Rare decay modes of the $K$ mesons in gauge theories
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(Received 4 March 1974)

Rare decay modes of the kaons such as $K \rightarrow \mu \bar{\mu}$, $K \rightarrow \pi \nu \bar{\nu}$, $K \rightarrow \gamma \gamma$, $K \rightarrow \pi \gamma$, and $K \rightarrow \tau \bar{\tau}$ are of theoretical interest since here we are observing higher-order weak and electromagnetic interactions. Recent advances in unified gauge theories of weak and electromagnetic interactions allow in principle unambiguous and finite predictions for these processes. The above processes, which are "induced" $|\Delta S| = 1$ transitions, are a good testing ground for the cancellation mechanism first invented by Glashow, Iliopoulos, and Maiani (GIM) in order to banish $|\Delta S| = 1$ neutral currents. The experimental suppression of $K_L \rightarrow \mu \bar{\mu}$ and nonsuppression of $K_L \rightarrow \gamma \gamma$ must find a natural explanation in the GIM mechanism which makes use of extra quark(s). The procedure we follow is the following: We deduce the effective interaction Lagrangian for $\lambda \rightarrow \pi -1 + f$ and $\lambda \rightarrow \pi - \gamma + \gamma$ in the free-quark model; then the appropriate matrix elements of these operators between hadronic states are evaluated with the aid of the principles of conserved vector current and partially conserved axial-vector current. We focus our attention on the Weinberg-Salam model. In this model, $K \rightarrow \mu \bar{\mu}$ is suppressed due to a fortuitous cancellation. To explain the small $K_L - K_S$ mass difference and nonsuppression of $K_L \rightarrow \gamma \gamma$, it is found necessary to assume $m_P^2 / m_{\pi^+} < 1$, where $m_P$ is the mass of the proton quark and $m_{\pi^+}$ the mass of the charged quark, and $m_{\pi^+} < 5$ GeV. We present a phenomenological argument which indicates that the average mass of charged pseudoscalar states lies below 10 GeV. The effective interactions so constructed are then used to estimate the rates of other processes. Some of the results are the following: $K_S \rightarrow \gamma \gamma$ is suppressed; $K_L \rightarrow \pi \gamma$ proceeds at a normal rate, but $K_L \rightarrow \pi \gamma$ is suppressed; $K_L \rightarrow \pi \bar{\pi}$ is very much forbidden, and $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ occurs with the branching ratio of $\sim 10^{-10}$; $K^+ \rightarrow \pi^+ e^+ \bar{\nu}$ has the branching ratio of $\sim 10^{-7}$, which is comparable to the presently available experimental upper bound. The predictions of other models are briefly discussed. Relevant renormalization procedures and computational details are discussed in appendixes.

I. INTRODUCTION

Recent developments$^1$ in the study of spontaneously broken gauge invariance have led to the possibility of a unified and renormalizable theory of weak and electromagnetic interactions. Most such theories necessitate the introduction of weakly coupled neutral currents, and their viability depends on their success in accounting for the observed suppression of $|\Delta S| = 1$, $|\Delta Q| = 0$ semi-leptonic decays. In nearly all models proposed thus far, this problem is dealt with by appealing to the Glashow-Iliopoulos-Maiani (GIM) mechanism,$^2$ which we shall briefly recall.

A criterion for renormalizability is that couplings be invariant under a group of gauge transformations. The charged Cabibbo currents, $J^\mu_L$, together with the neutral current $J^\mu_N$ defined by

$$J^\mu_N(x) = \bar{\psi}(x-y) \gamma^\mu \gamma^5 \psi(y) \delta(x-y)$$

+ Schwinger terms

satisfy the algebra of SU(2). In a gauge-invariant theory these currents must couple with equal strength to gauge bosons. However, if $J^\mu_L$ are the usual Cabibbo current operators, the neutral current defined in Eq. (1.1) contains a strangeness-changing hadronic part as well as a leptonic part. The gauge-invariant coupling then implies that a decay such as $K_L \rightarrow \mu \bar{\mu}$ should occur with a strength comparable to $K^+ \rightarrow \mu \nu$.

In the Weinberg-Salam model,$^3$ the remedy to this unwanted prediction is to modify the usual current

$$J^\mu_L = \bar{\psi} \gamma^\mu (1 - \gamma_5) (\lambda \cos \theta + \lambda \sin \theta)$$

by adding a coupling

$$J^\* = J^\*_{\mu} + \bar{\psi} \gamma^\mu (1 - \gamma_5) (\lambda \cos \theta - \lambda \sin \theta),$$

where $\theta$ is the Cabibbo angle; $\delta$, $\lambda$, and $\lambda$ are the usual quarks and $\phi'$ is a charmed quark with the charge of the proton quark. With this modification the current defined in (1.1) has no $|\Delta S| = 1$ component. It is generally assumed that $\phi'$ is much heavier than the other quarks to account for the fact that charmed particles have not been observed.

In the Weinberg-Salam model the electromagnetic current is included by extending the gauge group to SU(2)$\times$U(1); other constructions which satisfy low-energy phenomenology have also been proposed. For example, in the Georgi-Glashow model$^4$ the Cabibbo currents are modified by the
introduction of new particles in such a way that
\( J_0 \) is just the electromagnetic current and no other neutral
current need be introduced. However, when
higher-order processes are considered, an analog
of the GIM mechanism must be used in all models
of this type—increasing still further the number
of quarks.

Rare decay modes of \( K \) mesons, such as \( K_L \to \mu \bar{\mu} \)
or \( K_L \to \gamma \gamma \), are of immense theoretical interest
because here we are dealing with the workings
of higher-order weak and electromagnetic
interactions; and a renormalizable theory of weak
interactions provides in principle unambiguous and
finite predictions for these decays. There is a
general problem associated with higher-order
transitions and it arises in the following way. The
coupling constant \( g \) for charged bosons is compar-
able in strength to the electromagnetic coupling
\( e \). In the low-energy limit, where the boson
propagator reduces to \( m_w^{-2} \), first-order transitions
are effectively governed by the Fermi constant
\[ G_F / \sqrt{2} \sim g^2 / m_w^2 \, . \]

For second-order processes, however, massive
virtual bosons can be exchanged and the effective
second-order coupling strength is found to be
\[ g^4 / m_w^2 \sim G_F \alpha \, . \]

Empirically, the strength of second-order
processes involving a change of strangeness is char-
acterized by \( G_F^2 \Lambda^2 = G_F \alpha (\Lambda^2 / m_w^2) \), where \( \Lambda \)
is typically of the order of several GeV, as for the
\( K_L-K_S \) mass difference and the decay \( K_L \to \mu \bar{\mu} \). A
mechanism is thus required to suppress the con-
tribution of order \( G_F \alpha \).

Consider, for example, the decays
\[ K_L \to \gamma \gamma \, , \quad (1.4) \]
\[ K_L \to \mu \bar{\mu} \, . \quad (1.5) \]

Diagrams contributing to the decay amplitudes are
shown in Fig. 1. Setting \( g = e \) as is the case in
unified theories of weak and electromagnetic
interactions, both amplitudes are fourth order in the
coupling \( e \). Since the Feynman integrals are con-
vergent, and since \( m_w \) is the dominant mass oc-
curring in the propagators, each graph will give
a factor \( m_w^{-2} \). Then by virtue of the above dis-

\[ \begin{align*}
\alpha & = \frac{G_F}{m_w} \alpha \\
\Gamma & = \frac{2}{m_w} \Gamma(K_L - \gamma \gamma) \\
& \approx \frac{4}{m_w} \Gamma(K_L - \mu \bar{\mu}) \\
\end{align*} \]

As expected, the amplitude for (1.4) is suppressed
by roughly a factor \( \alpha \) with respect to the first-
order process \( K_L \to \mu \bar{\mu} \). However, (1.5) is sup-
pressed in amplitude by a factor of
\[ \alpha^2 \approx 5M^2 \alpha \, . \]

In fact a major contribution to the decay rate for
(1.5) is from the higher-order electromagnetic
process shown in Fig. 2. The imaginary part
of the amplitude is dominated by the graph of Fig. 2
with the two photons on their mass shell; this
contribution may be calculated in terms of the on
shell \( K_L \to \gamma \gamma \) coupling and is found to be
\[ \Gamma(K_L - \mu \bar{\mu})_{\text{absorption}} = 1.2 \times 10^{-5} \Gamma(K_L - \gamma \gamma) \, . \]

The rate (1.8) is known as the “unitarity bound”
for \( K_L \to \mu \bar{\mu} \), as it provides a lower limit for the
partial width:
\[ \Gamma(K_L - \mu \bar{\mu}) \geq \Gamma(K_L - \mu \bar{\mu})_{\text{absorption}} \, . \]

Thus a mechanism is required which suppresses
the rate for \( K_L \to \mu \bar{\mu} \) to the experimentally observed
level, and which leaves the rate for \( K_L \to \gamma \gamma \) es-
tsentially unaffected. The role of the GIM me-
chanism is illustrated in Fig. 3. As the product
of coupling constants [Eq. (1.3)] which enter in the
virtual transition
\[ \mathcal{M} = gW + gW \]

is equal in magnitude and opposite in sign with
respect to the similar process involving the \( \phi \)
[Eq. (1.2)], the graph of Fig. 3(a) will exactly can-
cel the graph of Fig. 1(a) in the limit of equal \( \phi \)
and \( \phi' \) masses. However, in the same limit the
graph of Fig. 3(b) will also cancel the graph of
Fig. 1(b), which is clearly not a desired result.

The problem can be posed most acutely in terms
of symmetry properties. In the Weinberg-Salam
model (as modified by GIM), the quarks transform

\[ \text{FIG. 1. Important diagrams for (a) } K_L \to \mu \bar{\mu} \text{ and (b) } K_L \to \gamma \gamma. \]

\[ \text{FIG. 2. Diagram which contributes to the absorptive part of the } K_L \to \mu \bar{\mu} \text{ amplitude.} \]
according to the fundamental representation of $SU(4)$, with components $(\theta^\prime, \theta, \Xi, \lambda)$. In analogy with $I$ spin, which mixes $\theta$ and $\Xi$, or $U$ spin, which mixes $\Xi$ and $\lambda$, there is an $SU(2)$ subgroup of $SU(4)$ which mixes $\theta$ and $\theta^\prime$. Let us call this group of transformations $P$ spin. Then in the limit of $\theta$ and $\theta^\prime$ degeneracy $P$ spin is a symmetry of the strong interactions. The electromagnetic current is a $P$-spin invariant. The lowest-order hadronic operator with $\Delta Q = 0$ and $|\Delta S| \neq 0$ which can be constructed from the current of Eq. (1.3) is of the form

$$\sin \theta \cos \theta \, \lambda (\bar{\Phi} \Phi - \bar{\Phi}^\prime \Phi^\prime) + H.c.$$  \hspace{1cm} (1.9)

The operator (1.9) is a $P$-spin vector; that is, the effective $|\Delta S| \neq 1, \Delta Q = 0$ hadronic operator has $|\Delta P| = 1$. Since the photon and leptons as well as the hadronic $P$-spin scalars, the transitions (1.4) and (1.5) are forbidden in the limit of $P$-spin invariance.

