representation to be complex with respect to SU(3) × SU(2) × U(1). I am assuming that the grand unifying symmetry is broken all the way down to SU(3) × SU(2) × U(1) at a momentum scale of $10^{15}$ GeV. I would therefore expect any subset of the LH fermion representation which is real with respect to SU(3) × SU(2) × U(1) to get a mass of the order of $10^{15}$ GeV from the interactions which cause the spontaneous breakdown. As a trivial example of this, consider an SU(5) theory in which the LH fermions are a 10, a 5 and two $\bar{5}$'s. In this theory there will be SU(3) × SU(2) × U(1) invariant mass terms connecting the 5 to some linear combination of the two $\bar{5}$'s. These ten (chiral) states will therefore correspond to 5 four-component fermions with masses of order $10^{15}$ GeV. The 10 and the orthogonal linear combination of the two $\bar{5}$'s will be left over as ordinary mass particles because they carry chiral SU(2) × U(1).

In general, it is only the “SU(3) × SU(2) × U(1) complex part” of the LH fermion representation that can avoid masses of order $10^{15}$ GeV and survive as the set of ordinary fermion fields.

4. Generalizations of one-family SU(5)

I now have the tools to generalize the SU(5) theory in a non-trivial way. I want to find a group and a representation for the LH fermions consistent with the rules set down in sect. 3 such that the ordinary mass fermions consist of several families of quarks and leptons, but I want to avoid a trivial solution in which a small representation is repeated several times, so I demand yet another principle.

*The third law of grand unification:* no irreducible representation should appear more than once in the representation of the LH fermions.

In the simplest SU(5), of course, this is just the restriction to one family.

One might hope to obtain more than one family without enlarging the gauge group by making use of more complicated representations of SU(5). Unfortunately, this does not work. Such representations, if they yield any ordinary mass fermions at all, predict ordinary mass fermions with peculiar SU(3) × SU(2) × U(1) properties *. Unless there are real surprises between presently accessible energies and 100 GeV, such representations are ruled out.

In enlarging the gauge group, I will restrict myself to groups which have the ordinary SU(5) theory as a subgroup. In fact, for reasons which will soon become clear, I will concentrate on larger unitary groups SU(N) into which SU(5) is em-

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* This is true because SU(3) × SU(2) × U(1) is a maximal subgroup of SU(5) (both are rank 4). Irreducible representations of SU(5) are uniquely labeled by their highest weight and thus by their SU(3) × SU(2) × U(1) properties.
bedded in the simplest possible way, with the $N$-dimensional representation of $SU(N)$ transforming like a 5 plus $N - 5$ singlets under the $SU(5)$ subgroup.

The virtue of this simplest embedding is that it makes it possible to satisfy the first law in a very simple way. All of the fundamental representations of $SU(N)$, the representations formed by taking antisymmetric combinations of the $N$-dimensional defining representation, contain only 3's, $\bar{3}$'s and singlets under the color $SU(3)$ subgroup. So if the anomaly of some set of such representations is zero, the total number of 3's must equal the total number of $\bar{3}$'s, so the representation will be real with respect to color $SU(3)$.

In fact, the $SU(3) \times SU(2) \times U(1)$ properties of the ordinary mass fermions arising from an anomaly-free set of fundamental representations of $SU(N)$ are automatically the same as those of some number of $SU(5)$ families. I can show this simply by considering the $SU(5)$ properties of the representation. The $SU(3) \times SU(2) \times U(1)$ properties of any representation essentially determine its $SU(5)$ properties because $SU(3) \times SU(2) \times U(1)$ is a maximal subgroup of $SU(5)$ (both have four commuting generators). The fundamental representations decompose under the $SU(5)$ subgroup of $SU(N)$ into a sum of $(5)$'s, $(\bar{5})$'s, $(10)$'s, $(\bar{10})$'s and singlets $(1)$'s. Call the total number of $(5)$'s in a given anomaly-free set of fundamental representations $N(5)$, the total number of $(\bar{5})$'s $N(\bar{5})$, etc. Since the anomaly is zero,

