# On Transition Probabilities in Double Beta-Disintegration

W. H. FURRY
*Physics Research Laboratory, Harvard University, Cambridge, Massachusetts*
(Received October 16, 1939)

The phenomenon of double $\beta$-disintegration is one for which there is a marked difference between the results of Majorana's symmetrical theory of the neutrino and those of the original Dirac-Fermi theory. In the older theory double $\beta$-disintegration involves the emission of four particles, two electrons (or positrons) and two antineutrinos (or neutrinos), and the probability of disintegration is extremely small. In the Majorana theory only two particles—the electrons or positrons—have to be emitted, and the transition probability is much larger. Approximate values of this probability are calculated on the Majorana theory for the various Fermi and Konopinski-Uhlenbeck expressions for the interaction energy. The selection rules are derived, and are found in all cases to allow transitions with $\Delta i = \pm 1, 0$. The results obtained with the Majorana theory indicate that it is not at all certain that double $\beta$-disintegration can never be observed. Indeed, if in this theory the interaction expression were of Konopinski-Uhlenbeck type this process would be quite likely to have a bearing on the abundances of isotopes and on the occurrence of observed long-lived radioactivities. If it is of Fermi type this could be so only if the mass difference were fairly large ($\epsilon \gtrsim 20$, $\Delta M \gtrsim 0.01$ unit).

## I. INTRODUCTION

THE probability of double $\beta$-disintegration was calculated some years ago by Goeppert-Mayer[1] on the basis of the Fermi theory.[2, 3] The result obtained was extremely small, corresponding to a lifetime of the order of $10^{25}$ years in the case of two isobars whose masses differ by 0.002 mass unit and whose atomic numbers differ by two units. Thus one can account for the large number of isobaric pairs with $\Delta Z = 2$, as compared to the scarcity of isobars with $\Delta Z = 1$. Although not strictly stable, the heavier isobar of a pair with $\Delta Z = 2$ may be supposed to be metastable, having a lifetime large compared to geologic time.

An inspection of the calculations shows that the results would not be changed by any significant factor by the use of an expression for the interaction energy involving derivatives of the neutrino wave function, as suggested by Konopinski and Uhlenbeck.[4] The same is true as regards the generalizations in structure of this expression,[5] which make it possible to obtain selection rules[6] for ordinary $\beta$-decay decidedly different from those originally given by Fermi. The original Fermi picture of the fundamental interaction processes concerned in $\beta$-decay has now been generally supplanted by a picture in which mesons play the role of mediaries between the heavy particles and the electrons and neutrinos. In this case also, as is evident from a consideration of the arguments of Yukawa,[7] the results of Goeppert-Mayer will remain unchanged.

The situation is, however, decidedly altered if one admits a change in the theory of the neutrino itself as an elementary particle. Such a change was suggested by Majorana[8] in a paper on the symmetry properties of the Dirac theory. Majorana's suggestions have been more generally developed in the case of the positron theory by Kramers,[9] and for the neutral particle by the writer.[10] Racah[11] has also discussed the application to the neutrino theory of $\beta$-decay.

The essential difference between the Majorana theory of the neutrino and the usual form of the Dirac theory is that in the former there are only

[1] M. Goeppert-Mayer, Phys. Rev. **48**, 512 (1935).
[2] E. Fermi, Zeits. f. Physik **88**, 161 (1934).
[3] For a review of the theory and its various modifications and applications up to the beginning of 1936, see pages 186–206 of the article by Bethe and Bacher, Rev. Mod. Phys. **8** (1936).
[4] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **48**, 7 (1935).
[5] Cf. reference 3, pp. 190–192.
[6] G. Gamow and E. Teller, Phys. Rev. **49**, 895 (1936).
[7] H. Yukawa, Proc. Phys.-Math. Soc. Japan, **17**, 55–56 (1935); H. Yukawa, S. Sakata, M. Kobayasa and M. Taketani, Proc. Phys.-Math. Soc. Japan **20**, 731–734 (1938).
[8] E. Majorana, Nuovo Cimento **14**, 171 (1937).
[9] H. A. Kramers, Proc. Amsterdam Akad. **40**, 814 (1937).
[10] W. H. Furry, Phys. Rev. **54**, 56 (1938), referred to as *N*.
[11] G. Racah, Nuovo Cimento **14**, 322 (1937).

two states for a given momentum, corresponding to the two possibilities for the spin; there are no negative energy neutrinos, and no "holes," or antineutrinos. If in applying the usual Dirac-Fermi theory to $\beta$-decay one assumes that the emission of a neutrino accompanies that of a positron, then the emission of an electron must be accompanied by the emission of an *antineutrino*, or else the *absorption* of a neutrino. In the Majorana theory the emission of *either* an electron *or* a positron is accompanied by *either* the emission *or* the absorption of a neutrino.

It can be shown that the use of the Majorana form of neutrino theory instead of the usual theory makes no difference in the case of ordinary $\beta$-decay.[12] For the double $\beta$-disintegration, however, there is a marked qualitative difference between the results of the two theories. In the ordinary form of the theory four particles must be emitted in such a process: two neutrinos (or antineutrinos) must accompany the emission of two positrons (or electrons). In the Majorana theory there can occur not only these four-particle disintegrations, but also disintegrations in which only the two charged particles—electrons or positrons—are emitted, unaccompanied by neutrinos. In these two-particle disintegrations the neutrino plays only a transitory or virtual part, such as is played by electron-positron pairs in certain hypothetical radiative processes.[13] Subject to the usual limitations on the meaning of such language, one can say that a (virtual) neutrino is emitted together with one of the electrons (or positrons), and reabsorbed when the other electron (or positron) is emitted.

In calculating the probability amplitude for such a process, one must integrate over the momentum space of the neutrino. This introduces a quantity which is a sort of "form factor" of the nucleus. In most cases the integral converges, giving a finite "form factor" without any arbitrary "cutting off" procedure; one has to resort to "cutting off" only for certain types of Konopinski-Uhlenbeck expressions for the interaction energy. The results obtained for the disintegration probability are, nevertheless, greater than those for the four-particle process by a factor which ranges from $10^5$ to $10^{15}$ or more, depending on the particular interaction expression used. The fact that the neutrino momentum is not limited by the amount of energy available in the transition also makes the probability less strongly dependent on the energy.