We know that $P$ spin is badly broken in nature. Charmed particles—if they exist—must be much heavier than the observed hadronic states. The question to which we address ourselves here is: How may $K_L \rightarrow \mu \mu$ remain strongly suppressed in the broken symmetry case while the suppression of $K_L \rightarrow \gamma \gamma$ essentially disappears?

A hint to the solution may be seen by considering low-energy phenomenology. Consider the pole diagram of Fig. 4(a). Both the transitions $K_L \rightarrow \pi^0$ and $\pi^0 \rightarrow \gamma \gamma$ are allowed by $P$ spin. Similar non-vanishing contributions occur via $\eta$ and $\eta'$ exchange. As $SU(4)$ is now the basic symmetry, we must add a fourth pseudoscalar meson: $\chi \rightarrow \bar{\Phi} \Phi'$. While in the $SU(4)$ limit the contribution of the $\chi$ necessarily cancels the others, in the physical world it can be considered negligible:

$$(m_{\pi}^2 - m_{\eta'}^2)^{-1} \ll (m_{\chi}^2 - (m_{\eta}, \eta')^2)^{-1}.$$  \hspace{1cm} (1.10)

Then we obtain a contribution to $K_L \rightarrow \gamma \gamma$ which is of the correct order:

$$\mathcal{A}(K_L \rightarrow \gamma \gamma) \sim \mathcal{A}(K_L \rightarrow \pi^0) \mathcal{A}(\pi^0 \rightarrow \gamma \gamma) - G_F \alpha.$$  \hspace{1cm} (1.11)

With this approximation the Feynman integral will be correct to order

$$(m_{\pi}^2/m_{\eta'}^2)^2 \ll 1,$$

where $m_{\pi}$ is a hadronic mass. If a process is truly fourth-order semiwake in the sense that two heavy bosons are exchanged, the integral will contain at least one more boson propagator giving an additional factor $m_{\pi'}^{-2}$, so that the resultant amplitude will be of order

$$K_L^2 m_{\pi}^2 - G_F^2 m_{\eta'}^2.$$  \hspace{1cm} (1.12)

However, if the process is second-order weak and second-order electromagnetic, there will in general be a graph [cf. Figs. 1(b) and 3(b)] in which only one heavy boson is exchanged. Once the approximation (1.11) is made, the remaining integral is independent of the boson mass; then the amplitude must be proportional to

$$g^2/m_{\pi}^2 e^2 \sim -\alpha G_F.$$  \hspace{1cm} (1.13)
(In the 't Hooft–Feynman gauge, the contributions of unphysical Higgs scalars are negligible compared to those of the vector bosons.)

As will be seen more explicitly in the following sections, the requirements that the amplitudes for (1.4) and (1.5) be of the correct order are respectively

\[ \frac{m_{\phi^2}^2 - m_{\phi^0}^2}{m_{\phi^0}^2} \ll 1 \]

and

\[ \frac{m_{\phi^+}^2 - m_{\phi^0}^2}{m_{\phi^0}^2} \sim 1 \text{ or } m_{\phi^+}^2 \gg m_{\phi^0}^2. \]

In other words, P-spin symmetry breaking must be small on the boson mass scale but very large on a hadronic mass scale.

The main body of this paper will be devoted to quantitative calculations within the Weinberg-Salam model of the processes \( K \rightarrow \mu \bar{\mu}, K \rightarrow \pi \nu \bar{\nu}, K \rightarrow \gamma \gamma, K \rightarrow \pi \gamma \gamma, \) and \( K \rightarrow ne\bar{e}. \) For this purpose we first estimate the matrix elements for the elementary processes \( \lambda = \lambda + l + l \) and \( \lambda = \lambda + \gamma + \gamma, \) and construct phenomenological interactions for free quarks. The matrix elements of such interactions between physical hadronic states can then be estimated, since the hadronic operators which appear are the familiar V-A currents. The neglect of strong-interaction effects may perhaps be justified in models in which quarks are confined in a finite region of space by any of the mechanisms that have recently been suggested (e.g., infrared catastrophe due to non-Abelian gauge fields which prevents the disassociation of “color”- neutral states into “colored” states, or the “bag” mechanism), and within this confinement, quarks are “almost free.”

Furthermore, in those theories in which the gauge group of strong interactions commutes with the gauge group of weak and electromagnetic interactions, the \( \lambda \gamma \)-gluon coupling gets transformed away by the wave-function renormalization of quark fields. [Off-shell corrections are expected to be of order \( G_F m_N^2 (m_{\phi^+}^2 - m_{\phi^0}^2)/m_{\phi^0}^2 \) and are not important.] Thus, effective two-body operators \( \lambda + q \rightarrow \bar{q} + l l \) or \( \gamma \gamma, \) which could contribute to \( K \rightarrow \pi \pi \) or \( \gamma \gamma, \) cannot be induced by gluon exchange in such theories. The contribution of two-body operators which are present in the free-quark model (via \( W^\pm \) exchange) will be discussed in the appropriate sections. A related, but differently motivated, estimate of higher-order weak interactions has been discussed by Appelquist, Bjorken, and Chanowitz.

In Sec. II, we evaluate the amplitude for \( K_L \rightarrow \mu \bar{\mu} \) to lowest order in \( G_F. \) In the Weinberg-Salam model, this amplitude is found to vanish by what appears to be a fortuitous cancellation between two kinds of diagrams. The general matrix element for the elementary process \( \lambda + \bar{N} - l + l \) is proportional to the quark mass difference

\[ \frac{\Delta m^2}{m_{\phi^0}^2} = m_{\phi^+}^2 - m_{\phi^0}^2. \]

This quantity is estimated by comparing the matrix element of the \( K_L \rightarrow K_S \) transition in this approximation with the \( K_L \rightarrow K_S \) mass difference, as the former is proportional to \( (\Delta m^2/m_{\phi^0}^2). \) The matrix element so deduced for \( \lambda + \bar{N} - l + l \) is then applied to estimate the decay rates for \( K \rightarrow \pi \pi \) and \( K \rightarrow n l. \) Present experimental limits are at a level of \( 10^{-4} - 10^{-5} \) with respect to \( K \rightarrow \pi \nu \bar{\nu} \) decay rates.

In Sec. III we present the analogous calculation for \( K \rightarrow \gamma \gamma. \) A phenomenological discussion of this amplitude is also given in terms of pole contributions [Fig. 4(a)]. The role of P spin is explicitly displayed and it is shown that a reasonable estimate is obtained in the limit of very high \( \theta' \) mass. The matrix elements for the elementary processes \( \lambda + \bar{N} - \gamma + \gamma \) and \( \lambda + \bar{N} - \gamma \) are used to estimate the rates for \( K \rightarrow \pi \gamma \gamma \) and \( K \rightarrow ne\bar{e}. \) We comment on the amplitude for \( K_S \rightarrow \gamma \gamma, \) which has the interesting property that the leading contribution vanishes in the free-quark model; we conclude that this decay will be dominated by the absorptive amplitude. The amplitudes for \( K' \rightarrow \pi^+ e\bar{e} \) and \( K_S - \pi^0 e\bar{e} \) are found to be comparatively large, being truly of order \( G_F. \) The presently available limit on \( K' \rightarrow \pi^+ e\bar{e} \) is rather stringent; an improvement of the experimental precision by an order of magnitude will severely test our approach.

Section IV contains a summary of our results and comparison with experiment. We also discuss the predictions of other models on the rare decay modes of \( K \) mesons.

Finally, the renormalization procedure and details of computations are outlined in appendixes.

In this paper, we shall assume that the mass of the physical Higgs scalar particle is sufficiently large so that its contribution to induced \( \Delta S = 1 \) transitions is negligible.

Note: In this paper we ignore completely the effects of CP violation. In theories of superweak CP violation, none of the estimates of this paper are affected thereby; in a recent paper, Ma dis discuss some of the material contained in the present paper.

II. SEMILEPTONIC DECAYS

A. \( K - \mu \bar{\mu} \) and \( K_L - K_S \) mass difference

As a prejudice to considering the decays \( K - \mu \bar{\mu}, \) we shall first discuss the elementary process
λ + K̄ → μ + μ̄, in the free-quark model. There are two classes of diagrams contributing to this process. One is the diagram in which a pair of W⁺ and W⁻ is exchanged between the quark and lepton lines. The second class of diagrams is generated by the "induced" AλZ coupling (Fig. 5). The evaluation of these diagrams, to the lowest order in m_w^2, is outlined in Appendixes B and C.

When the GIM mechanism is incorporated into the Weinberg-Salam model, the W⁺W⁻ contribution to the process λ + K̄ → l + l is, in the 't Hooft-Feynman gauge,

\[ -i \frac{G_F}{\sqrt{2}} \frac{\alpha}{\pi} \cos \theta_W \sin \theta_W \varepsilon \overline{\gamma}_A \left[ \frac{1}{2} (1 - \gamma_5) \right] \times \lambda \left[ \frac{1}{2} \overline{\mu} \gamma_A \left[ \frac{1}{2} (1 - \gamma_5) \right] \mu \right] - \frac{1}{2} \overline{\nu} \gamma_A \left[ \frac{1}{2} (1 - \gamma_5) \right] \nu \right], \]  

\[ \epsilon = (\Delta m^2 / m_e^2) (\sin^2 \theta_W) \ln(m_h^2 / m_e^2) - 1 \]  

(2.1)

and similarly for the electronic leptons, where m_e^2 = (38 GeV)^2, θ_W is the Cabibbo angle, and Δm^2 = m_e^2 - m_d^2 is the difference of masses.

\[ iT(λ + K̄ → l + l) ≈ \frac{G_F}{\sqrt{2}} \frac{\alpha}{\pi} \epsilon \cos \theta_W \sin \theta_W \varepsilon \overline{\gamma}_A \left[ \frac{1}{2} (1 - \gamma_5) \right] \lambda \left[ \frac{1}{2} \overline{\mu} \gamma_A \mu \right] - \frac{1}{2} \overline{\nu} \gamma_A \left[ \frac{1}{2} (1 - \gamma_5) \right] \nu \right]. \]  

(2.3)

where

\[ \epsilon ≈ \frac{\Delta m^2}{(38 \text{ GeV})^2} \ln(m_e^2 / m_d^2) \]  

(2.4)

is the suppression factor arising from the GIM mechanism.

In order to estimate the decay rate K_L–→μ̄μ we take the matrix element of (2.3) between the K_L state and the vacuum, and use the partially conserved axial-vector current (PCAC) principle. In this way we obtain

\[ T(K_L → μ̄μ) = 0, \]  

(2.5)

\[ T(K_L → μ̄μ) = -\sqrt{2} \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi} \cos \theta_W \sin \theta_W [i\kappa(p_L)^\alpha] \times \sin^2 \theta_W \overline{\mu} \gamma_A μ \]  

\[ = 0, \]  

where \( f_L \sin \theta_W ≈ 33 \text{ MeV} \). The \( T(K_L → μ̄μ) \) amplitude vanishes because \( (p_L)^\alpha \overline{\mu} \gamma_A μ = 0 \). The vanishing of this amplitude is due to the fortuitous cancellation of the axial-vector part \( \overline{\mu} \gamma_A μ̄ μ \) between the \( W⁺W⁻ \) and \( Z \) contributions, (2.1) and (2.2). Even when the effects of strong interactions are taken into account, it is probable that these two contributions cancel to a large extent (especially if strong interactions are described by an asymptotically free field theory). We suggest that the dominant mechanism for \( K_L–→μμ \) are the conventional ones:

\[ K_L–→γγ → μμ, K_S–→ππ → γγ → μμ. \]  

Then we expect

\[ \Gamma(K_L → μμ) \approx \Gamma(K_S → γγ) \]  

As we shall see in Sec. III, we expect \( \Gamma(K_S → γγ) \approx 2\Gamma(K_L → γγ) \).