$$N(5) = N(\bar{5}) = N(10) = N(\bar{10}).$$

Assume $N(10) \geq N(\bar{10})$ (if it isn't, then I can look instead at the complex conjugate representation). Then the $N(10)$ $(10)$'s will combine with $N(\bar{10})$ linear combinations of the $(10)$'s to form four component fermions with $SU(3) \times SU(2) \times U(1)$ invariant masses of order $10^{15}$ GeV. The $N(10) - N(\bar{10})$ orthogonal linear combinations carry the uncompensated chiral $SU(2) \times U(1)$ symmetry, so they are left light and correspond to ordinary fermions. Similarly, $N(5)$ $(5)$'s combine with $N(\bar{5})$ $(\bar{5})$'s, leaving $N(5) - N(\bar{5})$ $(\bar{5})$'s corresponding to ordinary fermions. Since $N(10) - N(\bar{10}) = N(5) - N(\bar{5})$ because of eq. (4.1) the ordinary mass fermion sector looks like $N(10) - N(\bar{10})$ ordinary $SU(5)$ families $(10) + (\bar{5})$.

Thus for sets of fundamental representations of $SU(N)$, my task consists of only two steps: (i) find a set of fundamental representations satisfying the third law (no repeated representations) which is complex but free of anomalies; (ii) count the number of $(10)$'s and $(\bar{10})$'s in the $SU(5)$ decomposition and subtract to obtain the number of families of ordinary mass fermions.

Proceeding in this way, I have identified several infinite sequences of generalizations of one-family $SU(5)$. To describe them, I will label each fundamental representation of $SU(N)$ by an integer $[m]$ where $m$ is the number of boxes in the corresponding Young tableau, or equivalently, the number of $(N)$'s which must be combined antisymmetrically to get the $[m]$ representation. Thus in $SU(5)$, $(5)$ is $[1]$.

* See previous footnote.
(10) is \([2], (\overline{10})\) is \([3]\) and \((\bar{5})\) is \([4]\). The one-family SU(5) theory has LH fermions transforming like \([2] + [4]\). In general, for SU(\(N\)), the dimension of \([m]\) is

\[
D(m) = \frac{N!}{m!(N - m)!},
\]

and the anomaly is \([8]\)

\[
A(m) = \frac{(N - 2m)(N - 3)!}{(N - m - 1)!(m - 1)!}.
\]

The following sequences have realizations (other than SU(5)) for SU(\(N\)) with \(N \leq 11\).

\[
\begin{align*}
\text{SU}(2k + 1): & \sum_{l=1}^{k} [2l], \\
\text{SU}(6k + 3): & \sum_{l=1}^{2k} [6k - 3l + 4], \\
\text{SU}(k^2) \text{ or } \text{SU}(k^2 + 1): & \left[\frac{k^2 - k + 2}{2}\right] + \left[\frac{k^2 + k + 2}{2}\right], \\
\text{SU}(3k + 2): & [k + 1] + \sum_{l=2k+2}^{3k+1} [l],
\end{align*}
\]

where \(k\) is any positive integer. The last of these is the most interesting, so I will discuss the others only briefly.

The first, eq. (4.4), is related to the natural embedding of SU(\(2k + 1\)) into the orthogonal group O(\(4k + 2\)). The \(4^k\) dimensional complex spinor representation of O(\(4k + 2\)) transforms under the SU(\(2k + 1\)) subgroup as \([2l]\) (where \([0]\) is the trivial one-dimensional representation). For \(k = 2\), this is the O(10) extension of SU(5): the 16 contains one SU(5) family and a singlet. Unfortunately, for \(k > 2\), there are no families at all. The larger orthogonal groups have O(10) subgroups under which the spinor representations transform like equal numbers of 16's and \(\overline{16}\)'s. This is easy to prove directly or by noting that the symmetric trace of five generators, which is non-zero for the 16 in O(10), is zero for all higherorthogonals \([7]\).

The other sequences, so far as I know, do not have any similar Lie-group-theoretic interpretation. The sequence shown in eq. (4.5) always gives one family, so it is not very interesting. The sequences shown in eq. (4.6) are very appealing since they involve only two irreducible representations, however I can show that the number of families is

\[
\frac{2(k^2 - 4)!}{m_+!m_-!}, \quad m_\pm = \frac{1}{2}(k^2 \pm k - 2).
\]
This is unacceptable because the number of families grows too fast. For $SU(9)$ and $SU(10)$ ($k = 3$) there is only one family. For $SU(16)$ and $SU(17)$ ($k = 4$) there are 22. With so many families, the color $SU(3)$ is not asymptotically free and it is not obvious what grand unification means.