In the following section we carry through the calculation for the simplest case, using the "scalar-type" interaction energy expression[14] and assuming that the nuclear spin does not change and that the transition is of even-even or odd-odd type. In Section III we consider the effects of using various other expressions for the interaction energy, and determine the selection rules governing changes in the nuclear spin $i$ and the even-odd or even-even, odd-odd character of the transitions.

Before we continue beyond the introduction it should be remarked that the Majorana form of the theory is not the only one which permits this new type of disintegration. The same sort of result can be obtained with the more usual theory if one introduces an interaction expression involving a linear combination of the neutrino wave function and its conjugate. Such an introduction has never been suggested, however, in the neutrino-antineutrino type of theory. The Majorana theory is the natural one to use for our purpose, both because, in the elimination of the antineutrino, it provides an independent motive for the use of such expressions, and because it provides, so to speak, a canonical form for them.

## II. Scalar-Type Interaction without Derivatives; $\Delta i=0$, Even-Even, Odd-Odd Transitions

We consider the transition of two neutrons in the nucleus into two protons, with the emission of electrons into states having wave functions $\psi_s$ and $\psi_t$ and energies $H_s$ and $H_t$. The formula for the probability amplitude must contain two terms, since either electron may be emitted first. These terms are of opposite signs because of the antisymmetry of the electronic wave functions; whether or not they actually cancel strongly depends on the interaction expression used.

[12] Reference 10, pp. 66–67; but see limitation stated in footnote 22 of reference 10.

[13] Cf. for example M. Delbrück, Zeits. f. Physik **84**, 144 (1933); N. Kemmer, Helv. Phys. Acta, **10**, 112 (1937); H. Euler, Ann. d. Physik **26**, 398 (1936).

[14] Cf. Eq. (208a) of reference 3.

The neutrino wave functions are taken to be expanded in terms of plane waves subject to a periodic boundary condition in a box of volume $V$, as expressed in Eqs. (30) and (31) of $N$. Only the neutrino amplitudes are regarded as quantized, and all other wave functions are treated as $c$-numbers.

Apart from a resonance factor, the square of whose absolute value may be set equal to $(2\pi t/\hbar)\delta(E_N+H_s+H_t-E_M)$, the probability amplitude is

$$a_{N\leftarrow M}=g^2\sum(\mathbf{k})\sum_L\int\Psi_N{}^\dagger\{\sum_i\beta_iQ_i{}^\dagger(\mathbf{r}_i|\mathbf{k})\}\Psi_Ld\tau$$
$$\cdot\int\Psi_L{}'^\dagger\{\sum_j\beta_jQ_j{}^\dagger(\mathbf{r}_j{}'|-\mathbf{k})\}\Psi_M{}'d\tau'$$
$$\cdot\left[\frac{(\psi_t{}^\dagger\beta a(\mathbf{k}))(\psi_s{}^\dagger\beta a(-\mathbf{k}))}{E_M-E_L-H_s-K_k}-\frac{(\psi_s{}^\dagger\beta a(\mathbf{k}))(\psi_t{}^\dagger\beta a(-\mathbf{k}))}{E_M-E_L-H_t-K_k}\right].\quad(1)$$

Here $g$ is Fermi's constant; $K_k$ is the neutrino energy in the virtual intermediate state; and $\psi_M$, $\psi_N$, and $\psi_L$ are the wave functions of the nucleus in the initial, final, and intermediate states. They are functions not only of the space coordinates of all the heavy particles, but also of the four-valued Dirac spin coordinates on which matrices such as $\beta$ operate, and of a two-valued fifth coordinate for each particle which indicates whether it is a neutron or a proton. The indicated integrations must be understood as including summation over the fourth and fifth coordinates. It is on the fifth coordinates that the Fermi operators $Q_i{}^\dagger$ act in such a way as to change a neutron into a proton; if the $i$th particle is already a proton, the operator $Q_i{}^\dagger$ gives the result zero. The operators $\beta_iQ_i{}^\dagger$ are symmetrized by summing over all the particles in the nucleus, and the factor $(\mathbf{r}_i|\mathbf{k})$ appears because the neutrino wave function must be given its value at the position of the heavy particle. The electron wave functions $\psi_s$ and $\psi_t$ are to be evaluated at the "outer edge of the nucleus," as is usual in calculations on $\beta$-decay.[1, 2]

According to the Majorana theory, the four-component neutrino amplitudes $a(\mathbf{k})$ are given by Eqs. (36), (50), and (53) of $N$. We have to substitute from these equations into (1), and then put

$$B_iB_j=0,\quad B_i{}^*B_j=0\quad(2)$$

and

$$B_i(\mathbf{k})B_j{}^*(\mathbf{k})=\delta_{ij},\quad B_i{}^*B_j{}^*=0.\quad(3)$$

Eq. (2) corresponds to the initial condition that no neutrinos are present, and (3) indicates our intention to calculate the process in which no neutrinos are emitted and not that in which two neutrinos are emitted. As indicated in $N$, the calculation can be simplified by using Eq. (51) of $N$ instead of (50), and then discarding all terms in $b_3$, $b_4$, $b_3{}^*$, and $b_4{}^*$. Our conditions (2) and (3) can be satisfied by using instead of Eq. (51) of $N$ just

$$b(\mathbf{k})\rightarrow b(\mathbf{k})\quad(4)$$

for the left-hand factor and

$$b(\mathbf{k})\rightarrow T^*(\mathbf{k})b^*(-\mathbf{k})\quad(4')$$

for the right-hand factor. Then, remembering that in the Majorana form of $\beta$-decay theory it is consistent to regard the factors $(2)^{-\frac{1}{2}}$ given by Eq. (53) of $N$ as absorbed in the disposable constant $g$, we obtain

$$(\psi_t{}^\dagger\beta a(\mathbf{k}))(\psi_s{}^\dagger\beta a(-\mathbf{k}))$$
$$=\sum_{i=1,\,2}(\psi_t{}^\dagger\beta S(\mathbf{k}))_i(\psi_s{}^\dagger\beta S(-\mathbf{k})T^*(-\mathbf{k}))_i$$
$$=\psi_t{}^\dagger\beta S(\mathbf{k})\cdot\tfrac{1}{2}(1+\rho_3)\cdot T^\dagger(-\mathbf{k})S'(-\mathbf{k})\beta'\psi_s{}^*.\quad(5)$$