To estimate the size of \( \epsilon \) we consider the \( K_L–→K_S \) mass difference. In Appendix F, we evaluate the effective Lagrangian for \( λ + K̄ → X + X \) in our approximation and obtain

\[ L_{eff} = -\frac{G_F}{\sqrt{2}} \frac{\alpha}{4\pi} \epsilon_0 \cos^2 \theta_W \sin^2 \theta_W \overline{X} \gamma_A \left[ \frac{1}{2} (1 - \gamma_5) \right] X^2 \]  

+ h.c.,

(2.7)

where \( \epsilon_0 \) is defined in Appendix F, (F4)–(F5). In order to estimate the magnitude of the \( K_RK̄ \) transition amplitude, we insert the vacuum state between two currents in all possible ways and use PCAC (it is admittedly a dubious procedure, but it will not mislead us as to the order of magnitude):
(This estimate is for a triplet-quark scheme; for the 3-color quarks, the above equation should be multiplied by $\frac{3}{2}$.)

The $K_L-K_S$ mass difference is given by

$$m_L-m_S=m_G(1+m_G^{-3})(R^0)-L_{\text{eff}}(K^0)\right)^{1/2},$$

$$m_G(1-m_G^{-3})(R^0)-L_{\text{eff}}(K^0)\right)^{1/2}$$

$$=\frac{1}{m_G} \langle R^0 \rangle - L_{\text{eff}}(K^0).$$

Thus,

$$\frac{m_L-m_S}{m_G} = \frac{G_F}{\sqrt{2}} f_k \frac{\alpha}{4\pi} \epsilon_0 \alpha \sin^2 \theta_C \cos^2 \theta C$$

$$= \epsilon_0 \times 5 \times 10^{-12}.$$  \hspace{1cm} (2.8)

Experimentally the left-hand side is about 0.7 $\times 10^{-14}$, so we have

$$\epsilon_0 = 1.4 \times 10^{-3}.$$  \hspace{1cm} (2.9)

Equation (2.8) is compatible [see Eqs. (F4)-(F5)] either with $m_\phi = m_\rho$ and large, and $m_\phi' = m_\rho$ = 1 GeV, or $m_\phi \ll m_\rho$ and $m_\phi' = 1.5$ GeV. We argue in Sec. III, in connection with the nonsuppression of the $K_L-\gamma \gamma$ rate, that the latter is the logically tenable alternative. In this case

$$\epsilon \approx \epsilon_0 \ln(\epsilon_0 \sin^2 \theta_W) \approx 10^{-2}.$$  \hspace{1cm} (2.9)

What is the meaning of the suppression factor $\epsilon$? The expression (2.4), couched as it is in the language of the free-quark model, is hard to interpret in the context of a realistic model. Nevertheless, it indicates the degree to which the GIM cancellation mechanism must be effective, and suggests that charmed meson states cannot be too massive. We suggest that in a more realistic model (which we shall discuss elsewhere) the suppression factor will take the form

$$\epsilon \approx \frac{m_\phi^2}{m_\phi'^2 \sin^2 \theta_W} \ln \left( \frac{m_\phi^2}{m_\rho} \right).$$

in the limit of chiral SU(3) × SU(3) symmetry, where $m_\phi$ is the average mass of the charmed pseudoscalar mesons. If this is correct, we expect $m_\phi$ to be less than, say, a few GeV. The experimental implications of the existence of charmed mesons have already been discussed by GIM, Snow, and others.

Finally, with the suppression factor $\epsilon$ of the order (2.9), the weak contribution to $K_L-\mu \mu$ would be well within the bounds implied by the experimental data even if the cancellation of the axial-vector part were not complete.

B. $K-\pi \nu \bar{\nu}$

The effective interaction derived in (A7) for the elementary process $\lambda + \Xi^{-} \rightarrow l + \bar{\nu}$ allows us to estimate the rates for $K^+ \rightarrow \pi^+ \nu + \bar{\nu}$. Noting that

$$\langle \pi^+ | \bar{\Xi} \gamma_{\mu}(1-\gamma_5) \lambda | K^+ \rangle = \langle (p_+)_{\mu} f_+ + (p_-)_{\mu} f_- \rangle$$

$$= \sqrt{2} \langle \pi^+ | \bar{\Xi} \gamma_{\mu}(1-\gamma_5) \lambda | K^0 \rangle$$

$$= -\sqrt{2} \langle \pi^+ | \bar{\lambda} \gamma_{\mu}(1-\gamma_5) \lambda | K^0 \rangle,$$

where $f_+$ and $f_-$ are the $K_{15}$ form factors, and $p_+ = p_k \pm p_\pi$, we find that

$$i T(K^+ \rightarrow \pi^+ \nu \bar{\nu}) = i \frac{G_F}{\sqrt{2}} \frac{\alpha}{4\pi} \epsilon \cos \theta_C \sin \theta_C$$

$$\times \left[ (p_+)_{\mu} f_+ + (p_-)_{\mu} f_- \right]$$

$$\times \left[ 3 \nu \gamma_{\mu} \left[ \frac{1}{2} (1-\gamma_5) \nu \right] \right]$$

$$= i T(K^0 \rightarrow \pi^0 \nu \bar{\nu})^2,$$  \hspace{1cm} (2.10)

and

$$i T(K^+ \rightarrow \pi^0 \nu \bar{\nu}) = 0.$$  \hspace{1cm} (2.11)

Thus, if we neglect the electron mass

$$\frac{\Gamma(K^+ \rightarrow \pi^+ \nu \bar{\nu})}{2 \Gamma(K^+ \rightarrow \pi^0 \nu \bar{\nu})} = \frac{\Gamma(K^+ \rightarrow \pi^+ \nu \bar{\nu})}{\Gamma(K^+ \rightarrow \pi^0 \nu \bar{\nu})}$$

$$= 2 \left( \frac{3 \alpha}{4\pi} \epsilon \cos \theta_C \right)^2,$$  \hspace{1cm} (2.12)

where

$$\Gamma(K_L-\pi \nu \bar{\nu}) = \Gamma(K_L-\pi^- \nu \bar{\nu}) + \Gamma(K_L-\pi^0 \nu \bar{\nu})$$

and the factor 2 on the right-hand side of (2.12) comes from summing over two kinds of neutrinos. Together with the limit on $\epsilon$ given in (2.6), we obtain

$$2 \left( \frac{3 \alpha}{4\pi} \epsilon \cos \theta_C \right)^2 \approx 10^{-9}$$

and

$$\frac{\Gamma(K^+ \rightarrow \pi^+ \nu \bar{\nu})}{\Gamma(K^+ \rightarrow \text{all})} \approx 10^{-10},$$  \hspace{1cm} (2.13)

$$\frac{\Gamma(K^+ \rightarrow \pi^0 \nu \bar{\nu})}{\Gamma(K^+ \rightarrow \text{all})} \approx 10^{-12},$$  \hspace{1cm} (2.14)

$$\frac{\Gamma(K^+ \rightarrow \pi^0 \nu \bar{\nu})}{\Gamma(K^+ \rightarrow \text{all})} \approx 10^{-4}.$$  \hspace{1cm} (2.15)

The last follows from the fact that, as implied in (2.11), the amplitude for $K_L \rightarrow \pi^0 \nu \bar{\nu}$ is at most of order $G_F \alpha \epsilon^2$.

The results of (2.10) and (2.11) based on the simple quark model, obtained by neglecting all but the one-body operator deduced by looking at the process $\lambda + \Xi^{-} \rightarrow l + \bar{\nu}$, may appear more suspect than those of Sec. II A. However, in this theory, there is a soft-pion theorem relating the $K \rightarrow \pi I$ and $K^0 \rightarrow \pi I$ amplitudes, and our results are consistent with it.

In the case $m_\phi, m_\mu \ll m_\rho'$, the chiral SU(2)
\begin{align*}
\times SU(2) &\text{ is a good symmetry, and the entire Lagrangian commutes with the generators } q_i^1 + q_i^2 \text{ of the right-handed chiral SU(2) group approximately (noncommuting pieces being of order } m_e/m_p), \text{ except for the couplings of the } Z \text{ meson and the photon to quarks. Thus, except for the } Z \text{-meson reducible diagrams, we have the soft-pion theorem}
\lim_{q \to 0} T^{\mu \nu}(K_{\xi, \lambda}(p) - \pi^0(q) \nu(r) \overline{\nu}(p - q - r)) \\
&= i \frac{1}{\sqrt{2} f_{\pi}} T^{\mu \nu}(K_{\xi, \lambda}(p) - \nu(r) \overline{\nu}(p - r)) , \quad (2.16)
\lim_{q \to 0} T^{\mu \nu}(K^+(p) - \pi^+(q) \nu(r) \overline{\nu}(p - q - r)) \\
&= i \frac{1}{f_{\pi}} T^{\mu \nu}(K^0(p) - \nu(r) \overline{\nu}(p - r)) , \quad (2.17)
\end{align*}

where } \nu \text{ and } \overline{\nu} \text{ are not in general on their mass shells and } T^{\mu \nu} \text{ denotes the single-particle irreducible amplitude. Equation (2.16) follows from the standard current-algebra manipulation.}

Let us now consider the } Z \text{-meson reducible diagrams. The effective } aKZ \text{ vertex of the form}
\begin{align*}
\left\langle \pi(q) \left| T \left\{ j_\mu^+(0) \exp \left[ i \int d^4 y L_{\nu}(y) \right] \right\} \right| K(p) \right\rangle
\end{align*}
where } L_{\nu} \text{ is the weak-interaction Lagrangian (} L_{\nu} \text{ includes the couplings of gauge bosons and Higgs scalars to fermions, and the interactions of weak boson fields; since we are interested in the matrix element to lowest nonvanishing order, we may ignore the couplings of the } Z \text{ meson and the photon to fermions, as well as any complications that might arise from the presence of the Feynman–DeWitt–Popov–Faddeev ghost fields). The part of } j_\mu^+ \text{ which does not commute with the right-handed chiral charge } q_i^1 + q_i^2 \text{ is } -\sin^2 \theta_w j_\mu^+ j_3^+ , \text{ where } j_3^+ \text{ is the electromagnetic current. Because } j_3^+ \text{ is conserved, the effective } K \pi \nu \text{ vertex must have the form}
\begin{align*}
\left\langle \pi(q) \left| T \left\{ j_\mu^+(0) \exp \left[ i \int d^4 y L_{\nu}(y) \right] \right\} \right| K(p) \right\rangle = (k^2 g_{\mu \nu} - k_\mu k_\nu) (p + q)^\mu G(t) , \quad (2.18)
\end{align*}