Finally, consider the last sequence, eq. (4.7). Here I can show that the number of families is

$$\frac{(3k - 3)!}{(k - 1)!(2k - 1)!}$$

(4.9)

For $SU(5)$ and $SU(8)$ ($k = 1$ and 2) there is only one family. $SU(11)$ ($k = 3$) has three families. $SU(14)$ ($k = 4$) has twelve. These two groups, $SU(11)$ and $SU(14)$ are the smallest examples I have found of generalizations of $SU(5)$ with interesting but manageable flavor structure.

Under the $SU(5)$ subgroup, the $SU(11)$ representation of eq. (4.7) transforms as follows:

- [4] (330-dimensional):
  $\bar{\mathbf{5}} + 6(\mathbf{10}) + 15(\mathbf{10}) + 20(\mathbf{5}) + 15(\mathbf{1})$,

- [8] (165-dimensional):
  $(\mathbf{10}) + 6(\mathbf{10}) + 15(\bar{\mathbf{5}}) + 20(\mathbf{1})$,

- [9] (55-dimensional):
  $(\mathbf{10}) + 6(\bar{\mathbf{5}}) + 15(\mathbf{1})$,

- [10] (11-dimensional):
  $(\bar{\mathbf{5}}) + 6(\mathbf{1})$.

(4.9)

Altogether there are sixteen $(\mathbf{10})$'s, thirteen $(\mathbf{10})$'s, twenty $(\mathbf{5})$'s, twenty-three $(\bar{\mathbf{5}})$'s and fifty-six singlets. Of these five hundred and sixty one LH fermion fields, only three families, forty-five fields, survive at ordinary masses.

Clearly the three ordinary mass families are embedded in a non-trivial way. The generators of the $SU(6)$ subgroup of $SU(11)$ which acts on the $SU(5)$ singlets form a kind of “flavor group”, but they connect the ordinary mass families not only to each other, but to the rich structure at $10^{15}$ GeV.

5. Questions

In this paper, I have presented what I believe to be a plausible set of guidelines to systematize the search for the gauge group of the world and the corresponding fermion representation. Even assuming that the basic concepts make sense, there are still many questions which must be answered to elevate the above analysis from recreational mathematics to physics.

I have already discussed the gauge hierarchy problem., but to be honest I should emphasize that nothing I have said in sects. 3 or 4 improves the situation. I think additional new ideas are needed.
The sequences described by eqs. (4.5)–(4.7) all have tantalizingly simple structures. It may be that they can be further embedded into yet more symmetrical theories. Any such elaboration would help me to decide whether such constructions might be relevant. At the moment, they all seem rather mysterious and ad hoc.

I have not discussed the nature of the breaking of the grand unified symmetry down to SU(3) × SU(2) × U(1), or the nature of the Higgs couplings to the fermions. For the large groups I have been led to consider, there are so many possibilities that I need some additional constraints to usefully attack these problems. A deeper understanding of the structure of the fermion representations would presumably help to limit the possibilities.

It may be that I have missed some beautiful theories by limiting my investigation to fundamental representations of SU(N) and trivial embeddings of SU(5) into SU(N). To study this question, I would need more efficient ways of finding representations which are real with respect to SU(3).

Finally, it is amusing to speculate that the structure of the theory at 10^{15} GeV might well involve new principles, not encountered in our experience with field theories at ordinary energies. For example, it may not be necessary to demand that the theory be renormalizable by power counting. The effective renormalizability of that fragment of the theory which survives at ordinary energies is more or less automatic simply because the only intrinsic mass scale in the theory is 10^{15} GeV [9]. This is easy to say, but it is not very interesting unless the constraint of renormalizability can be replaced by something else as restrictive or more restrictive. Especially at 10^{15} GeV where experimental information is rather severely limited, we need theoretical guidelines to help us in our search for understanding of the structure of the world.

References

   E. Eichten, K. Lane and S. Weinberg, private communication.