Here $\rho_3$ is a matrix originally defined by Dirac,[15] and quoted in $N$, Eq. (39). Then we can use the value of $T(\mathbf{k})$ as given in $N$, Eq. (43), and obtain

$$(\psi_t{}^\dagger\beta a(\mathbf{k}))(\psi_s{}^\dagger\beta a(-\mathbf{k}))=\psi_t{}^\dagger\beta S(\mathbf{k})$$
$$\cdot\tfrac{1}{2}(1+\rho_3)S^\dagger(\mathbf{k})A^\dagger\beta'\psi_s{}^*=\psi_t{}^\dagger M\psi_s{}^*.\quad(6)$$

It is readily verified that (6) has the same invariance properties as Eq. (56) of $N$ with respect to changes in the form of the matrices $\alpha_i$, $\beta$, and $S(\mathbf{k})$, and that if the neutrino mass is equal to zero $M$ is a symmetrical matrix, independently of such changes. We shall use the representation of $\alpha_i$, $\beta$ which was originally given by Dirac, and has usually been employed in explicit calculations; it is quoted in $N$, Eqs. (59), (62). We can accordingly substitute Eqs.

[15] P. A. M. Dirac, Proc. Roy. Soc. **A117**, 614 (1928).

(60)–(62) of $N$ into (6); we shall, however, replace both $\epsilon$ and $\epsilon+1$ by $k$, which corresponds to setting the neutrino mass equal to zero. The result is

$$M=-\tfrac{1}{2}k^{-1}\{k+(\boldsymbol{\alpha}\cdot\mathbf{k})\}\cdot i\alpha_y\beta=M'. \tag{7}$$

It is certainly correct to set the neutrino mass equal to zero, since important contributions to the result will come only from large values of the neutrino momentum. This step leads to a formal difficulty, however, because in $N$ the vector $\mathbf{k}$ represents the propagation vector in units $mc/\hbar$, where $m$ is the neutrino mass. We must, accordingly, change the units of $k$; this makes no difficulty, because (7) is homogeneous of degree zero in $k$. We shall from now on take $k$ to be expressed in c.g.s. units, and replace Eq. (31) of $N$ by

$$(\mathbf{r}|\mathbf{k})=V^{-\frac{1}{2}}\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{8}$$

When (7) is substituted into (1), we see that on account of the symmetry of the matrix $M$ the two numerators are equal. The minus sign due to the antisymmetry of the electronic wave function thus causes a strong cancellation, and we get

$$a_{N\leftarrow M}=g^2V^{-1}\sum(\mathbf{k})\sum_L\int\Psi_N{}^\dagger\{\sum_i\beta_iQ_i{}^\dagger\exp(i\mathbf{k}\cdot\mathbf{r}_i)\}\Psi_Ld\tau\cdot\int\Psi_L{}'^\dagger\{\sum_j\beta_jQ_j{}^\dagger\exp(-i\mathbf{k}\cdot\mathbf{r}_j{}')\}\Psi_N{}'d\tau$$
$$\cdot(H_s-H_t)(E_M-E_L-H_s-K_k)^{-1}(E_M-E_L-H_t-K_k)^{-1}\psi_t{}^\dagger M\psi_s{}^*. \tag{9}$$

The main difficulty in the calculation occurs in the evaluation of the factor

$$\Gamma=g^2\sum_L\int\Psi_M{}^\dagger\{\sum_i\beta_iQ_i{}^\dagger\exp(i\mathbf{k}\cdot\mathbf{r}_i)\}\Psi_Ld\tau\cdot\int\Psi_L{}'^\dagger\{\sum_j\beta_jQ_j{}^\dagger\exp(-i\mathbf{k}\cdot\mathbf{r}_j{}')\}\Psi_M{}'d\tau'\cdot f(E_L) \tag{10}$$

where

$$f(E_L)=(E_M-E_L-H_s-K_k)^{-1}(E_M-E_L-H_t-K_k)^{-1}.$$

For values of the neutrino energy small enough to allow us to put $\exp(i\mathbf{k}\cdot\mathbf{r}_i)=1$, the first factor of the summand is

$$\int\Psi_M{}^\dagger\{\sum_i\beta_iQ_i\}\Psi_Ld\tau. \tag{11}$$

This is just the sort of factor which occurs in calculations on single $\beta$-disintegration. It is usually assumed to have the value 1 if the angular momentum selection rule is obeyed, and, of course, the value 0 otherwise. Actually this is a decidedly crude assumption, as has been pointed out by Nordheim and Yost;[16] for moderately heavy nuclei it seems likely that the expression (11) cannot have a value larger than perhaps 0.1. Since, however, the value of $g$ given by Fermi was assigned to agree with experiment on the basis of setting an expression such as (11) equal to 1, we shall not make any very large net error if we use this value of $g$ and assume that there is at least one value of $L$ for which both of the first two factors of the summand in (10) can be set equal to 1. There still remains, however, the question as to how many terms in the summation will yield similarly appreciable contributions to the value of $\Gamma$.