where
\begin{align*}
k = p - q , \quad t = k^2 .
\end{align*}
The proof follows from the usual Ward identity, and is essentially identical to the one we present in Appendix D for the effective } aKZ \text{ vertex. On dimensional grounds, we have}
\begin{align*}
G(0) = O(G_\rho e) .
\end{align*}
On the other hand, the matrix element of the purely left-handed current } j_\mu^+ + (- \sin^2 \theta_w j_3^+) \text{, which is not conserved, is of the form}
\begin{align*}
\left\langle \pi(q) \left| T \left\{ j_\mu^+(0) + \sin^2 \theta_w j_3^+(0) \exp \left[ i \int d^4 y L_{\nu}(y) \right] \right\} \right| K(p) \right\rangle
&= (p + q)^\mu F_3(t) + (p - q)^\mu F_3(t) , \quad (2.20)
\end{align*}
Thus, Eqs. (2.16) and (2.17) should hold also for the full amplitudes, to lowest order in } (l/m^2) \text{ and } -m^2/m^2 \text{, i.e., neglecting contributions of order } (m^2/m^2) \text{ compared to } (l^2/m^2) \text{, where } m \text{ is the typical uncharmed hadron mass.}
The chiral charge } q_i^1 + q_i^2 \text{ does commute with the electromagnetic current, so that the soft-pion theorem (2.16) holds actually for the full amplitude. Let us parametrize the } K_{\xi, \lambda} - \pi^0 \nu\overline{\nu} \text{ amplitude by}
\begin{align*}
T(K(p) - \pi^0(q) \nu(r) \overline{\nu}(p - q - r))
&= [A(p + q)^\mu + B(p - q)^\mu + C \nu^\mu + i D \nu^{\mu \nu \rho \sigma} \rho_\nu q_\rho r_\sigma] \\
\times \nu \gamma_\mu (1 - \gamma_5) \nu , \quad (2.22)
\end{align*}
where } A, B, C, \text{ and } D \text{ are, in general, invariant functions of the momenta } p, q, \text{ and } r. \text{ In the soft-pion limit we have}

\begin{align*}
A_L + B_L = 0 ,
A_S + B_S = \left( \frac{f_{\pi}}{f} \right)^2 \frac{3}{4} \epsilon \cos \theta_C \sin \theta_C ,
\mathcal{C}_{L,S} = 0 ,
\end{align*}
where we have used the amplitudes for } K_{\xi, \lambda} - \nu\overline{\nu} \text{ (off shell) deduced from (2.3). The amplitude (2.10) is consistent with (2.22), (2.23) when the Callan–Treiman relation is taken into account. Note further that if we neglect the momentum dependence of the form factors, } CP \text{ invariance alone implies}
\begin{align*}
A_L = B_L = C_L = 0 ,
D_S = 0 ,
\end{align*}
These constraints follow from } CP \text{ invariance and the assumptions that the neutrino is left-handed and massless. In most models the form factor } D \text{ is identically zero and } K_{\xi, \lambda} - \pi^0 \nu \text{ is strongly suppressed.
III. ELECTROMAGNETIC DECAYS
A. \( K \to \gamma \gamma \): Quark model

We shall first discuss the free-quark-model calculations in analogy to our discussion of the leptonic decays. However, as the estimates we obtain appear a posteriori less reliable, we shall show in the case of \( K_L \to \gamma \gamma \) that a similar estimate can be obtained from low energy phenomenology. In both cases the numbers we use should be taken as order of magnitude estimates.

The first step in the calculation is to obtain an effective Lagrangian for the process \( \lambda + \pi \to \gamma + \gamma \). There are two classes of diagrams here. The first consists of those diagrams which are one-quark reducible (see Fig. 6). The \( \lambda \pi \gamma \) vertex is discussed in Appendix D (see also Appendix B). For the real-photon emission the transition charge form factor vanishes; the nonvanishing effects of these diagrams come from the \( \lambda - \pi \) transition magnetic moment as well as the off-shell correction due to the internal quark lines. These effects are, however, of order \( \epsilon \sim (\Delta m^2/m_\pi^2) \ln(m_\pi^2/m_\rho^2) \) compared to the main term we shall consider, and shall be ignored.

The second class of diagrams consists of one-particle irreducible ones. As is shown explicitly in Appendix E, the leading contribution comes from the Feynman diagrams in which only one heavy boson is exchanged (Fig. 7). From the point of view of spin dependence, the graphs of Fig. 7(a) are identical to those of Fig. 7(b), which are obtained from the former by means of a Fierz-Michel transformation,

\[
[\Re \gamma_\mu(1-\gamma_5)\lambda] [\Im \gamma^\mu(1-\gamma_5)\lambda] = [\Re \gamma_\mu(1-\gamma_5)\lambda] \times [\Im \gamma^\mu(1-\gamma_5)\lambda],
\]

where the anticommutativity of the quark fields is taken into account. Note that this transformation is possible only because the heavy boson propagator is \(-g_{\mu\nu}(p^2-m_F^2)^{-1}\) in the 't Hooft-Feynman gauge (see Appendix A). Then the effective interaction takes the form of a current operator

\[
J^\rho_\mu = \Re \gamma_\mu(1-\gamma_5)\lambda
\]

multiplying a closed-loop integral, where the integrand includes the \( W \) propagator. Apart from the \( W \) propagator, the integral involved is identical to that studied by Rosenberg and Adler. Recall that the latter is linearly divergent, and therefore its value depends on the choice of integration variables and on the way in which the integration is performed; this ambiguity is removed by requiring that the result be gauge-invariant with respect to both photons.

In our case the boson propagator makes the integral convergent and its value is unambiguous. However, if only, say, the \( \sigma' \)-quark loop is considered the result is not gauge-invariant. Other graphs must contribute gauge noninvariant terms of order \( m_\rho^{-2} \) which render the total amplitude gauge-invariant to that order. However, the gauge-noninvariant piece of the graphs in Fig. 7(a) is independent of the quark mass, and cancels out when the \( \sigma' \)- and \( \sigma'' \)-quark diagrams are summed. Thus for our purposes it is sufficient to consider only the one-\( W \)-exchange diagrams, as anticipated in the introduction. See Appendix E for estimates of order of magnitude of various diagrams; the integral is evaluated in Appendix A. To order \( m_\sigma^{-2} \), the effective operator for the transition is given by

\[
\left( J^\rho_\mu + J^\rho_\mu \right) T^\rho_{\rho_0}
\]

\[\text{FIG. 7. Leading contributions to } \lambda + \pi \to \gamma + \gamma. \text{ To leading order in } M_\pi^{-2}, \text{ the diagrams in (a) reduce to those of (b).}\]
where the amplitude $T_{\rho\circ}^\mu$ may be read off (A14).

To obtain the amplitude for $K_L \rightarrow \gamma \gamma$ we must take the matrix element of the current operator $\mathcal{O}$ between $K_L$ and the vacuum:

$$\langle 0 | J_\mu^{\circ} | K_L \rangle = (0 | J_\mu^{\circ} | K_L) = i \sqrt{2} f_K (p_K)_\mu .$$

(3.3)

Contracting $(p_K)_\mu = (k_1 + k_2)_\mu$ with $T_{\rho\circ}^\mu$, we obtain

$$
\mathcal{I} \mathcal{T}(K_L \rightarrow \gamma \gamma) = i F(\beta) \left( \frac{G_F}{\sqrt{2}} \right) \frac{2 \alpha}{\pi} \cos \theta_C \sin \theta_C \times \left( i \sqrt{2} f_K \right) A_\gamma ,
$$

(3.4)

where

$$A_\gamma = \mathcal{Q}^{2} \left[ F \left( \frac{m_\phi^2}{m_\pi^2} \right) - F \left( \frac{m_\phi^2}{m_\rho^2} \right) \right] ,$$

(3.5)

$Q$ being the charge of the $\phi$ quark. The function $F(\beta)$ is defined in (A12) and plotted in Fig. 8. For $\beta \rightarrow 1$, $F(\beta)$ converges rapidly to zero. If we assume that both quark masses $m_\phi$ and $m_\rho$ are very heavy compared to the kaon mass, we obtain a strong suppression of $K_L \rightarrow \gamma \gamma$ in contradiction with experiment. However, in the range $0 \leq \beta \leq 1$, $F(\beta)$ is a rapidly varying function with values in the range

$$0 \leq \frac{1}{2} \leq F(\beta) \leq \frac{1}{2} , \quad \text{for} \ 0 \leq \beta \leq 1 .$$

If $A_\gamma$ is treated as a phenomenological parameter as defined by Eq. (3.4), then using the experimental branching ratio

$$\frac{\Gamma(K_L \rightarrow \gamma \gamma)}{\Gamma(K_L \rightarrow \text{all})} = 4.9 \pm 0.4 \times 10^{-4} ,$$

(3.6)

we obtain the empirical determination

$$|A_\gamma| = 0.87 \pm 0.04 .$$

(3.7)

Assuming $m_\phi \gg m_\pi \gg m_\rho$, the free-quark model seems to be compatible with the non-suppression of the process $K_L \rightarrow \gamma \gamma$. (Another solution, which in fact fits the data better for $Q^2 = \frac{3}{2}$, is $m_\phi \approx m_\rho / 2$, $m_\phi \ll m_\rho / 4$. Such a low mass for $m_\phi$ is not really ruled out by the $K_L$-$K^0$ mass difference because of the approximation made in evaluating (2.7). However, this interpretation perhaps takes too literally the low mass region of Fig. 8.) Note incidentally that the expression (3.5) is valid in the color-quark scheme as well as in the one-triplet model.

To lowest order in $m_\rho^{-2}$, there is no term which leads to the $C P$ even state of $2\gamma$. This is most easily seen in Fig. 7 by noting that the $V V V$ vertex in question vanishes by Furry's theorem. However, phenomenologically, $K_L \rightarrow \gamma \gamma$ can proceed through the $2\pi$ intermediate state; we shall interpret the quark-model result to mean that the real

![FIG. 8. Re F(\beta) vs \beta. F(\beta) is defined in (A12). The figure is the result of a numerical integration of (A12). Courtesy of Professor Chris Quigg.](image)

data fragment of the $K_S \rightarrow \gamma \gamma$ amplitude is of order $\epsilon$ compared to the $K_L \rightarrow \gamma \gamma$ amplitude and negligible.

[Note further that $K_S \rightarrow \gamma \gamma$ is forbidden by $U$ spin in the exact SU(3) limit.] The $2\pi$ contribution to the absorptive part of the $K_S \rightarrow \gamma \gamma$ amplitude has been estimated. It is$^{20}$

$$\Gamma(K_S \rightarrow \gamma \gamma) \approx 2 \times 10^4 \ \text{sec} .$$

Thus we have

$$\frac{\Gamma(K_S \rightarrow \gamma \gamma)}{\Gamma(K_L \rightarrow \gamma \gamma)} \approx 2 .$$

(3.8)

B. $K_L \rightarrow \gamma \gamma$: Phenomenological analysis

In this section we shall estimate the amplitude for $K_L \rightarrow \gamma \gamma$ using the pole diagrams of Fig. 4(a). We define the following set of meson states in terms of their quark content:

$$\pi^0 = (\Phi^0 - \bar{\Phi}^0)/\sqrt{2} ,$$

$$\eta_8 = (\Phi^0 + \bar{\Phi}^0 - 2\lambda^0)/\sqrt{6} ,$$

$$\eta_8 = (\Phi^0 + \bar{\Phi}^0 + \lambda^0)/\sqrt{3} ,$$

$$\eta_c = \bar{\Phi}' \Phi' .$$

The isoscalar states can, of course, mix; we label the physical states by $\eta$, $X^0$, and $X_\circ$.

The effective nonleptonic weak interaction transforms as the first component of a $U$-spin vector and the third component of a $P$-spin vector:

$$\mathcal{L}_{\text{weak}} \sim (\bar{\Phi}^0 + \bar{\Phi}^0) (\Phi^0 - \Phi' \Phi' ) .$$

Neglecting symmetry-breaking corrections, the phenomenological coupling of $K_L$ to the neutral-
The meson system must be of the form
\[ \mathcal{L}_{\text{weak}} = cK \left( \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} + \frac{\eta_9}{\sqrt{3}} - \eta_c \right) \equiv cK \sum_i c_i P_i. \]  
(3.9)

The electromagnetic coupling is a scalar in both \( U \) spin and \( P \) spin:
\[ \mathcal{L}_{\text{em}} = 2F_{\mu\nu} \tilde{F}_{\mu\nu} \left[ A(\bar{\Phi}' \Phi') + B(\bar{\Phi} \Phi + \bar{\Phi}') \right]. \]

If we assume that the amplitude for \( P_1 - \gamma\gamma \) is determined by a single quark loop, we may interpret \( A \) and \( B \) as the squared charges of the quarks:
\[ \mathcal{L}_{\text{em}} = d F_{\mu\nu} F_{\mu\nu} \left[ A \left( \frac{\eta_8}{\sqrt{6}} + \frac{\eta_9}{\sqrt{3}} - \frac{\eta_c}{\sqrt{2}} \right) \right] \]
\[ \equiv d F_{\mu\nu} F_{\mu\nu} \sum_i d_i P_i. \]
(3.10)

It is obvious from (3.9) and (3.10) that pole contribution to \( K_\gamma - \gamma\gamma \) is forbidden in the limit of mass degeneracy, since \( \sum c_i d_i = 0 \). When mass splitting is present the amplitude takes the form
\[ \mathcal{A}(K_\gamma - \gamma\gamma) = c_i (m^2 - M^2)^{-1} \delta_i, \]
(3.11)

where \( M \) is the 3 \( \times \) 3 matrix which describes the isoscalar meson masses and their mixing:
\[ \mathcal{L}_{\text{mix}} = P_i M^2 \delta_i P_i. \]
(3.12)

In the Weinberg model SU(4) is broken by the quark mass splitting; to lowest order in the symmetry breaking the meson masses may be described by an effective Lagrangian of the form
\[ \mathcal{L}_{\text{mix}} = m_0 (\text{Tr} \pi^2) + m_1 \text{Tr} \tau^2 + \alpha \text{Tr} \pi \Delta M \pi + \beta \text{Tr} \pi \Delta M \pi, \]
(3.13)

where \( \pi \) is the 4 \( \times \) 4 representation of the pseudoscalar states (15-pi plus singlet) and \( \Delta M \) is a traceless diagonal matrix with two independent elements: \( m_\chi - m_\rho, m_\rho - m_\phi \). The matrix elements \( M^2 \delta_i \) in (3.12) can be extracted from (3.13).

Explicit evaluation of the amplitude (3.11) in the general case is quite involved. If we restrict ourselves to the case of fractionally charged quarks, we obtain
\[ \mathcal{A}(K_\gamma - \gamma\gamma) = h \left( \frac{N}{D} + \frac{1}{2\Delta m^2} \right), \]
(3.14)

where \( D = \text{Det}(m_{K}^2 - M^2) \)
\[ = (m_{K}^2 - m_{\eta}^2)(m_{K}^2 - m_{\chi^0}^2)(m_{K}^2 - m_{\pi}^2), \]

and
\[ N = -\frac{D}{2\Delta m^2} + 4(1 + \Delta m^2)\Delta m_{\rho}^2(2 + 8\gamma + 3\gamma^2), \]

where \( \gamma = \beta/\alpha \) and \( \Delta m_{\rho}^2/\Delta m^2 = (m_\rho - m_\phi)/(m_\chi - m_\rho) \).

The pion pole is exactly canceled by the residue at the \( \eta \) pole and we obtain
\[ \mathcal{A}(K_\gamma - \gamma\gamma) = \frac{m_{K}^2 - m_{\rho}^2}{3(m_{K}^2 - m_{\rho}^2)} \left( \frac{hK}{m_{K}^2 - m_{\chi^0}^2} \right), \]
(3.15)

with
\[ K = 4(2 + 8\gamma + 3\gamma^2)\Delta m_{\rho}^2/(m_{K}^2 - m_{\chi^0}^2). \]

While the result is finite and nonvanishing in the SU(3) limit, it vanishes in the \( P \)-spin limit since \( \Delta m_{\rho}^2 \to 0 \) does not require \( m_{K}^2 \to m_{\chi^0}^2 \):
\[ D - \Delta m^2(2\Delta m^2 + b\Delta m_{\rho}^2). \]

For the physical case of badly broken \( P \) spin we assume
\[ m_{\chi^0}^2 - m_{K}^2 \gg m_{\chi^0}^2 - m_{\pi}^2. \]

In this limit the \( X_{\gamma\rho} \) or \( \eta \) mixing is proportional to \( \gamma \); assuming small mixing we find
\[ m_{\chi^0}^2 \approx 2(1 + \gamma)\Delta m_{\rho}^2, \]
(3.16)

\[ K = -2(2 + 8\gamma + 3\gamma^2)/(1 + \gamma). \]

The amplitude (3.15) is very sensitive to the value of \( \gamma \) which also determines the \( X^0, \eta \) mixing parameter. The value \( \gamma \approx -0.6 \) corresponds to small mixing as experimentally determined.

For comparison of the amplitude (3.15) with experiment we must evaluate the effective coupling constant \( h \). If \( A_\pi \) is defined as the pion-pole contribution to (3.14),
\[ A_\pi = \mathcal{A}(K_\gamma - \pi^0 \pi^0)/m_{\pi}^2, \]
we have
\[ h = 2\Delta m^2 A_\pi. \]

and
\[ \mathcal{A}(K_\gamma - \gamma\gamma) = \frac{A_\pi K}{m_{K}^2 - m_{\chi^0}^2} \left( \frac{A_\pi K}{m_{K}^2 - m_{\pi}^2} \right), \]
(3.17)

The transition amplitude for \( K_\gamma - \pi^0 \) can be estimated using current algebra\(^1\):
\[ \mathcal{A}(K_\gamma - \pi^0) = -i\sqrt{2} f_\pi \mathcal{A}(K_\gamma - \pi^0 \pi^0). \]
(3.18)

Then using the decay rates for \( \pi - \gamma\gamma \) and \( K_\gamma - \pi^0 \pi^0 \)
we find
\[ |A_+/A_{\text{obs}}| = 2.3 \pm 0.1, \]  
(3.19)
where \( A_{\text{obs}} \) is the experimentally determined amplitude for \( K \to \pi \gamma \).

From Eq. (3.16) it is clear that the parameter \( K \) is very sensitive to the pseudoscalar mixing. A value of \( |K| \approx 0.5 \) would reproduce the experimental amplitude. This is achieved with \( \gamma \approx -0.3 \), which, however, gives too strong a mixing for the \( X^0, \eta \) system. We may also argue that a perturbative treatment of SU(4) breaking is meaningless, and simply neglect the contribution of \( \eta_0 \approx X^0 \). Using the experimentally determined masses and mixing angle for \( \eta \) and \( X^0 \) we obtain \( K \approx -1.5 \). If we also neglect the \( \eta, X^0 \) mixing, the amplitude (3.15) reduces to the \( X^0 \) pole with \( K = \frac{1}{2} \).

Considering the approximations and uncertainties
\[
T(K_L - \pi^0 \gamma \gamma) = 0, \\
iT(K^+ - \pi^+ \gamma \gamma) = iT(K_{\Sigma^0} - \pi^0 \gamma \gamma)
\]
\[
= -i \frac{G_F}{\sqrt{2}} \frac{2\alpha}{\pi} \cos \theta_C \sin \theta_C Q^2 \left[ F\left( \frac{m^2}{t} \right) - F\left( \frac{m^2}{t} \right) \right] \epsilon^+ \epsilon^+ \Lambda_{\Sigma^0 \pi^0} [f_{\Sigma^0}(t) + f_{\Sigma^0}(t)],
\]
(3.21)
where \( \Lambda_{\Sigma^0 \pi^0} \) is defined in Appendix A. The decay rate is proportional to the square of the divergence form factor
\[
f(t) = f_{\Sigma^0}(t) + \frac{t}{m^2 - m^2_{\Sigma^0}} f_{\Sigma^0}(t).
\]

In these decays the momentum transfer squared \( t \) varies from 0 to \( (m - m_{\Sigma^0})^2 = 0.55 m^2 \). Assuming that in this range \( f(t) \approx f(0) \), and \( F(m^2/t) = F = \text{constant} \), we have
\[
\Gamma(K_{\Sigma^0} - \pi^0 \gamma \gamma) = \frac{1}{2m^2} \left[ \frac{G_F}{\sqrt{2}} \frac{4\alpha}{\pi} Q^2 \sin \theta_C F(0) \right]^2 \\
\times \int_0^{0.55} dx (1 - x).
\]
(3.22)
To eliminate \( F \), we take the ratio \( \Gamma(K_{\Sigma^0} - \pi^0 \gamma \gamma)/\Gamma(K_L - \gamma \gamma) \):
\[
\frac{\Gamma(K_{\Sigma^0} - \pi^0 \gamma \gamma)}{\Gamma(K_L - \gamma \gamma)} \approx \frac{m^2_{\Sigma^0} f_{\Sigma^0}(0)}{f_{\Sigma^0}} \frac{1}{32\pi^2} \int_0^{0.55} dx (1 - x)
\]
\[
\approx 10^{-2}.
\]
(3.23)
The above estimate does not take into account the fact that \( F(t) \) vanishes as \( t \to 0 \). Depending on \( t \), the ratio of (3.23) may be much smaller, say \( \approx 10^{-3} \).

The effective interaction of Eq. (A14) allows us to estimate the rates for \( K \to \eta \gamma \). Using the matrix elements of the current operator (3.1),
\[
\sqrt{2} \langle \eta^0(q) | J^{\pi^0}_{\eta^0} | K^0(0) \rangle = -\sqrt{2} \langle \eta^0(q) | J^{\pi^0}_{\eta^0} | K^0(0) \rangle
\]
\[
= \langle \pi^0(q) | J^{\pi^0}_{\pi^0} | K^0(0) \rangle
\]
\[
= ([p + q])_{\pi^0} f_{\pi^0}(l) + ([p - q])_{\pi^0} f_{\pi^0}(l),
\]
\[
t = (p - q)^2
\]
we obtain
\[
\Gamma(K \to \eta \gamma) = \frac{1}{2m^2} \left( \frac{G_F}{\sqrt{2}} \frac{4\alpha}{\pi} Q^2 \sin \theta_C F(0) \right)^2 \\
\times \int_0^{0.55} dx (1 - x).
\]
(3.24)
Note that the estimates of (3.24) for \( K_L - \pi^0 \gamma \gamma \) and \( K_{\Sigma^0} - \pi^0 \gamma \gamma \) are consistent with the soft-pion theorems one can derive which connect the real parts of these amplitudes to those of \( K_{\Sigma^0} - \gamma \gamma \) and \( K_L - \gamma \gamma \). The derivation of these soft-pion theorems is analogous to that given in Sec. II B, and is based on the fact that the relevant part of the weak-interaction Lagrangian \( \mathcal{L}_w \) and the electromagnetic current \( j_{\mu}^I \) commute with the right-handed chiral charge \( Q^3 + Q^4 \) in the limit of the chiral SU(2) symmetry.