At first one might suppose that there would be a great many such values of $L$, because of the large number of energy levels possessed by any moderately heavy nucleus and because $\mathbf{k}$ is not restricted to small values, so that limitations on the angular momentum of the state $L$ do not appear in the usual way. Such a conclusion, however, would in all probability be erroneous. The nature of nuclear states is such that any appreciable excitation energy is practically certain to be distributed effectively among a considerable number of particles, so that a factor $\exp(i\mathbf{k}\cdot\mathbf{r}_i)$, which involves the coordinates of one particle only, is unlikely to be very effective in removing the orthogonality of two wave functions. The assumption that there is only one intermediate state $L$ which makes an important contribution to the sum is accordingly not an unreasonable one; we shall proceed in this way, with the reservation that our result is perhaps best regarded as a lower limit, from which the correct result should not differ by any very large factor.

[16] L. W. Nordheim and F. L. Yost, Phys. Rev. **51**, 942 (1937). Very recently (Phys. Rev. **56**, 519 (1939)) the matrix elements for a number of light radioactive elements have been calculated explicitly by Wigner.

In accordance with the above considerations, we shall use in our further calculations the simple formula

$$\Gamma = g^2 f(E_L)(4\pi\rho^3/3)^{-2} \iiint_{r<\rho} \exp(i\mathbf{k}\cdot\mathbf{r})d\tau \cdot \iiint_{r'<\rho} \exp(-i\mathbf{k}\cdot\mathbf{r}')d\tau'. \quad (12)$$

Here the integral is over the interior of a three-dimensional sphere of radius $\rho$, the nuclear radius. The factor $(4\pi\rho^3/3)^{-2}$ comes from normalization. The formula has been written on the assumption that the angular momentum of the state $L$ is the same as that of $M$ and $N$, but this assumption has actually rather little to do with our final result. If, for example, $i_L - i_M = i_L - i_N = \pm 1$, we should have to use some such expression as

$$\Gamma' = g^2 f(E_L)(4\pi\rho^3/3)^{-2}\cdot 3 \cdot \iiint_{r<\rho} \cos\theta \cdot \exp(i\mathbf{k}\cdot\mathbf{r})d\tau \cdot \iiint_{r'<\rho} \cos\theta' \exp(-i\mathbf{k}\cdot\mathbf{r}')d\tau', \quad (12')$$

and we shall verify later that this makes no decided difference.

Further simplification in Eq. (9) can easily be made. Since $k$ is in c.g.s. units, the energy of the neutrino is $K_k = \hbar ck$. The main contributions to (9) come from values of $k$ so large that the other energies can be neglected compared to $K_k$; the occurrence of a vanishing denominator is of course ruled out by the postulated impossibility of single $\beta$-disintegration for the nucleus in question. Accordingly we set

$$f(E_L) \doteq K_k^{-2} = (\hbar c)^{-2} k^{-2}. \quad (13)$$

The sum over the allowed values of $\mathbf{k}$ in the box of volume $V$ can be replaced by an integral in the usual way:

$$\sum(\mathbf{k}) \cdots = (2\pi)^{-3} V \int_0^\infty k^2 dk \iint d\Omega \cdots. \quad (14)$$

Substituting (12), (13) and (14) in (9), we obtain

$$a_{N\leftarrow M} = -(2\pi)^{-3}(g/\hbar c)^2 C^{0,0}\cdot(H_s - H_t)\psi_t^\dagger \cdot \tfrac{1}{2} i\alpha_y \beta \psi_s^*, \quad (15)$$

where

$$C^{0,0} = \int_0^\infty dk \iint d\Omega \left| (3/4\pi\rho^3) \iiint_{r<\rho} \exp(i\mathbf{k}\cdot\mathbf{r})d\tau \right|^2 = (9/4\pi\rho^6)\int_0^\infty dk \left| \iiint_{r<\rho} e^{ikz} d\tau \right|^2; \quad (16)$$

the superscripts 0,0 refer to the values of $i_L - i_M$ and $i_L - i_N$. The second term in (7) is omitted from (15), since it makes no contribution for even-even or odd-odd transitions.

Following Goeppert-Mayer, we express the energies of the electrons and the total available energy in terms of dimensionless quantities: $H_s = h_s mc^2$, $H_t = h_t mc^2$, $E_M - E_N = \epsilon mc^2$, where $m$ is the mass of an electron. Let us also put $p_s = (h_s^2 - 1)^{\frac{1}{2}}$, $p_t = (h_t^2 - 1)^{\frac{1}{2}}$, and make the abbreviation

$$R_s = (16\pi/|\Gamma(3+2S)|^2)(mc/h)^3(2mc\rho p_s/\hbar)^{2S} \cdot \exp(\pi\gamma h_s/p_s)|\Gamma(1+S+i\gamma h_s/p_s)|^2, \quad (17)$$

where $\gamma = Z/137$, $S = (1-\gamma^2)^{\frac{1}{2}} - 1$. We follow Goeppert-Mayer in replacing $R_s$ by the following approximation, valid because $|S| \ll 1$:

$$R_s \cong (16\pi/|\Gamma(3+2S)|^2)(mc/h)^3(2mc\rho/\hbar)^{2S} \cdot 2\pi\gamma h_s/p_s. \quad (17')$$

A similar definition and approximation hold for $R_t$.

When the resonance factor is included, the squared amplitude for the transition to two particular states $s$ and $t$ is $(2\pi t/\hbar)|a_{N\leftarrow M}|^2 \delta(\epsilon mc^2 - h_s mc^2 - h_t mc^2)$. The total probability per unit time for transitions in which one electron gets an energy between $h_s mc^2$ and $(h_s + dh_s)mc^2$ and the other receives the rest of the energy $\epsilon mc^2$ is then

$$P(h_s)dh_s = (2\pi/\hbar mc^2)(dh_t)^{-1} \sum_{dh_t} \sum_{dh_s} |a_{N\leftarrow M}|^2. \quad (18)$$

Here the symbol $\sum_{dh_t}$ means the sum over all states having energy between $h_t mc^2$ and $(h_t+dh_t)mc^2$; the result is of course proportional to $dh_t$. Using the Dirac wave functions for an electron in the Coulomb field, evaluated at $r=\rho$, we obtain, to the lowest order in $\gamma^2$:

$$\sum_{dh_s}\sum_{dh_t}|\psi_t^\dagger\cdot\tfrac{1}{2}i\alpha_y\beta\psi_s^*|^2 = \tfrac{1}{4}R_sR_tp_sp_t(h_sh_t-1)dh_sdh_t. \quad (19)$$

Then

$$P(h_s)dh_s = 2^{-7}\pi^{-5}g^4\hbar^{-5}mc^{-2}|C^{0,0}|^2 \times R_sR_tp_sp_t(h_sh_t-1)(h_s-h_t)^2dh_s. \quad (20)$$

The energy spectrum of the electrons is, of course, symmetrical around the value $\frac{1}{2}\epsilon mc^2$. The factor $(h_s-h_t)^2$ in (20), which makes the intensity vanish for the energy $\frac{1}{2}\epsilon mc^2$, comes directly from the cancellation due to the antisymmetry of the electronic wave function.