There is no analogous relation for \( K^+ - \pi^+ \gamma \gamma \) and \( K - \gamma \gamma \) because \( j_{\mu}^I \) does not commute with \( Q^3 + Q^4 \). Nevertheless, we see no reason why the amplitude for \( K^+ - \pi^+ \gamma \gamma \) should be suppressed, and we expect its order of magnitude to be given correctly by (3.24). In reality, \( K_{\Sigma^0} - \pi^0 \gamma \gamma \) can proceed through the \( 3 \pi \) intermediate states by the sequence \( K_L \to \pi^0(\pi^-) \to \pi^0(\gamma \gamma) \). Again, as for \( K_{\Sigma^0} - \gamma \gamma \), we interpret the quark-model result to mean that the dispersive part of the \( K_{\Sigma^0} - \pi^0 \gamma \gamma \) amplitude is of order \( \epsilon \) compared to the \( K_{\Sigma^0} - \pi^0 \gamma \gamma \) amplitude. We have estimated the unitarity correction to \( K_{\Sigma^0} - \pi^0 \gamma \gamma \) and found it negligible.\(^{21}\)

### D. \( K - \pi e\overline{e} \)

The effective \( \lambda \overline{\pi} \gamma \) vertex evaluated in Appendix D allows us to evaluate the rates for \( K - \pi e\overline{e} \). By far the largest contribution to the amplitude comes
from the transition charge-radius term in the effective \( \lambda \Phi \gamma \) vertex (see Appendix D):

\[
\frac{\Gamma}{\phi} (\rho \rightarrow \frac{1}{2} k, \rho \rightarrow \frac{1}{2} k) \text{renormalized } \lambda = \frac{1}{2} (1 - \gamma_5) \lambda (k^2 \Phi \rho - k \cdot k) \frac{1}{\Phi} (r^2)_{\Phi \lambda} + \cdots ,
\]

where \( m \) is the typical uncharmed hadron mass scale. The computation of Eq. (3.25) is outlined in Appendix D. Briefly, it comes about in the following way. To lowest order in \( m^2 \), the charge radius comes from the diagrams shown in Fig. 9(a). After the Fierz-Michel rearrangement of Sec. IIIA, one sees that the sum of the \( \phi^- \) and \( \phi^- \) - quark diagrams are convergent and gauge-invariant, even after the replacement

\[
i g_{\mu \nu} \frac{g^2}{m^2 - m^2} - i g_{\mu \nu} \frac{k^2}{m^2} = i g_{\mu \nu} g \frac{g_F}{m^2} \]  

is made. In this limit the relevant diagrams reduce to those of Fig. 9(b), precisely those of the current-current theory modified by the GIM mechanism. Since the mass of the unknown quark appears only through a logarithm, Eq. (3.25) may be expected to be rather stable against uncertainties due to strong interactions. The amplitudes for \( K \rightarrow \pi e \bar{e} \) are

\[
i T(K^+ (p) \rightarrow \pi^+(q) e \bar{e}) = i T(K_+ (p) \rightarrow \pi^0 (q) e \bar{e})
\]

\[
= \frac{g^2}{2 \cos \theta C} \sin \theta C \frac{2 \alpha}{3 \pi} m \Phi \rho \text{ f.s.} \text{ ln} \frac{m \Phi \rho^2}{m^2} ,
\]

\[
T(K_+ \rightarrow \pi^0 e \bar{e}) = 0 ,
\]

which give

\[
\Gamma(K' \rightarrow \pi^0 e \bar{e}) = \frac{2}{3 \pi} \frac{\alpha}{2 \cos \theta C} \sin \theta C \text{ f.s.} \text{ ln} \frac{m \Phi \rho^2}{m^2} .
\]

[The difference of the factor 2 between (2.12) and (3.28) is due to the difference in the lepton couplings: \( \bar{\nu} \gamma^\mu (1 - \gamma_5) \nu \) for the former, \( \bar{\nu} \gamma^\mu e \) for the latter.] Taking \( \delta = \frac{1}{2} \text{ ln}(m \Phi \rho^2/m^2) \approx 5 \) (2), we find that

\[
\Gamma(K' \rightarrow \pi^0 e \bar{e}) \approx 3 \times 10^{-6} \text{ (0.5 \times 10^{-6})} ,
\]

\[
\Gamma(K_+ \rightarrow \pi^0 e \bar{e}) \approx 10^{-8} \text{ (0.2 \times 10^{-6})} .
\]

The experimental upper bound is at the verge of contradicting the prediction (3.29).

As for the process \( K_+ \rightarrow \pi^0 e \bar{e} \), one-photon- and Z-exchange diagrams are absent by CP invariance; the \( W^- W^+ \) contribution is expected to be strongly suppressed as for the process \( K_+ \rightarrow \pi^0 \nu \bar{\nu} \).

IV. CONCLUDING REMARKS

We summarize in Table I (see Refs. 5, 6, 22–24) the main results of the present study and compare them to experiment. Owing to the extreme experimental difficulties in carrying out the necessary precision, the predictions of our treatment of the Weinberg-Salam model are well within the presently available experimental upper bounds in most cases. A notable exception is the rate of \( K^- \rightarrow \pi^- e \bar{e} \), where the experimental upper bound appears tantalizingly close to the limit of a reasonable range of theoretical uncertainties. We feel that the estimate (3.29) is of good standing; deviation of the experimental rate by a decade, say, would worry us. Measurement of this rate, not just setting of an upper bound, is clearly called for.

In most gauge models proposed so far, we believe that most of the results of this paper are also true.

(1) In the eight-quark version of the Georgi-Glashow model, most of the results on semileptonic decays also hold, as shown by Lee, Primack, and Treiman, and Lee and Treiman.25 For electromagnetic decays, the situation is murky. This is due to two factors. One is that in this model there might be an important contribution of the physical Higgs scalar to strangeness-changing transitions which we have not considered in this paper. The second is that in this model the \( \lambda \Phi \)
transition magnetic moment is large, being of the order of

\[ |\mathbb{M}_{t\pi}|_{\text{GIM}} = O\left( \frac{G_F}{\sqrt{2}} e(m_{\pi} + m_{\pi})m_{\rho}\sin\theta_C \right). \]

This is analogous to the anomalous magnetic moment of the muon in this model. It is of the order of \( (G_F/\sqrt{2}) \leq m_{\mu} m(Y') \). We do not know of a way of assessing the effects of the transition moment in the \( K\pi \) transitions reliably.

(2) In the Bars-Halpern-Yoshimura model, all almost of the present results should hold also. The reason is this: Since this model incorporates the field algebra, most integrals one encounters are superconvergent, so much so that they remain so even when the replacement (3.26) is made. The remaining integral is scaled by the typical hadronic mass \( m \). Thus the integral has the dependence on \( m_{\pi}^{-2} \), as a naive counting of the number of heavy boson propagators indicates: We expect the amplitude for \( K_L \rightarrow \mu \bar{\mu} \) to be of order \( G_F m_{\pi} \); the amplitudes for \( K_L \rightarrow \gamma \gamma \) of order \( G_F m_{\pi} \), etc.

(3) The situation in the Lee-Prentki-Zumino (LPZ) model should be almost identical to that in the Weinberg-Salam model.

### ACKNOWLEDGMENT


### APPENDIX A: FREE-QUARK MATRIX ELEMENTS

In this appendix, we collect the relevant matrix elements used in the text. These are the expressions correct to lowest order in \( m_{\pi}^{-2} \). The derivations are relegated to other appendices. In the following all calculations are done in the 't Hooft-Feynman gauge.

1. The effective \( \Sigma \lambda Z \) vertex:

\[
\left. i E^\Sigma(q, p) = \mathbb{M}(q)\gamma_\mu \left( \frac{i}{2}(1 - \gamma_5) \right) \lambda(p) \right|_{q^2} \right|_{E^\Sigma(q, p)} = \epsilon \left[ (g_1^2 + g_2^2)^{1/2} \cos\theta_C \sin\theta_C (\alpha/8\pi) \right. \\
\left. \times \epsilon_0 \left[ \ln(m_{\pi} m_{\rho} m_{\pi}^{-2}) - \frac{1}{1 - 1} \right] \right],
\]

where \( \epsilon_0 = \Delta m^2 / m_{\pi}^2 \sin^2\theta_W \), \( \Delta m^2 = m_{\rho}^2 - m_{\pi}^2 \).

2. The \( Z \)-meson propagator:

\[
-i \frac{1}{k^2 - m_Z^2} \approx i \frac{4}{(g_1^2 + g_2^2)t^2},
\]
where $\nu$ is the vacuum expectation value of the Higgs scalar field.

(3) The $Z'$-meson couplings to leptons:

$$\mathcal{L}_{Z'\ell} = \left( g^2 + g'^2 \right)^{1/2} \left( \bar{\nu}_\mu \gamma^\mu \left[ \frac{1}{2}(1 - \gamma_5) \right] \nu_\mu + \bar{\nu}_\mu \gamma^\mu \left[ -\frac{1}{2}(1 - \gamma_5) \right] \sin^2 \theta_w \mu \right),$$

(A3)

and similarly for the electronic leptons.

(4) The effective $\gamma\lambda\lambda$ vertex:

$$iS = \frac{G_F}{\sqrt{2}} \cos\theta_c \sin\theta_w \left[ \frac{1}{2}(1 - \gamma_5) \lambda \left[ -4\bar{\nu}_\mu \left[ \frac{1}{2}(1 - \gamma_5) \right] \nu + \bar{\nu}_\mu \gamma^\mu \left[ \frac{1}{2}(1 - \gamma_5) \right] \mu \right] \left(-\frac{1}{2}i\right) \right].$$

(5) The irreducible $I\lambda\lambda$ vertex:

$$F(\beta) = -\frac{1}{24\beta}, \quad \beta >> 1$$

- $\frac{1}{2}, \quad \beta << 1.$

(A13)

The part of (A8) relevant to $K_L - \gamma\gamma$ may be written as

$$i T_{\rho\omega} = -i \frac{G_F}{\sqrt{2}} \cos\theta_c \sin\theta_w \lambda \left[ \frac{1}{2}(1 - \gamma_5) \lambda \right] \left[ \frac{1}{2}\nu^\rho \nu^\omega \left( \frac{m^2}{m_{\rho\omega}^2 - m_{\rho\omega}^2} \right) \ln \left( \frac{m_{\rho\omega}^2}{m_{\rho\omega}^2 - 1} \right) \right].$$

(A6)

APPENDIX B: THE EFFECTIVE $\gamma\lambda\lambda$ VERTEX

In lowest order the effective $\lambda\lambda\lambda$ vertex is generated by the diagrams shown in Fig. 10(a)-

$$\lambda_i(\ell) \rightarrow \lambda_i + \lambda_j + \lambda_k$$

(a) (b) (c)

$$\lambda \rightarrow W^- + \lambda + \lambda$$

(d)

$$\lambda \rightarrow W^- + \phi^- + \phi^- + W^-$$

(e)

FIG. 10. Diagrams contributing to the effective $\lambda\lambda\lambda$ vertex.
10(c). The off-diagonal mass matrix element $\Sigma(p)$ and the irreducible $\lambda\lambda\gamma$ vertex $\Gamma^{(\lambda\lambda\gamma)}_\mu(q, p)$ are depicted in Fig. 10(d) and Fig. 10(e), respectively. The crosses in Fig. 10 represent the contributions of counterterms.

The renormalization counterterms arise in the following way. We renormalize the left-chiral fermion fields according to

$$\left(\begin{array}{c}
\theta' \\
\rho' \\
\lambda'
\end{array}\right)_L = Z_L^{-1/2} \left(\begin{array}{c}
\theta \\
\rho \\
\lambda
\end{array}\right)_L,$$

and similarly for the right-handed fermion fields,

$$\left(\begin{array}{c}
\theta' \\
\rho' \\
\lambda'
\end{array}\right)_R = Z_R^{-1/2} \left(\begin{array}{c}
\theta \\
\rho \\
\lambda
\end{array}\right)_R,$$

and renormalize coupling constants and gauge bosons according to

$$k' = k Z_1 Z_3^{-1/2},$$

$$k'' = k' Z_1 Z_3^{-1/2},$$

$$A_{\omega} = A_{\omega} Z_3^{-1/2},$$

$$B_{\omega} = B_{\omega} Z_3^{-1/2},$$

and choose the renormalization constants by suitably chosen conventions. We define the Weinberg angle in terms of renormalized constants:

$$\cos \theta_w = g' / (g^2 + g'')^{1/2}.$$

The relevant counterterms in Figs. 10(d) and 10(e) then take the form

$$T = \left[ 1 - \frac{1}{2} - (Q-1)\sin^2 \theta_w \right] L - (Q-1)\cos^2 \theta_w R,$$

$$\Gamma^{(\lambda\lambda\gamma)}_\mu(q, p) - \left[ \Sigma(q) T - T^* \Sigma(p) \right]$$

$$+ i\left[ -\frac{1}{2} - (g^2 + g'')^{1/2} / 2 \right] = 0,$$

where $\Gamma_\lambda$ is the irreducible $\lambda\lambda\gamma$ vertex (see Fig. 11). This follows from the Ward-Takahashi identities for proper vertices in the one-loop approximation, where the effects of the Feynman-DeWitt-Padeev-Popov ghost fields may be neglected. Alternatively it may be derived by the current-algebra technique applied to the source current of the $Z$ meson. In (B2) the matrix $T$ is

$$\left[ \frac{1}{2} (1 - \gamma_0) \right] \left[ \begin{array}{c}
\rho_\gamma \\
\phi_\gamma \\
\lambda_\gamma
\end{array}\right] \times \left[ \begin{array}{c}
\gamma \cdot \sigma \\
\omega \times \gamma
\lambda
\end{array}\right]$$

$$\left( \begin{array}{c}
\rho_\gamma \\
\phi_\gamma \\
\lambda_\gamma
\end{array}\right) = \left( \begin{array}{c}
\gamma \cdot \sigma \\
\omega \times \gamma
\lambda
\end{array}\right) \cdot \left( \begin{array}{c}
\gamma \cdot \sigma \\
\omega \times \gamma
\lambda
\end{array}\right)$$

FIG. 11. The $\lambda\lambda\gamma$ vertex.
\[ \Gamma_\mu(q, p) = \frac{g^4}{4} \left[ \frac{m_\rho^2}{v} \cos \theta_\rho \sin \theta_\rho \ 2(q - p) \cdot \gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \right. \]
\[ \times \int \frac{d^4 \gamma}{(2\pi)^4} \left( \frac{i}{\sqrt{v^2 - m_\rho^2}} \right)^2 \frac{\gamma^\mu}{v^2 - m_\omega^2} \]
\[- \left( m_\rho - m_\omega \right) \cdots. \]  
\[ (B7) \]

The above expression is valid as \( q - p \to 0 \). Thus if \( (q - p)^2 < m_\rho^2 \) or \( m_\rho^2 \), we have

\[ E_\mu^{(2)} = g^2 (g^2 + g'^2) \cos \theta_\rho \sin \theta_\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \lambda \]
\[ \times \frac{1}{16\pi^2} \frac{\Delta m^2}{m_\omega^2} \left( \ln \frac{m_\omega^2}{m_\rho^2} - 1 \right), \]  
\[ (B8) \]

where \( m_\rho \) is the larger of \( m_\rho \) and \( m_\rho' \).

Note further that the counterterms exhibited in \((B1)\) do not contribute to the effective vertex on the mass shell in \((B5)\). This is as it should be, since the counterterms in \((B1)\) can be completely eliminated from the Lagrangian if the strong interactions preserve the weak-interaction symmetry.

\[ iS(\lambda + \bar{\nu} - \mu + \bar{\mu}) \]
\[ = \frac{g^4}{4} \left[ \int \frac{d^4 \gamma}{(2\pi)^4} \left( \bar{\nu} \gamma^\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \right. \frac{i}{v \cdot \gamma - m_\rho} \gamma^\mu \left[ \frac{1}{2}(1 - \gamma_5) \right] \lambda \right. \]
\[ \left. \left( \bar{\mu} \gamma_\alpha \left[ \frac{1}{2}(1 - \gamma_5) \right] \right. \frac{i}{v \cdot \gamma - m_\rho} \gamma^\alpha \left[ \frac{1}{2}(1 - \gamma_5) \right] \lambda \right) \left( \frac{-i}{v^2 - m_\omega^2} \right)^2 \left( m_\rho - m_\omega \right) \left] \right. \]  
\[ iS(\lambda + \bar{\nu} - \mu + \bar{\mu}) \]
\[ = \frac{1}{4} g^4 \left[ \int \frac{d^4 \gamma}{(2\pi)^4} \left( \bar{\nu} \gamma^\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \right. \frac{i}{v \cdot \gamma - m_\rho} \gamma^\mu \left[ \frac{1}{2}(1 - \gamma_5) \right] \lambda \right. \]
\[ \left. \left( \bar{\mu} \gamma_\alpha \left[ \frac{1}{2}(1 - \gamma_5) \right] \right. \frac{i}{v \cdot \gamma - m_\rho} \gamma^\alpha \left[ \frac{1}{2}(1 - \gamma_5) \right] \lambda \right) \left( \frac{-i}{v^2 - m_\omega^2} \right)^2 \left( m_\rho - m_\omega \right) \left] \right. \]  
\[ (C1) \]

where we have neglected the muon mass. The Dirac algebra can be simplified by

\[ \gamma^\rho \gamma^\alpha \gamma^\beta = g^\alpha \beta \gamma^\rho + g^\beta \alpha \gamma^\rho - g^\rho \alpha \gamma^\alpha - i e^\rho \alpha \beta \gamma_\beta \gamma_5, \]
\[ e^\rho \alpha \beta \gamma_\alpha \gamma_\beta = -i \gamma^\rho \gamma^\beta, \]  
and

\[ iS(\lambda + \bar{\nu} - \mu + \bar{\mu}) = -i \left( \frac{1}{4} \right) \frac{g^4}{64\pi^2} \frac{1}{m_\omega^2} \left( \bar{\nu} \gamma^\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \lambda \right) \left( \bar{\mu} \gamma_\alpha \left[ \frac{1}{2}(1 - \gamma_5) \right] \lambda \right) \left( \frac{-i}{x^2 + y^2} \right)^2 \left( \frac{1}{x + y} \right), \]  
where

\[ y = m_\rho^2/m_\omega^2, \]
\[ and \quad y' = m_\rho'^2/m_\omega^2. \]  

\[ \text{APPENDIX D: THE EFFECTIVE } \lambda \mathbb{R} \gamma \text{ VERTEX} \]

If we write the irreducible \( \lambda \mathbb{R} \gamma \) vertex as \( \Gamma^{(\gamma)}(q, p) \), the effective \( \lambda \mathbb{R} \gamma \) vertex \( E^{(\gamma)}(q, p) \) defined similarly to \( E^{(\mu)}(q, p) \), i.e., by the sum of three diagrams \((\text{Figs. 10(a)}-\text{10(c)})\), where the \( Z \) line is replaced by the photon line, is

\[ E^{(\gamma)} = \Gamma^{(\gamma)} - e(q - 1) \gamma \mu L a, \]  
\[ (D1) \]

Equation \((B8)\) refers to on-shell \( n \) and \( \lambda \). The off-shell corrections are of the order \( p^2/m_\omega^2 \), where \( p \) is the off-shell momentum.

\[ \text{APPENDIX C: BOX DIAGRAMS FOR THE PROCESS } \lambda + \mathbb{R} \to \mu \bar{\nu} \]

In the Weinberg-Salam model there are two diagrams each \((\text{see Fig. 12})\) for \( \lambda + \mathbb{R} - \mu + \bar{\mu} \) and \( \lambda + \mathbb{R} - \nu + \bar{\nu} \). The second diagrams are generated from the first by replacing the \( Q \)-quark line by the \( \theta' \). The computation involved is very similar to that of Lee, Primack, and Treiman\(^5\) for the Georgi-Glashow model. The contributions of the diagrams in which one or both of the \( W \) lines are replaced by the unphysical Higgs scalars may be safely neglected, since such diagrams are down by at least one power of \( (m_\rho/m_\omega)^2 \).

In the limit of neglecting external momenta compared to the internal one we have

\[ \{ \gamma^\rho \gamma^\alpha \gamma^\beta \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \]
\[ = 4 \{ \gamma^\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \]  
\[ \{ \gamma^\rho \gamma^\alpha \gamma^\beta \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \]
\[ = 16 \{ \gamma^\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \}. \]  
The resulting expressions are

\[ \{ \gamma^\rho \gamma^\alpha \gamma^\beta \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \]
\[ = 4 \{ \gamma^\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \]
\[ \{ \gamma^\rho \gamma^\alpha \gamma^\beta \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \]
\[ = 16 \{ \gamma^\rho \left[ \frac{1}{2}(1 - \gamma_5) \right] \} \{ \gamma_\gamma \left[ \frac{1}{2}(1 - \gamma_5) \right] \}. \]

\[ (C2) \]
The proper vertex may be expanded in powers of external momenta. The structures of (D2) and (B4) restrict it to be of the form

\[ \Gamma^{(\gamma)}_\mu(p - \frac{1}{2} k, p + \frac{1}{2} k) = e(Q - 1) \gamma_\mu L a + i \sigma_{\mu\nu} k^\nu (m_{\pi} + m_{\omega}) \gamma_\lambda + \gamma^\nu (k^2 g_{\nu\omega} - k_{\nu} k_{\omega}) \frac{1}{6} (r^2)_{\rho\lambda} L + \cdots, \]

where higher-order terms in external momenta are neglected. Thus the effective photon vertex \( E^{(\gamma)}_\mu \) of (D1) consists of two terms—the transition magnetic-moment term \( \Re \gamma_\lambda \) and the transition charge-radius term \( \frac{1}{6} (r^2)_{\rho\lambda} \). We shall compute them in turn. The diagrams contributing to \( \Gamma^{(\gamma)}_\mu \) are shown in Fig. 13.