To get the total probability of disintegration per unit time, we must integrate (20) from $h_s=1$ to $h_s=\epsilon-1$. According to the approximation (17′) the quantity $R_sp_s/h_s$ is essentially independent of $h_s$. This enables us to write for the total probability of disintegration per unit time:

$$P_1=\int_1^{\epsilon-1}P(h_s)dh_s = 2^{-7}\pi^{-5}g^4\hbar^{-5}mc^{-2}|C^{0,0}|^2(R_sp_s/h_s)^2\cdot\int_1^{\epsilon-1}h_s(\epsilon-h_s) \times\{h_s(\epsilon-h_s)-1\}(\epsilon-2h_s)^2dh_s. \quad (21)$$

Then

$$P_1=2^{-7}\pi^{-5}g^4\hbar^{-5}mc^{-2}|C^{0,0}|^2(R_sp_s/h_s)^2\cdot\varphi(\epsilon-2), \quad (22)$$

where

$$\varphi(x)=(x^4/3)\{1+(11x/10)+(x^2/5)+(x^3/70)\}. \quad (23)$$

We shall now evaluate $P_1$ numerically for $Z=31$, the value used as an example by Goeppert-Mayer. Taking $\rho=8\times10^{-13}$ cm, we get from (17′) the value $R_sp_s/h_s=1.52\times10^{30}$ cm$^{-3}$. The integral (16) can be evaluated by a rather tedious elementary calculation, or more elegantly by means of Fourier's integral theorem:

$$\int_0^\infty dk\left|\iiint_{r<\rho}e^{ikz}d\tau\right|^2=\tfrac{1}{2}\int_{-\infty}^\infty dk\cdot\iiint_{r<\rho}e^{ikz}dzdxdy\cdot\iiint_{r'<\rho}e^{-ikz'}dz'dx'dy'=\pi\int\int\int\int\int dxdydx'dy'dz,$$

where in the last expression the limits for $x'$ and $y'$ depend on the value of $z$ in the same way as those for $x$ and $y$. Then

$$\int_0^\infty dk\left|\iiint_{r<\rho}e^{ikz}d\tau\right|^2=\pi\int_{-\rho}^{\rho}\pi^2(\rho^2-z^2)^2dz=16\pi^3\rho^5/15.$$

Thus

$$C^{0,0}=12\pi^2/5\rho. \quad (24)$$

A similar calculation shows that if (12′) were used instead of (12) we should have simply to replace $C^{0,0}$ by a quantity $C^{1,1}=9\pi^2/5\rho=(3/4)C^{0,0}$. Using $g=4\times10^{-50}$ erg cm$^3$, we obtain finally

$$P_1=1.07\times10^{-28}\varphi(\epsilon-2)\ \text{sec}^{-1}. \\ =3.4\times10^{-21}\varphi(\epsilon-2)\ \text{year}^{-1}. \quad (25)$$

Some values of $\varphi$ are as follows:

| $\epsilon=$ | 3 | 4 | 6 | 8 | 10 | 12 | 20 | |
|---|---|---|---|---|---|---|---|---|
| $\varphi(\epsilon-2)=$ | 0.77 | 22 | $8.1\times10^2$ | $7.7\times10^3$ | $4.1\times10^4$ | $1.54\times10^5$ | $5.9\times10^6$ | (26) |

For small values of $\epsilon$ the $P$ found here is about $10^6$ times that given by Goeppert-Mayer. For larger values of $\epsilon$ the ratio is not so large. This is natural, since for large $\epsilon$ there is plenty of energy available for the emission of neutrinos along with the electrons.

## III. Other Interaction Expressions

In the preceding section we calculated the value of $P$ by using the scalar-type expression for the interaction energy. We now want to consider what would be the effect of using any of the various other energy expressions which have been proposed.

Since we are interested only in the order of magnitude of $P$, we shall neglect the effect of using a different matrix instead of $\frac{1}{2}i\alpha_y\beta$ in calculating the expression corresponding to (19). This would at most replace the factor $(h_sh_t-1)$ by some factor such as $h_sh_t$ or $(h_sh_t+1)$, and for $\epsilon-2\gtrsim 1$ such a change amounts to only a factor of perhaps two in the answer. Hence we shall simply keep the factor $(h_sh_t-1)$ throughout.

To begin with we shall consider the two main differences in the size of $P$ which can arise from changes in the interaction energy expression. We shall then list the various expressions and indicate the main types of terms which they give in the integrand of the equation corresponding to Eq. (1) in the general case. This classification makes it possible to state the selection rules for the different types of interaction expression.