The computation of the transition magnetic-moment proceeds very similarly to that of the weak contribution to the anomalous magnetic moment in the Weinberg-Salam model. Figures 13(a) and 13(b) are nonexistent for the muon anomalous moment, since the neutrino is electrically neutral. Figure 13(a) contributes to the transition magnetic moment (in the 't Hooft-Feynman gauge):

\[ \Re \gamma_\lambda = -eQ \frac{G_F}{\sqrt{2}} \frac{1}{8\pi^2} \left( m_{\pi} + m_{\omega} \right) \frac{1}{2} \left( 1 + \gamma_5 \right) \ln \frac{m_{\omega}^2}{m_{\pi}^2} \cos \theta_c \sin \theta_c, \]

and Fig. 13(b) does not. Figures 13(e) and 13(f) do not contribute to the transition magnetic moment, and the contributions of Figs. 13(c) and 13(d) are of the order of

\[ \Re \gamma_\lambda = O \left( e \frac{G_F}{\sqrt{2}} \frac{1}{8\pi^2} \left( m_{\pi} + m_{\omega} \right) \frac{\Delta m^2}{m_{\omega}^2} \cos \theta_c \sin \theta_c \right) \]

without the logarithmic factor \( \ln(m_{\omega}^2/m_{\phi}^2) \).

In the 't Hooft-Feynman gauge, the leading contribution to the transition charge radius comes from Fig. 13(a). This could have been guessed at by a dispersion-theoretic argument and an estimate of coupling strengths. To lowest order in \( 1/m_{\pi}^2 \), the \( W \) exchange may be replaced by a \( V-A \) four-fermion interaction. In this limit we have

\[ (ieQ) \frac{G_F}{\sqrt{2}} \cos \theta_c \sin \theta_c [\Re \gamma_\lambda (1 - \gamma_5) \lambda] \]

\[ \times \left[ \text{Tr} \int \frac{d^4 r}{(2\pi)^4} \gamma^\nu \frac{i}{(r + \frac{1}{2} k) \cdot \gamma - m_{\phi}} \gamma_\mu \frac{i}{(r - \frac{1}{2} k) \cdot \gamma - m_{\phi}} - (m_{\phi} - m_{\pi} \gamma) \right]. \]

The integral to be evaluated is exactly the same as for the vacuum polarization in quantum electrodynamics,\(^3^0\) so that we get

\[ \text{FIG. 13.} \text{ Diagrams for the } \lambda \Re \gamma \text{ vertex } \Gamma^{(\gamma)}_\mu. \]

\[ \text{APPENDIX E: IRREDUCIBLE VERTEX FOR } \lambda + \Re \gamma \rightarrow \gamma + \gamma \]

Different kinds of diagrams representing the irreducible vertex for the process \( \lambda + \Re \rightarrow \gamma + \gamma \) are shown in Fig. 14. Let us first pursue the relative magnitude of each of these diagrams.

Take for example Figure 14(a). After the usual Feynman parametrization the relevant integrals take the form

\[ \text{FIG. 14. Some diagrams contributing to the irreducible } \lambda \Re \gamma \gamma \text{ vertex.} \]
\[ I_{abf}(p, k_1, k_2) = \int dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) \int \frac{d^4 R}{(2\pi)^4} \left[ R^2 + f(k_1, k_2, p) + x_1 m_w^2 + (x_2 + x_3 + x_4) m^2 \right]^2, \]

where \( m \) is the mass of the internal fermion line; \( s, t, \) and \( u \) are of the form

\[ s_\alpha = R_\alpha + Xk_1 + Yk_2 + Zp, \]

\( X, Y, \) and \( Z \) being linear homogeneous functions of the Feynman parameters \( x_1, \ldots, x_4. \) The electromagnetic gauge invariance applied to the irreducible vertex states that only the parts of \( I_{abf} \) proportional to three powers of external momenta carrying the Lorentz indices \( \alpha, \beta, \) and \( \gamma \) will survive when all diagrams are summed (the argument here is very similar to that of Rosenberg and Adler, except that the presence of an extra momentum \( p \) considerably complicates the details, which we do not feel worthy of presenting here), so we need consider only integrals of the form

\[ I = \int dx_1 dx_2 dx_3 dx_4 \delta(1 - x_1 - x_2 - x_3 - x_4) \frac{1}{[x_1 m_w^2 + (1 - x_1) m^2]^2} F(x_1, x_2, x_3, x_4) \]

in the low-energy limit (i.e., \( p, k_1, k_2 \to 0 \)). The integral \( I \) in (E1) may appear to be of order \( (m_w^2)^{-4} \) on dimensional grounds, but because of the divergence in the \( x_1 \) integration of the form

\[ \int \frac{dx_1}{x_1^2} \]

when \( m^2 \) is neglected, a more careful treatment

\[ -\left( \frac{i}{\sqrt{2}} \right)^2 (i\varphi \gamma^\mu) \cos \theta_c \sin \theta_c \mathcal{R} \gamma_\alpha \left[ \delta(1 - \gamma_\alpha) \right] \lambda \]

\[ \times \left[ \left( \text{Tr} \int \frac{d^4 \gamma}{(2\pi)^4} \gamma^\mu \left[ \frac{1}{2} (1 - \gamma_\beta) \right] \frac{i}{\gamma - \frac{1}{2} - m} \gamma_\beta \right) \right] \phi_{\nu - m} (\nu, k_1 - \sigma, k_2) \left( m \to m \right) \]

\[ \text{where the relevant part of the interaction Lagrangian is} \]

\[ L_I = \frac{\lambda}{\sqrt{2}} \left[ \lambda_{\mu \nu} \left( \frac{1}{2} (1 - \gamma_\beta) \right) (\theta \sin \theta_c + \theta' \cos \theta_c) W^\mu \right. \]

\[ + \left. \mathcal{R} \gamma_\mu \left[ \frac{1}{2} (1 - \gamma_\beta) \right] (\theta \cos \theta_c - \theta' \sin \theta_c) W^\nu \right) \]

\[ + \text{H.c.}. \]

\[ \text{(F2)} \]

The fourth-order term contains

\[ \lambda_{\mu \nu} \gamma_\alpha \left[ \frac{1}{2} (1 - \gamma_\beta) \right] S_{\mu \nu} (x_1 - x_2; m_\phi) \gamma_\beta \gamma_\gamma \left[ \frac{1}{2} (1 - \gamma_\tau) \right] \mathcal{R} (x_3) \]

\[ \times \bar{\lambda} (x_2) \gamma_\alpha \left[ \frac{1}{2} (1 - \gamma_\beta) \right] S_{\mu \nu} (x_2 - x_3; m_\phi) \gamma_\beta \gamma_\gamma \left[ \frac{1}{2} (1 - \gamma_\tau) \right] \mathcal{R} (x_3) \]

\[ \times i D_{\mu \nu} (x_1 - x_2; m_w^2) i D_{\alpha \beta} (x_2 - x_3; m_w^2) - (m_\phi - m_\tau), \]

\[ \text{(F3)} \]
where the factor 4! in the expansion of the exponential is canceled by the number of ways (\(\frac{1}{4!}\)) the Wick contraction of \(\langle \mathcal{L} \rangle^4\) yields the term on the right-hand side of (F3). The integral implied in (F3) can best be evaluated in momentum space. The relevant integral is of the form

\[
\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m_\phi^2)^2} \frac{1}{(k^2-m^2)} = \frac{\delta}{16\pi m^2} \frac{\epsilon_0}{m_W^2 \sin^2 \theta_W} \left\langle \bar{\psi}(1-\gamma_5) \right\rangle \left\langle \gamma_\alpha \bar{\psi}(1-\gamma_5) \right\rangle,
\]

where \(\epsilon_0 = \frac{\delta}{\left(38 \text{ GeV}\right)^2}\). (F5)

Consequently the effective Lagrangian is

\[
\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{4\pi} \epsilon_0 \sin^2 \theta_W \left\langle \bar{\psi}(1-\gamma_5) \right\rangle \times \left[ \bar{\chi} \gamma_\mu \left[ \bar{\psi}(1-\gamma_5) \right] \chi \right],
\]

where

\[
\delta = m_{W^\pm}^2 \sin^2 \theta_W.
\]
η'(958) branching ratio, linear matrix element, and dipion phase shift*

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(Received 28 February 1974)

The “world” data on η'(958) in regard to the branching ratio $R = (\pi^+ \pi^- \gamma / \pi^+ \pi^- \eta)$, ηγ denoting η(549) → all neutrals, and the dipion mass spectrum in the η' → π+ π- η decay are reviewed. We find that $R = 0.97 \pm 0.08$, independent of energy, exclusive of the data of Aguiller-Benitez et al. Under the assumption that $J^P = 0^-$, the best value of the α parameter of the linear matrix element is $\alpha = -0.08 \pm 0.03$. Possible connections between the dipion mass spectrum and $\delta_{\eta\gamma}$, the $l=J=0$ dipion phase shift, are discussed.

Many studies of the η'(958) have been made. We comment here on the branching ratio $R = (\pi^+ \pi^- \gamma / \pi^+ \pi^- \eta)$, where ηγ represents η(549) → all neutrals, and on the linear matrix element of the η'(958). We base these comments on the data presented in Refs. 1–6; some additional data are not used.7 We obtain the current “world” averages of $R$ and $\alpha$, the parameter characterizing the linear matrix element. We also comment on possible connections with $\delta_{\eta\gamma}$, the $l=J=0$ dipion phase shift. Additional comments on the η'(958) are given in Ref. 8.

BRANCHING RATIO

The data of Refs. 1–6 on the branching ratio $R = (\pi^+ \pi^- \gamma / \pi^+ \pi^- \eta)$ are given in Table I and are shown in Fig. 1 versus the momentum of the incident beam in the experiments. Included is the branching ratio of the $M(953)$ of Ref. 3. The values of the η' branching ratio $R$ from Refs. 1, 2, and 4–6 are consistent; the weighted average is $R = 0.97 \pm 0.08$ and is indicated also in Fig. 1. The value of $R$ for the η'(958) from Ref. 3 is in disagreement with this average value. Since there is no evidence that the $\pi^+ \pi^- \gamma$ and $\pi^+ \pi^- \eta$ states at 958 MeV are different,3 we must attribute the disagreement between $R$ and $R$ (Ref. 3) to some unknown systematic effect in that experiment. The evidence that the $M(953)$ is not identical to η' (958) rests in the $M(\pi^+ \pi^-)$ distribution of the $\pi^+ \pi^- \gamma$ decay mode. The data in the $\rho^0$ region of both the η' (958) and $M(953)$ data samples have the characteristic $\sin^2 \theta_{\pi\gamma}$ distribution of η'(958) events. Therefore, the evidence for an $M(953)$ should be obtained from the $M(\pi^+ \pi^-)$ mass distribution, Fig. 46(a) of Ref. 3, where events containing a $\rho^0(765)$ were excluded. The data yield 37±10 events outside the $\rho$ band, of which about 8–15 should be attributed to the tail of the $\rho^0$ in the $\eta' - \rho^0 \gamma$ decay. Thus, the evidence for an $M(953)$ is about 2 to 3 standard deviations, attributing all the other "M" events to η'(958); then the branching ratio $R$ of the η' in the "M" sample is about $0.8 \pm 0.2$.  

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