### Differences in order of magnitude of $P$

The first important difference which may come from the use of a different interaction expression is the absence of the strong cancellation due to the antisymmetry of the electronic wave function which occurred in Section II. If the matrix[17] $M$ is not purely symmetric, the main contribution to $a_{N\leftarrow M}$ will come from the antisymmetric part of $M$, $M_A=\frac{1}{2}(M-M')$. Eq. (15) is then replaced by

$$a_{N\leftarrow M}=(2\pi)^{-3}(g^2/\hbar c)D^{0,0}\psi_t{}^\dagger M_A\psi_s{}^* \quad (27)$$

and Eq. (20) by

$$P(h_s)dh_s=2^{-7}\pi^{-5}g^4\hbar^{-3}m^{-1}c^{-4}|D^{0,0}|^2 \times R_sR_tp_sp_t(h_sh_t-1)dh_s, \quad (28)$$

where

$$D^{0,0}=\int_0^\infty 2kdk\int\int d\Omega \times\left|(3/4\pi\rho^3)\int\int\int_{r<\rho}\exp(i\mathbf{k}\cdot\mathbf{r})d\tau\right|^2. \quad (29)$$

[17] In general it is actually a matrix *function* which has to be used, as is indicated below in Eq. (45). In the case treated in Section II this function could conveniently be split into factors, only one of which is a matrix (Cf. Eq. (46)). In any case it is only the symmetry or antisymmetry under the interchange of two four-valued indices which comes into question in determining the presence or absence of the cancellation.

The spectrum now has a maximum at $h_s=\frac{1}{2}\epsilon$, instead of vanishing there.

The total probability of disintegration per unit time is now

$$P_2=2^{-7}\pi^{-5}g^4\hbar^{-3}m^{-1}c^{-4}|D^{0,0}|^2 \times(R_sp_s/h_s)^2\cdot\chi(\epsilon-2), \quad (30)$$

where

$$\chi(\epsilon-2)=\int_1^{\epsilon-1}h_s(\epsilon-h_s)\{h_s(\epsilon-h_s)-1\}dh_s$$

or

$$\chi(x)=x^2\{1+(7x/6)+(x^2/3)+(x^3/30)\}. \quad (31)$$

Integration of (29) by elementary methods gives

$$D^{0,0}=18\pi/\rho^2. \quad (32)$$

Then

$$P_2=1.43\times10^{-24}\chi(\epsilon-2)\text{ sec.}^{-1} =4.6\times10^{-17}\chi(\epsilon-2)\text{ year}^{-1}. \quad (33)$$

| $\epsilon=$ | 3 | 4 | 6 | 8 | 10 | 12 | 20 | |
|---|---|---|---|---|---|---|---|---|
| $\chi(\epsilon-2)=$ | 2.5 | 19.7 | $2.1\times10^2$ | $9.8\times10^2$ | $3.1\times10^3$ | $7.9\times10^3$ | $1.05\times10^5$ | (34) |

Another important effect on the disintegration probability is obtained if we use a Konopinski-Uhlenbeck form of interaction expression. The use of such an expression is subject to certain theoretical difficulties,[18] and it has been suggested[19] that the experimental results can perhaps be fully explained without using an interaction expression involving derivatives; nevertheless it is not without interest to see what such expressions give for the double $\beta$-disintegration.

The use of a Konopinski-Uhlenbeck expression means essentially the insertion of a factor $k^2$ into the integrand in (16) or (29), and the replacement of the constant $g$ by a constant which differs from it by the dimensions of a length. The length by which $g$ is multiplied to give the new constant should be $(2\pi)^{-1}$ times a typical de Broglie wave-length of the neutrino in an ordinary $\beta$-disintegration, in order that the probabilities for single $\beta$-disintegration may be kept the same. Hence we shall replace $g$ by $(\hbar/2mc)g$. The total disintegration probability per unit time is then

$$P_3=2^{-11}\pi^{-5}g^4\hbar^{-1}m^{-3}c^{-6}|F^{0,0}|^2 \times(R_sp_s/h_s)^2\cdot\varphi(\epsilon-2), \quad (35)$$

[18] Cf. M. Fierz, Helv. Phys. Acta **10**, 123, 284 (1937).
[19] H. A. Bethe, F. Hoyle and R. Peierls, Nature **143**, 200 (1939).

where

$$F^{0,0}=\int_0^\infty k^2dk\int\int d\Omega \times\left|(3/4\pi\rho^3)\int\int\int_{r<\rho}\exp\,(i\mathbf{k}\cdot\mathbf{r})d\tau\right|^2 \quad (36)$$

if $M$ is a pure symmetric matrix. If $M$ is asymmetric we get

$$P_4=2^{-11}\pi^{-5}g^4\hbar m^{-5}c^{-8}|G^{0,0}|^2 \times(R_sp_s/h_s)^2\chi(\epsilon-2), \quad (37)$$

where

$$G^{0,0}=\int_0^\infty 2k^3dk\int\int d\Omega \times\left|(3/4\pi\rho^3)\int\int\int_{r<\rho}\exp\,(i\mathbf{k}\cdot\mathbf{r})d\tau\right|^2. \quad (38)$$

The integral (36) can be evaluated elegantly by using Fourier's integral theorem:

$$F^{0,0}=(3/4\pi\rho^3)^2\int_{-\infty}^{\infty}dk_x\int_{-\infty}^{\infty}dk_y\int_{-\infty}^{\infty}dk_z \cdot\int\int\int_{r<\rho}\exp\,(i\mathbf{k}\cdot\mathbf{r})dxdydz \cdot\int\int\int_{r'<\rho}\exp\,(-i\mathbf{k}\cdot\mathbf{r}')dx'dy'dz'$$

$$=(3/4\pi\rho^3)^2\cdot(2\pi)^3\cdot\int\int\int_{r<\rho}dxdydz=6\pi^2/\rho^3. \quad (39)$$

Numerical computation then gives:

$$P_3=2.28\times10^{-22}\varphi(\epsilon-2)\text{ sec.}^{-1} =7.17\times10^{-15}\varphi(\epsilon-2)\text{ year}^{-1}. \quad (40)$$

The integral (38) diverges logarithmically, so that in order to obtain a finite value it is necessary to "cut off" at some value of $k$. It is generally assumed that the value at which the theory becomes unreliable and the cutting off should be done is about $mc^2/e^2$. In evaluating $G^{0,0}$ it is particularly convenient to cut off at a value which makes $k\rho=\pi$. One then readily finds by elementary methods that

$$G^{0,0}=(72\pi/\rho^4)\left\{\int_0^\pi \sin^2 y\cdot dy/y-\tfrac{1}{2}\right\} \cong 162/\rho^4. \quad (41)$$

The cut-off occurs rather beyond the first maximum of the integrand. The range $0<k<\pi/\rho$ includes the essential contributions to the integrals $C^{0,0}$, $D^{0,0}$, and $F^{0,0}$, so that their values are genuinely independent of whether one cuts off or not.

Using the value (41), one obtains

$$P_4=3.99\times10^{-18}\chi(\epsilon-2)\text{ sec.}^{-1} =1.26\times10^{-10}\chi(\epsilon-2)\text{ year}^{-1}. \quad (42)$$

We have now listed four typical formulas for the disintegration constant $P$. Next we must classify the various possible expressions for the interaction energy in such a way as to show what value of $P$ each would give.

**Classification of interaction energy expressions**

By methods analogous to those used in deriving Eq. (7) one can readily calculate the value of $a_{N\leftarrow M}$ corresponding to any particular expression for the interaction energy. The formulas obtained are, however, for the most part very long and cumbersome. Accordingly we shall indicate only the typical characteristics of the dominant terms, which suffice to determine the order of magnitude of the disintegration constant and to determine the selection rules.

Usually in calculations using the neutrino theory of $\beta$-decay one neglects terms involving the small components of the wave functions of the heavy particles compared to those which involve only the large components. This is done because the small components are of order $\bar{v}/c$ compared to the large, $\bar{v}$ being a typical velocity of a heavy particle, and $\bar{v}/c$ is only about 0.2. For our present purposes we can neglect such terms usually, but not always; if the terms involving only the large components give purely symmetric matrices $M$ and other terms give asymmetric matrices, then the latter will be more important because of the absence of any strong cancellation. This situation arises with Fermi's original "vector-type" interaction expression.

Usually only the product of the time components is used in calculations, because the product of the space components brings in small components of the heavy particle wave functions. In our present case, however, such a procedure gives a symmetric matrix $M$, and the space components appear in the actual dominant terms.

To avoid useless multiplication of the number of types listed, we shall write the typical expressions in terms of just the matrices 1 and $\boldsymbol{\sigma}$ for the heavy particles, whose wave functions may then be thought of as two-component Pauli wave functions Any occurrence of the small components in the dominant terms will then be indicated by a suitable power of $(\bar{v}/c)$. Explicitly, so far as the heavy particles are concerned we set

$$\beta\rightarrow 1,\quad \gamma_5=\rho_1\rightarrow(\bar{v}/c)$$
$$\beta\boldsymbol{\sigma}\rightarrow\boldsymbol{\sigma},\quad \boldsymbol{\alpha}\rightarrow(\bar{v}/c)\boldsymbol{\sigma},\quad \beta\boldsymbol{\alpha}\rightarrow(v/c)\boldsymbol{\sigma}. \tag{43}$$

We use the abbreviations:

$$\begin{aligned}\Psi_L{}^\dagger\{\sum_i Q_i{}^\dagger(\mathbf{r}_i|-\mathbf{k})\}\Psi_M&=a_1,\\ \Psi_L{}^\dagger\{\sum_i\boldsymbol{\sigma}_iQ_i{}^\dagger(\mathbf{r}_i|-\mathbf{k})\}\Psi_M&=\mathbf{b}_1,\\ \Psi_N{}^\dagger\{\sum_i Q_i{}^\dagger(\mathbf{r}_i|\mathbf{k})\}\Psi_L&=a_2,\\ \Psi_N\{\sum_i\boldsymbol{\sigma}_iQ_i{}^\dagger(\mathbf{r}_i|\mathbf{k})\Psi_L&=\mathbf{b}_2.\end{aligned} \tag{44}$$

The quantity $a_{N\leftarrow M}$ can always be expressed in the form

$$a_{N\leftarrow M}=g^2\sum(\mathbf{k})\int d\tau\int d\tau'\sum_L\sum_{lm}\psi_{tl}{}^*\times\left\{\frac{\Lambda_{lm}{}^L(\mathbf{r},\mathbf{r}';\mathbf{k})}{E_M-E_L-H_s-K_k}-\frac{\Lambda_{ml}{}^L(\mathbf{r},\mathbf{r}';\mathbf{k})}{E_M-E_L-H_t-K_k}\right\}\psi_{sm}{}^*. \tag{45}$$

In the case treated in Section II the matrix function $\Lambda$ was simply

$$\Lambda(\mathbf{r},\mathbf{r}';\mathbf{k})=a_2(\mathbf{r};\mathbf{k})M(\mathbf{k})a_1(\mathbf{r}';\mathbf{k}), \tag{46}$$

with $M$ given by Eq. (7). Eq. (9) gives the result of substituting (46) and (44) in (45), provided (43) is taken into account.

We now list the types of expression which can appear as the dominant terms of the matrix function $\Lambda$:

*Symmetric Terms:*

$$S_1=a_2\cdot i\alpha_y\beta a_1, \tag{47a}$$

$$S_2=a_2(\boldsymbol{\alpha}\cdot\mathbf{n})i\alpha_y\beta a_1\quad(\mathbf{n}=\mathbf{k}/k). \tag{47b}$$

Many other types of symmetric term occur, but only in combination with antisymmetric terms which give the dominant contributions to the result.

*Antisymmetric Terms:*

$$A_1=i([\mathbf{b}_1\times\mathbf{b}_2]\cdot\boldsymbol{\sigma})i\alpha_y\beta, \tag{48a}$$

$$A_2=i([\mathbf{b}_1\times\mathbf{b}_2]\cdot\mathbf{n})\rho_1 i\alpha_y\beta, \tag{48b}$$

$$A_3=i([\mathbf{b}_1\times\mathbf{b}_2]\cdot\mathbf{n})(\mathbf{n}\cdot\boldsymbol{\sigma})i\alpha_y\beta, \tag{48c}$$

$$A_4=\beta([\mathbf{b}_1\times\mathbf{n}]\cdot[\mathbf{b}_2\times\mathbf{n}])i\alpha_y\beta, \tag{48d}$$

$$A_5=i([(a_1\mathbf{b}_2-a_2\mathbf{b}_1)\times\mathbf{n}]\cdot\boldsymbol{\sigma})i\alpha_y\beta. \tag{48e}$$

We give the different expressions for the interaction energy the same designations as used by Bethe and Bacher.[3] The types of the dominant terms in $\Lambda$ and the order of magnitude of the disintegration constant $P$ are given in the following list:

*Fermi Expressions:*

Scalar: $\Lambda\sim S_1,\ S_2;\quad P\approx P_1,$ (49a)

Vector: $\Lambda\sim(\bar{v}/c)A_5;\quad P\approx(\bar{v}/c)^2P_2,$ (49b)

Tensor: $\Lambda\sim A_1,\ A_2;\quad P\approx P_2,$ (49c)

Pseudovector: $\Lambda\sim A_1,\ A_2;\quad P\approx P_2,$ (49d)

Pseudoscalar:

$$\Lambda\sim(\bar{v}/c)^2S_1,\ (\bar{v}/c)^2S_2;\quad P\approx(\bar{v}/c)^4P_1. \tag{49e}$$

*Konopinski-Uhlenbeck Expressions:*

Vector: $\Lambda\sim qS_1,\ qS_2;\quad P\approx P_3,$ (50a)

Tensor: $\Lambda\sim qA_2,\ qA_3;\quad P\approx P_4,$ (50b)

Pseudovector:

$$\Lambda\sim qA_1,\ qA_2,\ qA_3,\ qA_4;\quad P\approx P_4. \tag{50c}$$

Here

$$q=k^2(\hbar/2mc)^2 \tag{51}$$

is the essential factor which distinguishes the Konopinski-Uhlenbeck from the Fermi formulas, and which was taken into account in calculating $P_3$ and $P_4$.

**Selection rules**

The expressions

$$\sum_L \psi_t^\dagger S_j \psi_s^* \quad \text{and} \quad \sum_L \psi_t^\dagger A_j \psi_s^*$$

are all scalars with respect to the three-dimensional rotation-reflection group: so far as transformation properties are concerned we can think of $i\alpha_y\beta\psi_s^*$ as replaced by a function $\psi_\sigma$.[20] To find the selection rules corresponding to the expressions (47) and (48) we may inspect the transformation properties of either the factors involving $a_n$, $\mathbf{b}_n$, and $\mathbf{n}$ or the matrix factors, apart from the factors $i\alpha_y\beta$. We obtain:

$S_1$ and $A_4$: Scalars: $\Delta i=0$; odd-odd and even-even transitions.

$A_2$: Scalar, but with reflection character $-1$: $\Delta i=0$; odd-even and even-odd transitions.

$S_2$: Polar vector: $\Delta i=0$ or $\pm 1$, but not $0\rightarrow 0$; odd-even and even-odd transitions.

$A_1$, $A_3$, and $A_5$: Axial vectors: $\Delta i=0$ or $\pm 1$, but not $0\rightarrow 0$; odd-odd and even-even transitions.

The selection rules for each interaction expression can then be read off from (49) or (50). The selection rules in general bear more resemblance to those of Gamow and Teller than to those of Fermi, since some $\Delta i=\pm 1$ transitions are always allowed. Transitions forbidden by these rules can occur only if at least one electron is emitted in a state with $j\neq\frac{1}{2}$, and their probabilities will be decidedly smaller than those given by (49) and (50). For $\epsilon-2\gtrsim 2$ the probabilities will be decreased by a factor of roughly $\{(\epsilon-2)\rho mc/2\hbar\}^2$ for a change in parity type of the transition or for each extra unit of $\Delta i$. For smaller values of $\epsilon-2$ the decrease will be more drastic.

[20] Cf. discussion in $N$, p. 66; also W. H. Furry, Phys. Rev. **51**, 125 (1937), Eq. (8) with $A=i\alpha_y\beta$. The free electron functions form a complete set.

**Dependence of transition probability on atomic number**

The numerical results which we have given were all computed for $Z=31$, $\rho=8\times 10^{-13}$ cm. The quantity $(R_s p_s/h_s)$ is roughly proportional to $Z$, and $\rho$ may be taken to be proportional to $Z^{\frac{1}{3}}$. Approximate relations for changing the results to different values of $Z$ or a different estimate of the nuclear radius are, accordingly:

$$P_1 \propto Z^2\rho^{-2} \propto Z^{4/3}, \quad P_2 \propto Z^2\rho^{-4} \propto Z^{\frac{2}{3}},$$
$$P_3 \propto Z^2\rho^{-6} \propto \text{cons.}, \quad P_4 \propto Z^2\rho^{-8} \propto Z^{-\frac{2}{3}}. \quad (52)$$

The results as calculated are for emission of two negative electrons. In the case of positron emission the probabilities will be smaller, because $R_s$ contains an additional factor $\exp(-2\pi\gamma h_s/p_s)$, and $R_t$ a similar factor. These factors decidedly complicate the evaluation of the integrals from which the functions $\varphi$ and $\chi$ have to be determined. It is evident, however, that for $\epsilon-2\gtrsim 2$ the decrease will be by a factor not much smaller than $e^{-4\pi\gamma}=e^{-0.092Z}$. As is the case for the forbidden transitions, the decrease is more decided for smaller values of $\epsilon-2$.

## IV. Conclusion

We have seen that the phenomenon of double beta-disintegration is one for which there is a decided difference between the results of the Majorana theory and those of the older theory of the neutrino. According to the older theory it seemed certain that double beta-disintegration could never be capable of observation because of its extremely minute probability, but the Majorana theory indicates that this is by no means necessarily the case. Indeed, if the interaction expression were of Konopinski-Uhlenbeck type this process would be quite likely to have a bearing on the abundances of isotopes and on the occurrence of observed long-lived radioactivities. If it is of Fermi type this could be so only if the mass difference were fairly large ($\epsilon\gtrsim 20$, $\Delta M\gtrsim 0.01$ unit).